On the Number of Ground States of the Edwards-Anderson Spin Glass Model

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joint with

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The Edwards-Anderson Model

Let $G_N = (V_N, E_N)$ be a graph on $N$ vertices.

We define the Ising spin glass Hamiltonian on $\Sigma_N = \{-1, +1\}^N$:

$$H_{N,J}(\sigma) = - \sum_{(x,y) \in E_N} J_{xy} \sigma_x \sigma_y .$$

where $J = (J_{xy}; (x, y) \in E_N)$ i.i.d. of law $\nu$ Gaussian (say)
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where $J = (J_{xy}; (x, y) \in E_N)$ i.i.d. of law $\nu$ Gaussian (say)

- Covariance $\int \nu(dJ) H_{N,J}(\sigma) H_{N,J}(\sigma') = \sum_{(x,y) \in E_N} \sigma_x \sigma_y \sigma'_x \sigma'_y$
- Edge overlap $R_N(\sigma, \sigma') = \frac{1}{|E_N|} \sum_{(x,y) \in E_N} \sigma_x \sigma_y \sigma'_x \sigma'_y$

- Sherrington-Kirkpatrick model: $G_N$ is the complete graph.
- Edwards-Anderson model: $G_N$ is a box of $\mathbb{Z}^d$.

”Describe” the minima of $H_{N,J}$ for $N$ large.
The Gibbs Measure of the SK model

\[ G_{\beta,N,J}(\sigma) = \frac{\exp -\beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)} \quad \text{as } N \to \infty \]

The order parameter is

\[ x_\beta(q) = \lim_{N \to \infty} \int \nu(dJ) \ G_{\beta,N,J}^{\times 2} \{ R_N(\sigma,\sigma') \leq q \} \]

More and more things are proved:

- **Parisi formula**: free energy is a variational formula over c.d.f. \( x \).
- **Phase transition**: for \( \beta > \beta_c = 1 \), \( x_\beta(q) \) has more than one jump.
- **Parisi Ultrametricity Conjecture**:
  Infinite number of pure states with ultrametric overlaps

\[ G_{\beta,N,J}^{\times 3} \left\{ R_N(\sigma,\sigma') \geq \min\{ R_N(\sigma',\sigma'') ; R_N(\sigma'',\sigma') \} \right\} \to 1 \quad \text{in } \nu\text{-prob.} \]
The Gibbs Measure of the EA model on $\mathbb{Z}^d$

$$G_{\beta,N,J}(\sigma) = \frac{\exp -\beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)}$$ as $N \to \infty$?

General results applies

- **DLR equations:**
  $$\mathcal{G}_d(\beta, J) = \text{set of Gibbs measures on } \{-1, +1\}^{\mathbb{Z}^d} \text{ at } \beta \text{ and } J$$

- **Pure states:** elements of $\text{ext } \mathcal{G}_d(\beta, J)$.
- **$\mathcal{N}_d(\beta) = |\text{ext } \mathcal{G}_d(\beta, J)|$** is a constant $\nu$-a.s.!
- **High Temp./Low $\beta$:** $\mathcal{N}_d(\beta) = 1$. 
The Gibbs Measure of the EA model on $\mathbb{Z}^d$

$$G_{\beta,N,J}(\sigma) = \frac{\exp - \beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)}$$ as $N \to \infty$?

Low temperature: (almost)-everything is unknown
- $d < d_c$: No phase transition $\mathcal{N}_d(\beta) = 1$?
- $d \geq d_c$, $\beta$ large: Phase transition $\mathcal{N}_d(\beta) > 1$?

Droplet Scenario:
Phase transition of Ising-type
  - $\mathcal{N}_d(\beta) = 2$

RSB Scenario:
Phase transition of SK-type
  - $\mathcal{N}_d(\beta) = \infty$
  - Ultrametric overlaps
Ground States of EA model for finite $N$

Instead of studying the pure states, we study the **ground states**:

$$\beta \to \infty \text{ then } N \to \infty .$$

Let

$$\sigma^*_N(J) = \arg \min_{\sigma \in \Sigma_N} H_{N,J}(\sigma)$$

- The minimizer (ground state) is unique because $\nu$ is continuous.

$$H_{N,J}(\sigma) = - \sum_{(x,y) \in E_N} J_{xy} \sigma_x \sigma_y .$$

- Typically, $\sigma^*_N(J)$ do not satisfy all constraints
  (satisfied $\leftrightarrow \sigma_x \sigma_y = \text{sgn} \ J_{xy}$)

  Odd number of negative $J$'s in a cycle $C$

  $\iff$

  $\forall \sigma$, Odd number of unsatisfied edges on $C$. 
Definition
\( \sigma \in \{-1, +1\}^{\mathbb{Z}^d} \) is a ground state for \( J \) \( \text{iff} \) for any finite set \( B \) of vertices:

\[
\sum_{(x,y) \in \partial B} J_{xy} \sigma_x \sigma_y \geq 0 \text{ flip energy}.
\]

In words, a ground state locally minimizes the Hamiltonian.
Ground States of the EA model on $\mathbb{Z}^d$

**Definition**
\[ \sigma \in \{-1, +1\}^{\mathbb{Z}^d} \text{ is a ground state for } J \text{ iif for any finite set } B \text{ of vertices:} \]
\[ \sum_{(x,y) \in \partial B} J_{xy} \sigma_x \sigma_y \geq 0 \text{ flip energy}. \]

$\mathcal{G}(J) \subset \{-1, +1\}^{\mathbb{Z}^d}$: the set of ground states on $\mathbb{Z}^d$ for couplings $J$

- $\sigma \in \mathcal{G}(J) \leftrightarrow -\sigma \in \mathcal{G}(J)$ Ground State Pairs
- $|\mathcal{G}(J)|$ is a constant $\nu$-a.s., say $N_d$
Ground States of the EA model on $\mathbb{Z}^d$

Conjecture

For $d = 2$, there is only one ground state pair ($N_d = 2$).
(Is there a $d_c$ where $N_d > 2$ for $d > d_c$?)
Ground States of the EA model

Study probability measures on $\mathcal{G}(J)$ to get information on the set.

Weak limit of finite-volume ground states

- Look at the sequence of $\sigma_N^*(J)$ as $N$ grows.
- Record the values it takes in a fixed box.
Ground States of the EA model

Study probability measures on $\mathcal{G}(J)$ to get information on the set.

**Weak limit of finite-volume ground states**

1. Sequence $(G_N) \to \{-1, +1\}^\mathbb{Z}_d$ ($G_N$ with b.c.)
2. The ground state $\sigma^*_N(J)$ is unique (up to flip).
3. Take $\kappa_N = \nu(dJ)\delta_{\sigma^*_N(J)}$.
4. A subsequence converges weakly to $\kappa$.
   
   $\kappa$ samples $J$ and a ground state $\sigma$.

5. $\kappa_J$, the conditional measure given $J$ is supported on ground states.
Ground States of the EA model

Study probability measures on $\mathcal{G}(J)$ to get information on the set.

Uniform measure on $\mathcal{G}(J)$

1. Well defined if $N_d < \infty$.
2. For $A \subset \{-1, +1\}^{\mathbb{Z}^d}$

$$\mu_J(A) = \frac{|\mathcal{G}(J) \cap A|}{N_d}$$
Some Rigorous Results

There are rigorous results on the half-plane $\mathbb{Z} \times N$ (free b.c. at the bottom).

Theorem (A-Damron-Newman-Stein ’10)

If $(G_N)$ are finite boxes (free b.c. vertical, periodic b.c. horizontal),

- $G_N \to \mathbb{Z} \times N$
- the measure $\kappa_N$ converges weakly to $\kappa$;
- $\kappa_J$ is supported on two flip-related ground states $\sigma^* \nu$-a.s.

Are there other ground states on the half-plane? Other b.c.?
Some Rigorous Results

There are rigorous results on the half-plane $\mathbb{Z} \times \mathbb{N}$ (free b.c. at the bottom).

**Theorem (A-Damron ’11)**

*For the half-plane $\mathbb{Z} \times \mathbb{N}$, either $\mathcal{N} = 2$ or $\mathcal{N} = \infty \nu$-a.s.*

For the disordered ferromagnet ($J_{xy} > 0$ \nu-a.s.)

- Wehr ’97: $\mathcal{N} = 2$ or $\infty$ on $\mathbb{Z}^d$ for any $d$.
- Wehr ’& Woo ’98: $\mathcal{N} = 2$ for the half-plane $\mathbb{Z} \times \mathbb{N}$. 
Techniques of Proof

Can be used on $\mathbb{Z}^d$ and the half-plane.

- $\kappa_J$ constructed from finite graphs with periodic b.c. and the uniform measure $\mu_J$ are translation-covariant

$$\kappa_{TJ}(A) = \kappa_J(T^{-1}A)$$
$$\mu_{TJ}(A) = \mu_J(T^{-1}A) .$$

(!) Hard to construct translation-covariant measures on ground states!

- $M = \nu(dJ) \mu_J \times \mu_J$ is translation-invariant (same for $\kappa_J$).
Consider the interface

\[ \sigma \Delta \sigma' = \{(x,y) \in E : \sigma_x \sigma_y \neq \sigma'_x \sigma'_y\} . \]

\[ \sigma \Delta \sigma' = \emptyset \iff \sigma = \sigma' \text{ or } \sigma = -\sigma' . \]

\[ M = \nu(dJ) \mu_J \times \mu_J . \]

Study \( \sigma \Delta \sigma' \) as a random interface under the measure \( M \).
Figure: An example of interface between ground states on the half-plane. The edges in $\sigma \Delta \sigma'$ are the thick ones.
Let $\sigma$ and $\sigma'$ be distinct ground states.

On a general graph:
- $\sigma \Delta \sigma'$ cannot have dangling ends (or 3-branching points).
- $\sigma \Delta \sigma'$ cannot contain loops.

On $\mathbb{Z}^2$ (when sampled from translation-invariant $M$)
- $\sigma \Delta \sigma'$ has positive density;
- No 4-branching points (TI+Burton-Keane argument);
- $\Rightarrow$ the interface is the union of doubly-infinite self-avoiding paths partitioning the plane into topological strips.
The Newman-Stein Theorem on $\mathbb{Z}^2$

For $\mathbb{Z}^2$:

Theorem (Newman-Stein ’01)

Let $M = \nu(dJ)(\kappa_J \times \kappa'_J)$ be a TI measure where $\kappa_J$ and $\kappa'_J$ are constructed from finite-volume ground states with periodic b.c.

$$M \{\sigma \Delta \sigma' \neq \emptyset \text{ and } \sigma \Delta \sigma' \text{ is not connected} \} = 0.$$  

$\Rightarrow$ $\sigma \Delta \sigma'$ is a doubly-infinite self-avoiding path of positive density.

- OPEN: Rule out the existence of this path to show uniqueness on $\mathbb{Z}^2$.  

The Newman-Stein Theorem: Idea of Proof

Suppose $M \{\sigma \Delta \sigma' \neq \emptyset \text{ and } \sigma \Delta \sigma' \text{ is not connected} \} > 0$.

- The interface partition the plane into topological strips.
- Consider rungs $R$ between connected components of the interface.

\[
E(R) = \sum_{(x,y) \in R} J_{xy} \sigma_x \sigma_y .
\]

\[
I = \inf_{R: D_1 \to D_2} E(R).
\]

- Show that $I \leq 0$ and $I > 0$ both have zero probability.
Ground States on the Half-Plane

Back on the half-plane and consider

$M = \nu(dJ) \ \kappa_J \times \kappa'_J$

$\kappa_J$ and $\kappa'_J$ are weak limits of ground states on $G_N \to \mathbb{Z} \times \mathbb{N}$ with horizontal periodic b.c. and vertical free b.c.

$M = \nu(dJ) \ \mu_J \times \mu_J$

where $\mu_J$ is the uniform measure on ground states (ok for $N < \infty$).

Horizontal TI but not vertical TI

We show by contradiction that

$M\{\sigma \Delta \sigma' \neq \emptyset\} = 0$ .

This implies

1. If $N < \infty$, then $N = 2$.
2. $\kappa_J$ is supported on a flip-related pair and $\kappa'_J = \kappa_J$. 
Interfaces in the Half-Plane

Proposition

If \( M \{ \sigma \Delta \sigma' \neq \emptyset \} > 0 \), then for any edge \( e \),
\( M \{ e \in \sigma \Delta \sigma' \} > 0 \).

Interface touches the boundary with positive probability!

\( \sigma \Delta \sigma' \) cannot touch the boundary twice.

Horizontal TI: One tethered path \( \Rightarrow \) infinitely many.
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Interface touches the boundary with positive probability!

- \( \sigma \Delta \sigma' \) cannot touch the boundary twice.
- Horizontal TI: One tethered path \( \Rightarrow \) infinitely many.
Tethered paths are distinct.

How many “tethered paths” do we see at height $k$?

$N_{n,k}$: Number of tethered paths intersecting $[-n, n] \times \{k\}$

- Horizontal TI

$$\lim_{n \to \infty} \frac{1}{n} M[N_{n,0}] = c > 0.$$  

- $N_{n,0} - N_{n,k} \leq 2k$.

- $\inf_{n \geq 1} \frac{1}{n} M[N_{n,k}] = \frac{1}{n} \lim_{n \to \infty} M[N_{n,k}] = c$.

At all heights, we see many tethered paths.
First step of the contradiction

Construct a measure on \( \mathbb{Z}^2 \) from the one on the half-plane.

Take \( T \) a vertical translation.

\[
M_{\mathbb{Z}^2} = \lim_{k \to \infty} \frac{1}{k} \sum_{l=1}^{k} T^{-l} M \quad \text{(subseq.)}
\]

- \( M_{\mathbb{Z}^2} \) is supported on ground states in \( \mathbb{Z}^2 \).
- It is TI in \( \mathbb{Z}^2 \) by construction.

Because we see many tethered paths...

Proposition

If \( M_{\mathbb{Z} \times \mathbb{N}} \{ \sigma \Delta \sigma' \neq \emptyset \} > 0 \), then \( M_{\mathbb{Z}^2} \{ \sigma \Delta \sigma' \text{ is not connected} \} > 0 \).
Second step of the contradiction

Mimic the Newman-Stein argument for ground states on $\mathbb{Z}^2$

\[ M_{\mathbb{Z}^2}\{\sigma\Delta\sigma' \neq \emptyset \text{ and } \sigma\Delta\sigma' \text{ is not connected}\} = 0. \]

We conclude that $M_{\mathbb{Z} \times \mathbb{N}}\{\sigma\Delta\sigma' \neq \emptyset\} = 0$.

- In the case of the uniform measure, the theorem has to be considerably adapted but the same idea works.
In increasing difficulty?

- $\mathcal{N} = 2$ or $\infty$ on $\mathbb{Z}^d$?
- $\mathcal{N} = 2$ on the half-plane and on $\mathbb{Z}^2$?
- Describe the unique ground state pair.
- Show there is no phase transition on $\mathbb{Z}^2$: $\mathcal{N}_2(\beta) = 1$ for all $\beta$.
- Show there exists $d_c$ such that $\mathcal{N}_d(\beta) > 1$ for $d \geq d_c$, $\beta$ large.
- If so, does $\mathcal{N}_d(\beta) = \infty$?
Open Questions

In increasing difficulty?

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Thank you!