Paulo Ribenboim, at the time of his retirement

by

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At this, the third meeting of the Canadian Number Theory Association, we have had the opportunity to enjoy a lot of good mathematics, while celebrating the career of one of Canada's most distinguished academics. Paulo Ribenboim, author of more than a hundred journal articles and of thirteen books, a Fellow of the Royal Society of Canada, and colleague and advisor to so many mathematicians, is retiring after a career spanning over forty years and three continents. In this preface we present a biography of Paulo, together with a sampling of a few of his many interesting results.

Paulo was born in 1928, into a middle class family in Recife, Brazil. In the picture on the left you can see him at a very young age with his two brothers. After high school, he did his military service (see the picture below), where his mathematical talents were recognized and he found himself teaching calculus. It amuses Paulo to think that one or two of his students were to become military rulers of Brazil – what did they learn from him in that calculus class ?

After finishing in the army, Paulo entered university in Rio de Janeiro, graduating with a B.Sc. in 1948. Following one more year in Rio, at the Instituto di Rio de Janeiro (the forerunner of today's IMPA), Paulo received a fellowship to study in France. So, in 1950, he went to Nancy, to work under Dieudonné's supervision. There he was close to Laurent Schwarz and Roger Godement; and had many colleagues and friends who went on to become important mathematicians, such as Grothendieck, Malliavin, Malgrange, Lions, P.M. Cohn and H. Bauer.

This time in Nancy was important for Paulo in other ways too, for it was there that he met his lovely wife Huguette. The photographs on the left were both taken in Nancy, soon after they met.

Paulo found himself fascinated by the Bourbaki program, and returned to Rio in 1952 determined to pursue these studies. Soon after, he helped bring his friend Grothendieck over to Brazil for a couple of years, during which time Grothendieck wrote many of his important papers on topological vector spaces and had a lasting impact on the development of Brazilian mathematics.

Under Dieudonné's influence, Paulo had found himself particularly interested in the theory of valuations and, in 1954, headed back to Europe to study at the Mathematisches Institut in Bonn, under the direction of Krull. As you can see from the accompanying photograph, Paulo soon found himself in the full swing of things in Bonn, and started proving noteworthy results in this fast developing area.

So what are valuations ? The simplest example is the 'p-adic valuation', that function which gives the exact power of prime p dividing a given rational number; thus $v_p(a/b)$ is the power of p dividing a minus the power of p dividing b (so $v_2(16) = 4$, $v_2(1/2) = -1$, etc.). There are two obvious properties of such functions: For all a and b, we have

$$v(ab) = v(a) + v(b) \quad \text{and} \quad v(a+b) \geq \min\{v(a), v(b)\}.$$

So a valuation is a homomorphism $v\colon K \to \Gamma \cup \{\infty\}$, from a field K to an additive, abelian group Γ which is totally ordered. We call $R = \{r \in K :\ v(r) \geq 0\}$ the *valuation ring* of v, so that $P = \{r \in K :\ v(r) > 0\}$ is the unique maximal ideal in R, and define $\bar{K} = R/P$ to be the *residue field* of K. In 1932, Krull wrote a seminal paper in which he developed a general theory of such 'valuations', based on the properties that we have just described.

We shall start by examining how Γ differs from being a subset of the real numbers, $\mathbf{R}$. Suppose that $0 < a < b$ belong to Γ: is it true that there exists a positive integer n such that $b < an$? (Of course this would be the case for any subset of $\mathbf{R}$). It turns out that not all totally ordered, additive abelian groups Γ satisfy this condition; for example, if $\Gamma = \mathbf{Z} \times \mathbf{Z}$, with lexicographic ordering (that is $(x_1, y_1) < (x_2, y_2)$ if and only if $x_1 < x_2$, or $x_1 = x_2$ and $y_1 < y_2$), then $n(0,1) < (1,0)$ for *all* positive integers n. We thus introduce the notion of *archimedean equivalence*, so that for $a, b \in \Gamma$,

$$a \sim b \text{ if and only if } a < mb \text{ and } b < na \text{ for some positive integers } m \text{ and } n.$$

The *height* of a valuation is the totally ordered set of such archimedean equivalence classes, or simply their number if this set is finite. Notice that any height 1 valuation can be embedded into the reals.

Now, Krull noted that if R is a height 1 valuation ring then (i) all ideals are primary and there is just one non-zero prime ideal, (ii) R is an integral domain, and (iii) R is completely integrally closed, and he asked whether the converse holds, which would lead to a nice classification of height 1 valuation rings. However, in 1955, Ribenboim exhibited a counterexample * as well providing various (iv)th conditions under which the converse would indeed hold (for instance, if R is Noetherian). This was Paulo's first major result.

Suppose that one is given a set of valuations $v_i : K \to \Gamma_i$, for $i = 1, 2, \ldots, n$. In analogy with the Chinese Remainder Theorem (CRT) one might ask whether, for given $k_i \in K$ and $\gamma_i \in \Gamma_i$, there exists $x \in K$ for which

$$v_i(x - k_i) = \gamma_i \quad \text{for } i = 1, 2, \ldots, n.$$

To stress the importance of such a question we quote Schilling who wrote, 'Some of the more important tools in the theory of algebraic numbers and the arithmetic theory of covering varieties are furnished by appropriate generalizations of the CRT'. Krull showed that such an x exists if no two of the associated valuation rings have a common non-trivial prime ideal (corresponding to coprime moduli in CRT), leaving the more general question open; indeed Krull felt that a general solution to this problem was one of the most challenging open questions in the area. However, in 1957 Ribenboim solved this, giving an elegant 'if and only if' condition for whether such an x exists, depending only on the images of $P_i = \{r \in K : v_i(r) > 0\}$ under v_i. This condition and its proof are considered to be very clever, and stand as an important and attractive result in the subject of valuation rings.

Another nice result in this area, due to Ribenboim, is the generalization of Hasse's Theorem to valuation rings: In 1925, Hasse showed that given any algebraic number field K and 'partition' of a given integer n as $\sum_i e_i f_i$, and given any prime ideal $\mathbf{p}$ of K, there exists a field extension L of degree n over K such that $\mathbf{p}$ factors into prime ideals of L as $\prod_i p_i^{e_i}$, where p_i has relative degree f_i. In 1959, Paulo showed that this result can be extended to fields K that admit suitable valuations of finite height.

Ribenboim proved many other results on valuations and wrote a monograph [4] that quickly became a standard reference in the area. He is proud of the part that this monograph plays in Ax and Kochen's 1965 proof of Artin's Conjecture (that, for any given d, there exists a prime p_d such that any homogenous form f of degree d in n ($> d^2$) variables, has a non-trivial p-adic zero for all primes $p \geq p_d$), which employs a non-standard model of the integers via valuation theory.

Getting back to Paulo's biography, he left Bonn in 1956, returning to Brazil until 1959. In 1957 he received the first Ph.D. ever granted by a Brazilian university (in São Paulo). From 1959 to 1962 Paulo was a Fulbright Fellow at Urbana-Champaign, and in 1962, moved to Queen's University, here in Canada, where he still works today.

During Paulo's first decade at Queen's, the mathematics department was small, yet there were students and faculty from all over the world. The atmosphere was friendly and

* This appears in Bourbaki's *Commutative Algebra*, Chapter 6

there was an active departmental colloquium, as well as a steady stream of visitors from across Canada, the United States and Europe. Paulo invited a variety of visiting faculty members, including J. Neukirch, W. Scharlau and W.D. Geyer. Paulo was an enthusiastic organizer and, with his wife Huguette, provided a congenial atmosphere in their home for regular informal gatherings that brought together mathematicians with very different specialties.

In the early 1970s, under such an influence, Queen's became a center for research in Algebra and a 'place to be' for anyone in the area. In particular, Paulo was then working on generalizations and variants of Hilbert's 17th Problem. To remind the reader, in 1927 Artin solved Hilbert's 17th problem, which was to show that every positive definite rational function over the reals, equals the sum of squares of rational functions. Paulo and Gondard studied under what special conditions the same could be said of polynomial functions (this is not true in general). Also, in 1974, they proved the following delightful result, using methods from model theory: Let K be a real closed field, and m and n positive integers. A symmetric n-by-n matrix $M(X_1, \ldots, X_m)$, with entries in $K(X_1, \ldots, X_m)$, is said to be *positive definite* if the quadratic form associated to $M(x_1, \ldots, x_m)$ is positive definite whenever $M(x_1, \ldots, x_m)$ is well-defined for $x_1, \ldots, x_m \in K$. They proved that if M is positive definite then it is the sum of squares of symmetric matrices with entries from $K(X_1, \ldots, X_m)$.

In recent years, Paulo's research has mostly been in number theory, though with excursions into questions involving equations in groups, generalized power series, and ascending chain conditions.

In a series of interesting papers, Paulo has shown how to apply Baker's and Faltings' Theorems to a wide variety of Diophantine and related questions. For instance, in 1985, he and Powell used Faltings' Theorem to show that there are only finitely many coprime integer solutions to any one of the equations, $x^{2n} \pm y^{2n} = z^2$, $x^{2n} + y^{2n} = z^3$ or $x^4 - y^4 = z^n$, for any given $n > 3$ (while preparing this article, the author noted that their proof may be modified to prove that any equation $x^p \pm y^p = z^q$ has finitely many coprime integer solutions provided $2/p + 1/q < 1$).

An old problem of elementary number theory is to prove that no value occurs infinitely often in a (non-degenerate) linear recurrence (for second order recurrences, Beukers has proved that no value occurs more than three times, except in a few simple cases). A substantial generalization of this is to show that there are only finitely many pairs of distinct elements of any (greater than first order) linear recurrence whose product is a square. In 1989 Ribenboim proved this true of the Fibonacci and Lucas sequences, listing all such products ($F_1F_2 = 1^2$, $F_1F_{12} = F_2F_{12} = 12^2$, $F_3F_6 = 4^2$, $L_1L_3 = 2^2$ and $L_0L_6 = 6^2$). Recently, with McDaniel, he has generalized the qualitative result to all Lucas-type sequences.

Paulo is admired for many of the qualities that he has brought to his career as a mathematician. Besides his research, some of which we have discussed above, he is reknowned for being a marvellous author who enriches and enlivens our subject, and for his gentle yet exacting dealings with students and colleagues. His kind, gentlemanly manner makes him an excellent advisor of students, and he has succeeded in getting the most out of them.

In his educational writings, to whit thirteen books and over thirty surveys and 'technical reports', he has always kept in mind the varied backgrounds of his reading public, succeeding in reaching out and explaining difficult concepts to non-specialists as well as specialists, novices as well as experts. His book on valuations is the standard text in that area. *13 Lectures on Fermat's Last Theorem* is written in a very distinctive style, with a charm and lightness of touch, yet with a precision, that is guaranteed to excite any reader. His recent *Book of Prime Number Records*, recommended by Choice Magazine as one of the five best science books of 1990, is being printed in Japanese and French, as well as English, and allows any interested reader to obtain an excellent insight into many aspects of prime numbers. An abridged, 'pocket-sized' version [10] was published to coincide with the start of the conference.

In 1965 Ribenboim graduated his first doctoral student, Malcolm Griffin, and has now had a total of thirteen doctoral students (including the author of this article). Add to this list his seven other students with varying degrees, and we find academics now working from Canada to Brazil, from Norway to Morocco, from India to France, and from Singapore to the United States. This is typical of Paulo, a truly international mathematician, who has spread his pleasure of mathematics, his infectious enthusiasm, to every corner of the globe. He is widely sought after as a lecturer, and has travelled extensively inspiring people everywhere: He has held academic positions in eleven countries, and given lectures in five South American countries, fifteen Western and six Eastern European countries, half a dozen countries in Africa and the Middle East, as well as a couple in the Far East, including such far-flung locations as Colombia, Curaçao, Egypt, Finland, Iran, Sudan, Taiwan and Tunisia. His invited addresses include a UNESCO conference in Buenos Aires in 1959, a 'Mathematics in the Third World' meeting in Khartoum in 1978, the 'Gauss Symposium' in Brazil in 1989, as well as a televised address in Québec in 1987.

In conclusion let us wish Paulo well in his retirement, in the hope that the forthcoming years might remain as productive and worthwhile as his career has been so far.

Remark: *This article is derived from my talk at the conference on August 23rd. I would like to thank Ernst Kani and Jan Minaç, as well as Huguette and Paulo Ribenboim, for their help in the presentation of the mathematics, and also of the personal photographs and biography.*

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