On the Number of Ground States of the Edwards-Anderson Spin Glass Model

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joint with
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1. Comm. Math. Phys. 300 (2010)
2. arXiv:1110.6913 (2011)

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The Edwards-Anderson Model

Let $G_N = (V_N, E_N)$ be a graph on N vertices.

We define the Ising spin glass Hamiltonian on $\Sigma_N = \{-1,+1\}^N$:

$$H_{N,J}(\sigma) = -\sum_{(x,y)\in E_N} J_{xy}\sigma_x\sigma_y \ .$$

where $J = (J_{xy}; (x,y) \in E_N)$ i.i.d. of law ν Gaussian (say)

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- Covariance $\int \nu(dJ)\ H_{N,J}(\sigma)H_{N,J}(\sigma') = \sum_{(x,y)\in E_N} \sigma_x\sigma_y\ \sigma'_x\sigma'_y$
- Edge overlap $R_N(\sigma,\sigma') = \frac{1}{|E_N|}\sum_{(x,y)\in E_N} \sigma_x\sigma_y\ \sigma'_x\sigma'_y$

- Sherrington-Kirkpatrick model: G_N is the complete graph.
- Edwards-Anderson model: G_N is a box of $\mathbb{Z}^d$.

"Describe" the minima of $H_{N,J}$ for N large.

The Gibbs Measure of the SK model

$$G_{\beta,N,J}(\sigma) = \frac{\exp -\beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)} \quad \text{as } N \to \infty \text{ ?}$$

The order parameter is

$$x_\beta(q) = \lim_{N\to\infty} \int \nu(dJ)\ G^{\times 2}_{\beta,N,J}\{R_N(\sigma,\sigma') \le q\}$$

More and more things are proved:

- Parisi formula: free energy is a variational formula over c.d.f. x.
- Phase transition: for $\beta > \beta_c = 1$, $x_\beta(q)$ has more than one jump.
- Parisi Ultrametricity Conjecture:
  Infinite number of pure states with ultrametric overlaps

$$G^{\times 3}_{\beta,N,J}\Big\{R_N(\sigma,\sigma') \ge \min\{R_N(\sigma',\sigma''); R_N(\sigma'',\sigma')\}\Big\} \to 1 \quad \text{in } \nu\text{-prob.}$$

The Gibbs Measure of the EA model on $\mathbb{Z}^d$

$$G_{\beta,N,J}(\sigma) = \frac{\exp -\beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)} \quad \text{as } N \to \infty \ ?$$

General results applies

- DLR equations:

$$\mathcal{G}_d(\beta, J) = \text{set of Gibbs measures on } \{-1,+1\}^{\mathbb{Z}^d} \text{ at } \beta \text{ and } J$$

- Pure states: elements of ext $\mathcal{G}_d(\beta, J)$.
- $\mathcal{N}_d(\beta) = |\text{ext } \mathcal{G}_d(\beta, J)|$ is a constant ν-a.s.!
- High Temp./Low β: $\mathcal{N}_d(\beta) = 1$.

The Gibbs Measure of the EA model on $\mathbb{Z}^d$

$$G_{\beta,N,J}(\sigma) = \frac{\exp -\beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)} \quad \text{as } N \to \infty \ ?$$

Low temperature: (almost)-everything is unknown

- $d < d_c$: No phase transition $\mathcal{N}_d(\beta) = 1$?
- $d \geq d_c$, β large: Phase transition $\mathcal{N}_d(\beta) > 1$?

Droplet Scenario:
Phase transition of Ising-type

- $\mathcal{N}_d(\beta) = 2$

RSB Scenario:
Phase transition of SK-type

- $\mathcal{N}_d(\beta) = \infty$
- Ultrametric overlaps

Ground States of EA model for finite N

Instead of studying the pure states, we study the the ground states:

$$\beta \to \infty \text{ then } N \to \infty \ .$$

Let

$$\sigma_N^*(J) = \arg\min_{\sigma \in \Sigma_N} H_{N,J}(\sigma)$$

- The minimizer (ground state) is unique because ν is continuous.

$$H_{N,J}(\sigma) = -\sum_{(x,y)\in E_N} J_{xy}\sigma_x\sigma_y \ .$$

- Typically, $\sigma_N^*(J)$ do not satisfy all constraints (satisfied $\leftrightarrow$ $\sigma_x\sigma_y = \text{sgn } J_{xy}$)

Odd number of negative J's in a cycle $\mathcal{C}$

$$\Longleftrightarrow$$

$\forall\sigma$, Odd number of unsatisfied edges on $\mathcal{C}$.

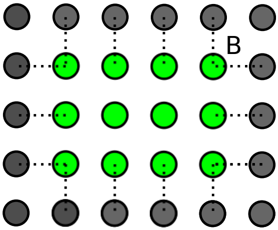
Ground States of the EA model on $\mathbb{Z}^d$

Definition

$\sigma \in \{-1,+1\}^{\mathbb{Z}^d}$ *is a ground state for* J *iif for any finite set* B *of vertices:*

$$\sum_{(x,y)\in\partial B} J_{xy}\sigma_x\sigma_y \geq 0 \quad \textit{flip energy} .$$

In words, a ground state locally minimizes the Hamiltonian.



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$\mathcal{G}(J) \subset \{-1,+1\}^{\mathbb{Z}^d}$: the set of ground states on $\mathbb{Z}^d$ for couplings J

- $\sigma \in \mathcal{G}(J) \Leftrightarrow -\sigma \in \mathcal{G}(J)$ Ground State Pairs
- $|\mathcal{G}(J)|$ is a constant ν-a.s., say $\mathcal{N}_d$

Ground States of the EA model on $\mathbb{Z}^d$

Conjecture

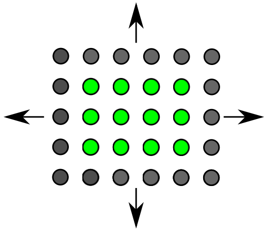
For $d = 2$, there is only one ground state pair ($\mathcal{N}_d = 2$).
(Is there a d_c where $\mathcal{N}_d > 2$ for $d > d_c$?)

Ground States of the EA model

Study probability measures on $\mathcal{G}(J)$ to get information on the set.

Weak limit of finite-volume ground states

- Look at the sequence of $\sigma_N^*(J)$ as N grows.
- Record the values it takes in a fixed box.



Ground States of the EA model

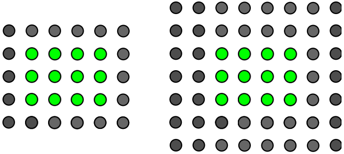
Study probability measures on $\mathcal{G}(J)$ to get information on the set.

Weak limit of finite-volume ground states

1. Sequence $(G_N) \to \{-1,+1\}^{\mathbb{Z}^d}$ (G_N with b.c.)
2. The ground state $\sigma^*_N(J)$ is unique (up to flip).
3. Take $\kappa_N = \nu(dJ)\delta_{\sigma^*_N(J)}$.
4. A subsequence converges weakly to κ.

$$\kappa \text{ samples } J \text{ and a ground state } \sigma.$$

5. κ_J, the conditional measure given J is supported on ground states.



Ground States of the EA model

Study probability measures on $\mathcal{G}(J)$ to get information on the set.

Uniform measure on $\mathcal{G}(J)$

1. Well defined if $\mathcal{N}_d < \infty$.
2. For $A \subset \{-1, +1\}^{\mathbb{Z}^d}$

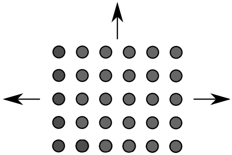
$$\mu_J(A) = \frac{|\mathcal{G}(J) \cap A|}{\mathcal{N}_d}$$

Some Rigorous Results

There are rigorous results on the half-plane $\mathbb{Z} \times \mathbb{N}$ (free b.c. at the bottom).

Theorem (A-Damron-Newman-Stein '10)

If (G_N) are finite boxes (free b.c. vertical, periodic b.c. horizontal),



$G_N \to \mathbb{Z} \times \mathbb{N}$

- *the measure κ_N converges weakly to κ;*
- *κ_J is supported on two flip-related ground states σ^* ν-a.s.*

Are there other ground states on the half-plane ? Other b.c. ?

Some Rigorous Results

There are rigorous results on the half-plane $\mathbb{Z} \times \mathbb{N}$ (free b.c. at the bottom).

Theorem (A-Damron '11)

For the half-plane $\mathbb{Z} \times \mathbb{N}$, *either* $\mathcal{N} = 2$ *or* $\mathcal{N} = \infty$ ν*-a.s.*

For the disordered ferromagnet ($J_{xy} > 0$ ν-a.s.)

- Wehr '97: $\mathcal{N} = 2$ or ∞ on $\mathbb{Z}^d$ for any d.
- Wehr '& Woo '98: $\mathcal{N} = 2$ for the half-plane $\mathbb{Z} \times \mathbb{N}$.

Techniques of Proof

Can be used on $\mathbb{Z}^d$ and the half-plane.

- κ_J constructed from finite graphs with periodic b.c. and the uniform measure μ_J are translation-covariant

$$\kappa_{TJ}(A) = \kappa_J(T^{-1}A)$$
$$\mu_{TJ}(A) = \mu_J(T^{-1}A) \ .$$

(!) Hard to construct translation-covariant measures on ground states!

- $M = \nu(dJ)\ \mu_J \times \mu_J$ is translation-invariant (same for κ_J).

Techniques of Proof

- Consider the interface

$$\sigma \Delta \sigma' = \{(x,y) \in E : \sigma_x \sigma_y \neq \sigma'_x \sigma'_y\} \ .$$

- $$\sigma \Delta \sigma' = \emptyset \iff \sigma = \sigma' \text{ or } \sigma = -\sigma' \ .$$

- $M = \nu(dJ)\ \mu_J \times \mu_J$.

  Study $\sigma \Delta \sigma'$ as a random interface under the measure M.

Interface between Ground States

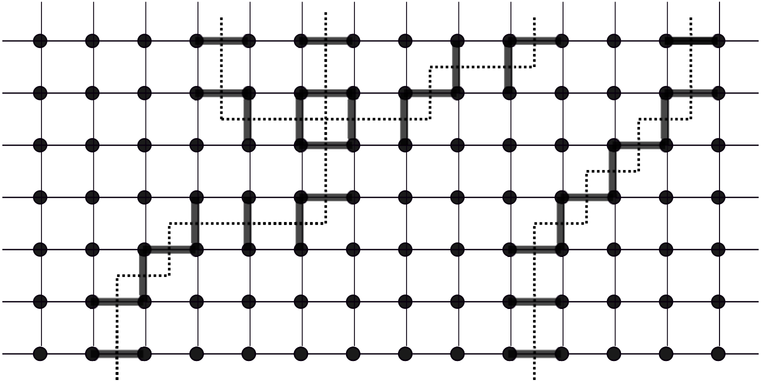


Figure: An example of interface between ground states on the half-plane. The edges in $\sigma\Delta\sigma'$ are the thick ones.

Interface of Ground States: Elementary

Let σ and σ' be distinct ground states.

On a general graph:

- $\sigma \Delta \sigma'$ cannot have dangling ends (or 3-branching points).
- $\sigma \Delta \sigma'$ cannot contain loops.

On $\mathbb{Z}^2$ (when sampled from translation-invariant M)

- $\sigma \Delta \sigma'$ has positive density;
- No 4-branching points (TI+Burton-Keane argument);
- $\Rightarrow$ the interface is the union of doubly-infinite self-avoiding paths partitioning the plane into topological strips.

The Newman-Stein Theorem on $\mathbb{Z}^2$

For $\mathbb{Z}^2$:

Theorem (Newman-Stein '01)

Let $M = \nu(dJ)(\kappa_J \times \kappa'_J)$ be a TI measure where κ_J and κ'_J are constructed from finite-volume ground states with periodic b.c.

$$M \left\{\sigma \Delta \sigma' \neq \emptyset \text{ and } \sigma \Delta \sigma' \text{ is not connected}\right\} = 0 \ .$$

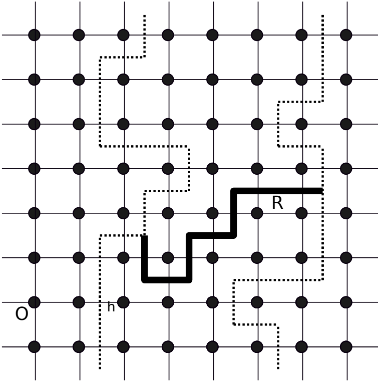
$\Rightarrow$ $\sigma\Delta\sigma'$ is a doubly-infinite self-avoiding path of positive density.

- OPEN: Rule out the existence of this path to show uniqueness on $\mathbb{Z}^2$.

The Newman-Stein Theorem: Idea of Proof

Suppose $M\{\sigma\Delta\sigma' \neq \emptyset$ and $\sigma\Delta\sigma'$ is not connected$\} > 0$.

- The interface partition the plane into topological strips.
- Consider rungs R between connected components of the interface.



$$E(R) = \sum_{(x,y)\in R} J_{xy}\sigma_x\sigma_y \ .$$

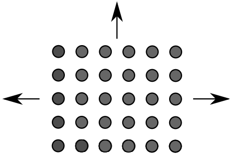
$$I = \inf_{R:D_1\to D_2} E(R).$$

- Show that $I \leq 0$ and $I > 0$ both have zero probability.

Ground States on the Half-Plane

Back on the half-plane and consider

- $M = \nu(dJ)\ \kappa_J \times \kappa'_J$
  κ_J and κ'_J are weak limits of ground states on $G_N \to \mathbb{Z} \times \mathbb{N}$ with horizontal periodic b.c. and vertical free b.c.



- $M = \nu(dJ)\ \mu_J \times \mu_J$
  where μ_J is the uniform measure on ground states (ok for $\mathcal{N} < \infty$).

Horizontal TI but not vertical TI

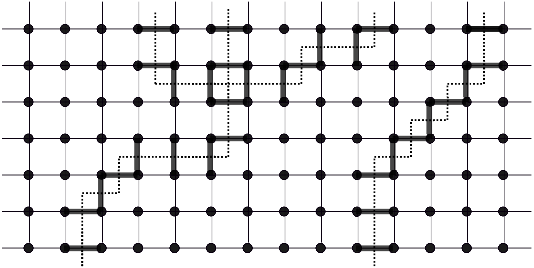
We show by contradiction that

$$M\{\sigma\Delta\sigma' \neq \emptyset) = 0 \quad .$$

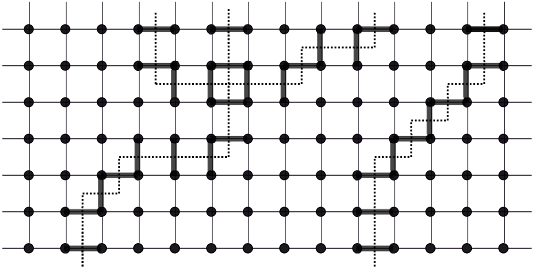
This implies

1. If $\mathcal{N} < \infty$, then $\mathcal{N} = 2$.
2. κ_J is supported on a flip-related pair and $\kappa'_J = \kappa_J$.

Interfaces in the Half-Plane



Interfaces in the Half-Plane



Proposition

If $M\{\sigma\Delta\sigma' \neq \emptyset\} > 0$*, then for any edge* e*,* $M\{e \in \sigma\Delta\sigma'\} > 0$.

Interface touches the boundary with positive probability!

- $\sigma\Delta\sigma'$ cannot touch the boundary twice.
- Horizontal TI: One tethered path $\Rightarrow$ infinitely many.

Density of Tethered Paths

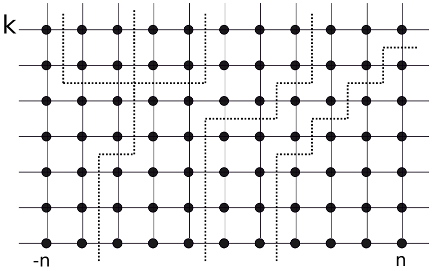
Tethered paths are distinct.
How many "tethered paths" do we see at height k ?

$N_{n,k}$: Number of tethered paths intersecting $[-n,n] \times \{k\}$

- Horizontal TI

$$\lim_{n\to\infty} \frac{1}{n} M[N_{n,0}] = c > 0 \ .$$

- $N_{n,0} - N_{n,k} \leq 2k$.
- $\inf_{n\geq 1} \frac{1}{n} M[N_{n,k}] = \frac{1}{n} \lim_{n\to\infty} M[N_{n,k}] = c$.



At all heights, we see many tethered paths.

First step of the contradiction

Construct a measure on $\mathbb{Z}^2$ from the one on the half-plane.

Take T a vertical translation.

$$M_{\mathbb{Z}^2} = \lim_{k \to \infty} \frac{1}{k} \sum_{l=1}^{k} T^{-l} M \quad \text{(subseq.)}$$

- $M_{\mathbb{Z}^2}$ is supported on ground states in $\mathbb{Z}^2$.
- It is TI in $\mathbb{Z}^2$ by construction.

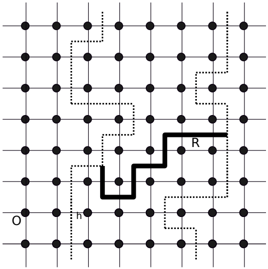
Because we see many tethered paths...

Proposition

If $M_{\mathbb{Z}\times\mathbb{N}}\{\sigma\Delta\sigma' \neq \emptyset\} > 0$, *then* $M_{\mathbb{Z}^2}\{\sigma\Delta\sigma' \text{ is not connected}\} > 0$.

Second step of the contradiction

Mimic the Newman-Stein argument for ground states on $\mathbb{Z}^2$



Theorem

$$M_{\mathbb{Z}^2}\{\sigma\Delta\sigma' \neq \emptyset \textit{ and } \sigma\Delta\sigma' \textit{ is not connected}\} = 0 \ .$$

We conclude that $M_{\mathbb{Z}\times\mathbb{N}}\{\sigma\Delta\sigma' \neq \emptyset\} = 0$.

- In the case of the uniform measure, the theorem has to be considerably adapted but the same idea works.

Open Questions

In increasing difficulty ?

- $\mathcal{N} = 2$ or ∞ on $\mathbb{Z}^d$?
- $\mathcal{N} = 2$ on the half-plane and on $\mathbb{Z}^2$?
- Describe the unique ground state pair.
- Show there is no phase transition on $\mathbb{Z}^2$: $\mathcal{N}_2(\beta) = 1$ for all β.
- Show there exists d_c such that $\mathcal{N}_d(\beta) > 1$ for $d \geq d_c$, β large.
- If so, does $\mathcal{N}_d(\beta) = \infty$?

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Thank you!