ON THE EXPONENTIAL SUM OVER K-FREE NUMBERS

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1. Introduction

Applications of the Hardy-Littlewood method to diophantine problems, with variables restricted to inconvenient sets $\mathcal{A}$, can frequently be much simplified provided that the L^1-mean of the exponential sum generated by $\mathcal{A}$ is suitably small (see §6 for a discussion of this idea). It is therefore a matter of some interest to determine in what circumstances such L^1-means are indeed small. It seems that the only non-trivial exponential sums whose L^1-means have been investigated hitherto are those corresponding to the von Mangoldt function, Λ, and the Möbius function, μ. Thus, denoting $e^{2\pi iz}$ by $e(z)$, Vaughan (1988) has shown that

$$x^{1/2} \ll \int_0^1 \left| \sum_{1\le n\le x} \Lambda(n)e(n\alpha)\right| d\alpha \le (1+o(1))(x\log(2x))^{1/2},$$

and Balog and Perelli (to appear) have very recently established the bounds

$$\exp\left(c\frac{\log x}{\log\log x}\right) \ll \int_0^1 \left| \sum_{1\le n\le x} \mu(n)e(n\alpha)\right| d\alpha \ll x^{1/2},$$

for a suitable positive number c. In each case the upper bounds, which are believed to lie close to the truth, follow from the corresponding L^2-means via the Cauchy-Schwarz inequality. Within the framework of the Hardy-Littlewood method, such a belief implies that a positive proportion of the L^2-mean must be concentrated on suitably chosen minor arcs. Our primary concern in this paper is to establish bounds for the L^1- and L^2-means of the exponential sum formed with the k-free numbers, which is to say the numbers not divisible by any kth power of a prime. Our conclusions are in marked contrast with the above cases. When $k\ge 2$, the L^1-mean of this exponential sum is substantially smaller than the bound derived from the L^2-mean via the Cauchy-Schwarz inequality, and the L^2-mean is concentrated on the major arcs of the Hardy-Littlewood dissection.

In order to describe our first theorem we require some notation. Let $\mu_k(n)$ be the characteristic function of the k-free numbers, and define $f_k(\alpha) = f_k(\alpha;x)$ by

$$f_k(\alpha;x) = \sum_{1\le n\le x} \mu_k(n)e(n\alpha). \tag{1.1}$$

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Theorem 1. *Let k be an integer with $k \geq 2$. Then for each positive number ε,*

$$x^{\frac{1}{2k}} \ll_k \int_0^1 |f_k(\alpha; x)| d\alpha \ll_{k,\varepsilon} x^{\frac{1}{k+1}+\varepsilon}. \tag{1.2}$$

We expect that the lower bound in (1.2) reflects the true order of magnitude of the L^1-mean of $f_k(\alpha)$. For comparison, the bound $\int_0^1 |f_k(\alpha; x)| d\alpha \ll x^{1/2}$ is immediate from the L^2-mean, on applying Schwarz's inequality. Thus, while the upper bound provided by (1.2) is considerably sharper than this trivial bound, it falls short of what we believe to be true.

Theorem 1 plainly implies that the exponential sum $f_k(\alpha; x)$ can be large only on a very thin set.

Corollary 1. *Suppose that δ is a positive number, and denote by $\mathcal{C}_{k,\delta}(x)$ the set of $\alpha \in [0, 1)$ satisfying the condition $|f_k(\alpha; x)| > x^{1/(k+1)+\delta}$. Then the Lebesgue measure of $\mathcal{C}_{k,\delta}(x)$ is $O_{k,\varepsilon}(x^{\varepsilon-\delta})$.*

In applications of the Hardy-Littlewood method involving k-free numbers it is desirable to have available strong estimates for $f_k(\alpha; x)$ with α restricted to suitable minor arcs. In Theorem 2 below we provide a non-trivial upper bound for the L^2-mean of $f_k(\alpha)$ restricted to those α lying on suitable minor arcs, obtaining an estimate sharper than would be obtained on the expectation of "square-root cancellation". When Q and x are real numbers with $1 \leq Q \leq x$, we define these minor arcs $\mathfrak{m}(Q; x)$ to be the set of real numbers $\alpha \in [0, 1)$ such that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a, q) = 1$ and $|q\alpha - a| \leq Qx^{-1}$, one has $q > Q$.

Theorem 2. *Suppose that $1 \leq Q \leq \frac{1}{2}x^{1/2}$. Then for each positive number ε,*

$$\int_{\mathfrak{m}(Q;x)} |f_k(\alpha; x)|^2 d\alpha \ll_{k,\varepsilon} x^{1+\varepsilon} Q^{\frac{1}{k}-1} + x^{\frac{2}{k}-1+\varepsilon} Q^{3-\frac{2}{k}}.$$

When γ is a positive number satisfying $\gamma < \frac{1}{2}$, and $Q = x^\gamma$, Theorem 2 implies that

$$\int_{\mathfrak{m}(Q;x)} |f_k(\alpha; x)|^2 d\alpha = o(x).$$

Also, by using the asymptotic formula for the number of k-free numbers up to x,

$$\int_0^1 |f_k(\alpha; x)|^2 d\alpha = \sum_{1 \leq n \leq x} \mu_k(n) = \frac{x}{\zeta(k)} + O(x^{1/k}). \tag{1.3}$$

Thus the L^2-mean of $f_k(\alpha; x)$ must be concentrated on the major arcs $\mathfrak{M}(Q) = [0, 1) \setminus \mathfrak{m}(Q; x)$, confirming the assertion made in our opening paragraph.

We record the optimal bound stemming from Theorem 2 in Corollary 1 below.

Corollary 1. *Write $\theta = (2k - 2)/(4k - 3)$, and define*

$$\beta_k = \frac{1}{4} + \frac{5k - 4}{4k(4k - 3)}. \tag{1.4}$$

Then for each positive number ε,

$$\int_{\mathfrak{m}(x^\theta;x)} |f_k(\alpha;x)|^2\,d\alpha \ll_{k,\varepsilon} x^{2\beta_k+\varepsilon}. \tag{1.5}$$

Note that the exponent β_k is a decreasing function of k with $\beta_2 = 2/5$ and $\lim_{k\to\infty}\beta_k = 1/4$.

There are several interesting consequences of Corollary 1, both of a theoretical nature, and of use in applications of the Hardy-Littlewood method. We start with a corollary which, in certain circumstances, leads to a sharper conclusion than that provided by Corollary 1 to Theorem 1 in the case $k = 2$. In particular, this corollary shows that $f_2(\alpha;x)$ is almost always rather smaller than $x^{1/2}$.

Corollary 2. *Suppose that δ is a positive number, and denote by $\mathcal{B}_\delta(x)$ the set of $\alpha \in [0,1)$ satisfying the condition $|f_2(\alpha;x)| > x^{2/5+\delta}$. Then when $\delta < \frac{1}{10}$, the Lebesgue measure of $\mathcal{B}_\delta(x)$ is $O_\varepsilon(x^{\varepsilon-2\delta})$.*

The bound (1.5) provided by Corollary 1 is of sufficient strength to permit the successful treatment, through the use of the Hardy-Littlewood method, of binary additive problems involving k-free numbers. By way of illustration, let $r_k(n)$ denote the number of representations of n as the sum of two positive k-free numbers, and let $\mathfrak{S}_k(n)$ be the natural singular series for this additive problem, that is,

$$\mathfrak{S}_k(n) = \prod_{p^k\mid n}\left(\frac{p^k-1}{p^k-2}\right)\prod_p\left(1-\frac{2}{p^k}\right),$$

where the products are over prime numbers.

Corollary 3. *For each positive number ε,*

$$r_k(n) = \mathfrak{S}_k(n)n + O_{\varepsilon,k}(n^{2\beta_k+\varepsilon}).$$

Sums of square-free numbers and similar problems involving k-free numbers were first investigated by Evelyn and Linfoot in a series of papers (see Linfoot & Evelyn (1929) and Evelyn & Linfoot (1931), and also Estermann (1931) and Mirsky (1947, 1948)). In cases in which there are s summands, with $s \geq 3$, they used the circle method to obtain an asymptotic formula for the number of representations in the proposed manner. In the case $s = 2$, however, their circle method approach fails, and they remark that "though the analytic method is still applicable in principle, there seems little hope of its furnishing a practicable proof in this case". Indeeed, for the binary problem Evelyn and Linfoot were forced to make use of an elementary argument which yields the formula $r_k(n) = \mathfrak{S}_k(n)n + O_{\varepsilon,k}(n^{2/(k+1)+\varepsilon})$. Thus, while Corollary 3 yields a weaker error term, it at least addresses the problem implicit in the above remark of Evelyn and Linfoot. Moreover the flexibility of the circle method offers certain advantages over the elementary methods. For example, one can obtain an asymptotic formula for the number of representations of a given integer as the sum of a positive k-free number and a positive l-free number. It is not clear in general how to approach such a problem via the elementary methods, the inherent difficulty being one of obtaining suitable bounds for the number of integral solutions (x, y) of equations of the shape $ax^k + by^l = n$. Furthermore, our methods should yield sharper error terms than those of Evelyn and Linfoot when the number of summands exceeds two.

As a final application of Corollary 1 we provide a result concerning square-free numbers represented as sums of three cubes. Denote by $\mathcal{N}(x)$ the number of integers, n, with $1 \le n \le x$, such that n is square-free and n is the sum of three cubes of positive integers.

Corollary 4. *One has* $\mathcal{N}(x) \gg x^{11/12}$.

We remark that the exponent $\frac{11}{12}$ could be replaced by one very slightly larger by making full use of Theorem 1.2 of Wooley (1995). However, in the present circumstances expedience seems preferable to precision. Again, Corollary 4 seems to be difficult to establish through elementary methods.

Our argument for bounding from below the L^1-mean of $f_k(\alpha)$ is based on the treatment given by Vaughan (1988) for the exponential sum formed with the von Mangoldt function $\Lambda(n)$. In its most general form this treatment establishes Theorem 3 below. In order to describe this theorem we require some notation. We consider a sequence of complex numbers $(\lambda_n)_{n\in\mathbb{N}}$, and define

$$g(\alpha; \boldsymbol{\lambda}) = \sum_{1\le n\le x} \lambda_n e(n\alpha). \tag{1.6}$$

In what follows we adopt the convention, for each real number b, of defining $\sin(bt)/t$ to be b when $t = 0$.

Theorem 3. *Suppose that Q and x are real numbers with $1 \le Q \le x^{1/2}$, and that ε_q $(1 \le q \le Q)$ are non-negative numbers with $\varepsilon_q \le (2qQ)^{-1}$. Then one has*

$$\int_0^1 |g(\alpha; \boldsymbol{\lambda})|\, d\alpha \ge \sum_{1\le q\le Q} \max_{1\le a\le q} |S_{q,a}|,$$

where

$$S_{q,a} = \sum_{d|q} \mu(q/d) d \sum_{\substack{1\le n\le x\\ n\equiv a \pmod d}} \lambda_n \frac{\sin(2\pi(n-a)\varepsilon_q)}{\pi(n-a)}.$$

The conclusion of Theorem 3 implies, roughly speaking, that dense sequences with small L^1-means must be very evenly distributed in the arithmetic progressions with moduli of size up to the square-root of the length of the sequence. Since Theorem 3 does not itself make this observation transparent, we provide a quite general corollary which makes the situation clearer.

Corollary 1. *Suppose that Q and x are real numbers with $1 \le Q \le x^{1/2}$. When $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers, write $\tilde{\lambda}(x) = \max_{1\le n\le x} |\lambda_n|$. Also, let $\delta(x)$ be any positive function decreasing to zero monotonically. Then*

$$\int_0^1 |g(\alpha; \boldsymbol{\lambda})| d\alpha \ge \frac{2\delta(x)}{x} \sum_{1\le q\le Q} \max_{1\le a\le q} |E_{q,a}(x; \boldsymbol{\lambda})| + O\left(\tilde{\lambda}(x)\delta(x)^3 Q \log Q\right),$$

where

$$E_{q,a}(x;\boldsymbol{\lambda}) = \sum_{d|q} \mu(q/d) d \sum_{\substack{1\le n\le x \\ n\equiv a \pmod{d}}} \lambda_n .$$

It transpires that when $(\boldsymbol{\lambda})$ is the characteristic function of a sequence such as the prime numbers, the integers having only small prime factors, or the k-free numbers, it is a simple matter to apply Theorem 3 to deduce that the L^1-mean of $g(\alpha;\boldsymbol{\lambda})$ is large. We defer to §6 a more thorough discussion of lower bounds for L^1-means. For the moment we content ourselves with a corollary concerning smooth numbers. We denote by $\mathcal{A}(P,R)$ the set of R-smooth numbers up to P, that is

$$\mathcal{A}(P,R) = \{n \in [1,P]\cap\mathbb{Z} : p|n,\ p \text{ prime} \Rightarrow p \le R\}.$$

Corollary 2. *When η is a real number with $0<\eta<1/2$, and x is large, one has*

$$\int_0^1 \left| \sum_{n\in\mathcal{A}(x,x^\eta)} e(n\alpha) \right| d\alpha \gg_\eta \frac{x^{1/2}}{\log x}.$$

Sections 2, 3 and 4 of this paper are devoted to the proof of the upper bound recorded in Theorem 2. We exploit two observations in this proof. First we make use of the high level of distribution of k-free numbers amongst arithmetic progressions to gain control of the contribution of the major arcs to the L^2-mean of $f_k(\alpha)$. This we achieve in §3 through the use of sums of Barban-Davenport-Halberstam type, the necessary prerequisites being covered in §2. It transpires that the major arcs in the Hardy-Littlewood dissection of the L^2-mean of $f_k(\alpha)$ contribute the leading term in the asymptotic formula (1.3). Thus a non-trivial estimate for the corresponding minor arc contribution may be obtained by merely subtracting the major arc contribution from the expression on the right hand side of (1.3). This second idea we pursue in §4, where we complete the proof of Theorem 2. In §5 we provide brief proofs of the corollaries to Theorem 2. We turn our attention in §6 to lower bounds for L^1-means of exponential sums, proving Theorem 3 and its corollaries, and also providing a proof of the lower bound in (1.2). Finally, in §7, we describe an essentially elementary argument which completes the proof of Theorem 1.

Throughout, the implicit constants in Vinogradov's notation $\ll$ and $\gg$, and in Landau's O-notation, will depend at most on k and ε. Also, we write $\|x\|$ for $\min_{y\in\mathbb{Z}} |x-y|$, and write $p^t\|n$ when $p^t|n$ but $p^{t+1}\nmid n$.

2. The distribution of k-free numbers in arithmetic progressions

In advance of our analysis of the major arcs in the Hardy-Littlewood dissection of the L^2-mean of $f_k(\alpha)$, we must gain some understanding of the distribution of k-free numbers in arithmetic progressions. For our purposes an average result analogous to the Barban-Davenport-Halberstam theorem suffices, and such a result may be established by generalising the treatment given in §§1 and 4 of Warlimont (1969) for the squarefree case. In order to describe this average result we require some notation. When p is a prime number, k is a natural number, and l and m are non-negative integers, define the function $g_k(p^l,p^m)$ by

$$p^l(1-p^{-k})g_k(p^l,p^m)=\begin{cases}0, & \text{when } l\ge m\ge k,\\ 1, & \text{when } m<\min\{l,k\},\\ 1-p^{l-k}, & \text{when } l=m<k.\end{cases} \tag{2.1}$$

Further, when d and q are positive integers with $d|q$, and the canonical factorisation of q into prime powers is $q=\prod_{p|q}p^{e_p}$, we extend the definition of g_k by taking

$$g_k(q,d)=\prod_{p|q}g_k(p^{e_p},(d,p^{e_p})). \tag{2.2}$$

Finally, when d, q and a are positive integers, we denote by $N_k(d;q,a)$ the number of solutions of the congruence $md^k\equiv a \pmod q$ with $1\le m\le q$.

We first record a useful property of the function $N_k(d;q,a)$.

Lemma 2.1. *Suppose that a, q and k are positive integers with $k\ge 2$, and that z is a positive real number. When $1\le i<k$, define*

$$\pi_i=\prod_{p^i\|(a,q)}p,$$

and define also

$$\pi_k=\prod_{p^k|(a,q)}p.$$

Then for each positive number ε,

$$q^{-1}\sum_{1\le d\le z}\frac{\mu(d)}{d^k}N_k(d;q,a)=\zeta(k)^{-1}g_k(q,(q,a))+O\left(q^{\varepsilon-1}z^{1-k}\prod_{i=1}^k\pi_i^{i-1}\right).$$

Proof. Clearly $N_k(d;q,a)=(d^k,q)$ or 0 according to whether (d^k,q) does, or does not, divide a. Furthermore, $(d^k,q)|a$ if and only if $(d^k,q)|(q,a)$. Thus it suffices to prove the lemma when $a|q$, which we assume henceforward. Moreover, we then have

$$\sum_{1\le d\le z}\frac{\mu(d)}{d^k}N_k(d;q,a)=\sum_{\substack{1\le d\le z\\(d^k,q)|a}}\frac{\mu(d)}{d^k}(d^k,q). \tag{2.3}$$

We note for future reference that these observations imply that

$$N_k(p;p^l,p^m)=\begin{cases}0, & \text{when } m<\min\{l,k\},\\ p^k, & \text{when } l\ge m\ge k,\\ p^l, & \text{when } l=m<k.\end{cases} \tag{2.4}$$

The general term in the sum on the right of (2.3) is bounded by

$$\frac{\mu(d)^2}{d^k}\prod_{i=1}^{k}(d,\pi_i)^i,$$

and when $Z \geq 1$, the pairwise coprimality of the π_i may be exploited to obtain

$$\begin{aligned}\sum_{Z<d\leq 2Z}\frac{\mu^2(d)}{d^k}\prod_{i=1}^{k}(\pi_i,d)^i &\leq Z^{-k}\sum_{e_1|\pi_1}\cdots\sum_{e_k|\pi_k}\frac{Z}{e_1\dots e_k}\prod_{i=1}^{k}e_i^i\\ &\ll q^{\varepsilon}Z^{1-k}\prod_{i=1}^{k}\pi_i^{i-1}.\end{aligned}$$

Thus we deduce that the series on the right of (2.3), when completed to infinity, is absolutely convergent, and that

$$\sum_{1\leq d\leq z}\frac{\mu(d)}{d^k}N_k(d;q,a) = \sum_{\substack{d=1\\(d^k,q)|a}}^{\infty}\frac{\mu(d)}{d^k}(d^k,q)+O\left(q^{\varepsilon}z^{1-k}\prod_{i=1}^{k}\pi_i^{i-1}\right). \tag{2.5}$$

Suppose next that the canonical prime factorisations of q and (q,a) are

$$q=\prod_{p|q}p^{l_p} \quad\text{and}\quad (q,a)=\prod_{p|q}p^{m_p}.$$

Then by (2.4) the series on the right of (2.5) has an Euler product

$$\prod_{\substack{p|q\\ \min\{k,l_p\}\leq m_p}}\left(1-p^{-k}(p^k,p^{l_p})\right)\prod_{p\nmid q}\left(1-p^{-k}\right),$$

and the lemma then follows from (2.1) and (2.2).

We now arrive at the object of the discussion in this section. When a and q are positive integers, define $E_k(x;q,a)$ by the relation

$$\sum_{\substack{1\leq n\leq x\\ n\equiv a \pmod q}}\mu_k(n) = \frac{x}{\zeta(k)}g_k(q,(q,a))+E_k(x;q,a). \tag{2.6}$$

Also, define

$$\Upsilon_k(x,Q)=\sum_{1\leq q\leq Q}\sum_{a=1}^{q}|E_k(x;q,a)|^2. \tag{2.7}$$

We estimate $\Upsilon_k(x,Q)$ by using an argument similar to that used by Warlimont (1969) in the case $k=2$. We note that Warlimont estimates a sum analogous to $\Upsilon_2(x,Q)$, save that the second summation in the definition (2.7) is subject to the additional condition $(a,q)=1$. The latter condition simplifies the ensuing argument considerably.

Lemma 2.2. *Suppose that Q and x are positive real numbers, and that k is an integer with $k \geq 2$. Then for each positive number ε,*

$$\Upsilon_k(x,Q) \ll \begin{cases} x^{2/k+\varepsilon}Q^{2-2/k}, & \text{when } 1 \leq Q \leq x, \\ Q^2 \log(2Q), & \text{when } Q > x. \end{cases}$$

Proof. We begin by disposing of the case in which $Q > x$. When $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ it follows from (2.1) and (2.2) that $g(q,(q,a)) \ll q^{-1}$, and hence we deduce from (2.6) that $E(x;q,a) \ll 1 + x/q$. Thus by (2.7),

$$\Upsilon_k(x,Q) \ll \sum_{1\leq q\leq Q} \sum_{a=1}^{q} (1+x/q)^2 \ll Q^2 + x^2 \log(2Q),$$

and the desired conclusion follows in the case $Q > x$.

Suppose next that $Q < x$, and take z to be a parameter satisfying $1 \leq z \leq x^{1/k}$ to be chosen later. Write

$$\sum_{\substack{1\leq n\leq x \\ n\equiv a \pmod q}} \mu_k(n) = S_1(q,a) + S_2(q,a), \tag{2.8}$$

where

$$S_1(q,a) = \sum_{\substack{1\leq md^k\leq x \\ md^k\equiv a \pmod q \\ d\leq z}} \mu(d) \quad \text{and} \quad S_2(q,a) = \sum_{\substack{1\leq md^k\leq x \\ md^k\equiv a \pmod q \\ d>z}} \mu(d). \tag{2.9}$$

On recalling the notation defined in the first paragraph of this section, it follows from (2.9) that

$$S_1(q,a) = \sum_{1\leq d\leq z} \mu(d) \sum_{\substack{1\leq m\leq x/d^k \\ md^k\equiv a \pmod q}} 1 = \sum_{1\leq d\leq z} \left(\frac{x}{q}\frac{\mu(d)}{d^k} N_k(d;q,a) + O(1) \right),$$

whence by Lemma 2.1 (and employing the notation of the statement of that lemma),

$$S_1(q,a) - \zeta(k)^{-1} x g_k(q,(q,a)) \ll z + q^{\varepsilon-1} x z^{1-k} \prod_{i=1}^{k} \pi_i^{i-1}. \tag{2.10}$$

Consequently, on recalling (2.6) and (2.8),

$$|E_k(x;q,a)|^2 \ll |S_2(q,a)|^2 + z^2 + q^{2\varepsilon-2} x^2 z^{2-2k} \prod_{i=1}^{k} \pi_i^{2i-2}. \tag{2.11}$$

Next we observe that by (2.9),

$$\sum_{a=1}^{q} |S_2(q,a)|^2 \leq \sum_{\substack{1\leq md^k\leq x \\ d>z}} \sum_{\substack{1\leq ne^k\leq x \\ e>z \\ md^k\equiv ne^k \pmod q}} 1,$$

and hence

$$\sum_{1\le q\le Q}\sum_{a=1}^{q}|S_2(q,a)|^2 \le \sum_{\substack{1\le md^k\le x\\ d>z}}\ \sum_{\substack{1\le ne^k\le x\\ e>z}}\ \sum_{\substack{1\le q\le Q\\ q|(md^k-ne^k)}} 1$$
$$\ll Qx^{1+\varepsilon}z^{1-k}+x^{2+\varepsilon}z^{2-2k}, \tag{2.12}$$

on using a standard estimate for the divisor function. The π_i, defined in Lemma 2.1, satisfy $\pi_1\pi_2^2\dots\pi_k^k|(q,a)$. Therefore,

$$\sum_{1\le q\le Q}\sum_{a=1}^{q} q^{2\varepsilon-2}\prod_{i=1}^{k}\pi_i^{2i-2} \ll \sum_{d_1,\dots,d_k}\ \sum_{\substack{1\le q\le Q\\ d_1d_2^2\dots d_k^k|q}} q^{2\varepsilon-1}\prod_{i=1}^{k} d_i^{i-2}$$
$$\ll \sum_{d_1,\dots,d_k} Q^{2\varepsilon}\prod_{i=1}^{k} d_i^{-2} \ll Q^{2\varepsilon}.$$

Consequently, on recalling (2.7), (2.11) and (2.12),

$$\Upsilon_k(x,Q)\ll Q^2z^2+Q^{2\varepsilon}x^{2+\varepsilon}z^{2-2k}+Qx^{1+\varepsilon}z^{1-k},$$

and the lemma follows in the case $Q\le x$ on taking $z=(x/Q)^{1/k}$.

We conclude this section by noting, for future reference, that on taking $z=x^{1/k}$, it follows from (2.8), (2.9) and (2.10) that when q is squarefree,

$$\sum_{\substack{1\le n\le x\\ n\equiv 0 \pmod{q^{k-1}}}} \mu_k(n) = \frac{x}{\zeta(k)} g_k(q^{k-1},q^{k-1}) + O(x^{1/k})$$
$$= \frac{x}{\zeta(k)q^{k-1}}\prod_{p|q}\left(\frac{1-p^{-1}}{1-p^{-k}}\right) + O(x^{1/k}). \tag{2.13}$$

3. The major arc contribution to the L^2-mean

We now address the problem of determining the major arc contribution to the L^2-mean of $f_k(\alpha)$. Henceforth it will be convenient to regard k as a fixed integer with $k\ge 2$, and to drop the suffix k from various of our notations. We define the major arcs in our Hardy-Littlewood dissection as follows. We take x to be the basic parameter, a sufficiently large real number, and let Q be a real number with $1\le Q\le \frac{1}{2}x^{1/2}$. When $a\in\mathbb{Z}$ and $q\in\mathbb{N}$, define the major arc $\mathfrak{M}(q,a)$ by

$$\mathfrak{M}(q,a)=\{\alpha\in[0,1)\ :\ |q\alpha-a|\le Qx^{-1}\}.$$

Further, define $\mathfrak{M}$ to be the union of the major arcs $\mathfrak{M}(q,a)$ with $0\le a\le q\le Q$ and $(a,q)=1$. Notice that the latter arcs $\mathfrak{M}(q,a)$ are disjoint. We approximate $f(\alpha)$ for $\alpha\in\mathfrak{M}$ with the function $f^*(\alpha)$ defined as follows. Write

$$G(q)=\sum_{b=1}^{q} g(q,(q,b))e(b/q) \quad\text{and}\quad I(\beta)=\sum_{1\le n\le x} e(n\beta), \tag{3.1}$$

and define $f^*(\alpha)$ by

$$f^*(\alpha) = \begin{cases} \zeta(k)^{-1}G(q)I(\alpha - a/q), & \text{when } \alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Finally, write

$$\Delta(\alpha) = f(\alpha) - f^*(\alpha). \tag{3.3}$$

Before proceeding to the central topic of this section, which concerns the estimation of $\int_{\mathfrak{M}} |\Delta(\alpha)|^2 d\alpha$, it is useful to record some estimates involving $G(q)$.

Lemma 3.1. *When R is a positive real number, one has*

$$\sum_{R<q\le 2R} q|G(q)|^2 \ll R^{1/k-1}\log^2(2R) \quad \textit{and} \quad \sum_{1\le q\le R} q|G(q)| \ll R^{1/k}\log^2(2R). \tag{3.4}$$

Further,

$$\sum_{q=1}^{\infty} \phi(q)|G(q)|^2 = \zeta(k). \tag{3.5}$$

Proof. Definition (3.1) yields

$$G(q) = \sum_{d|q} g(q,d) \sum_{\substack{b=1 \\ (b,q)=d}}^{q} e(b/q) = \sum_{d|q} \mu(q/d)g(q,d).$$

Thus G is a multiplicative function, and for each prime number p and natural number l one has $G(p^l) = g(p^l,p^l) - g(p^l,p^{l-1})$. It therefore follows from (2.1) that

$$G(p^l) = \begin{cases} -p^{-k}(1-p^{-k})^{-1}, & \text{when } 1 \le l \le k, \\ 0, & \text{when } l > k. \end{cases} \tag{3.6}$$

The explicit values provided by (3.6), together with the multiplicative property of G, imply that for each positive number y,

$$\sum_{1\le q\le y} q^{2-1/k}|G(q)|^2 \le \prod_{p\le y}\left(1 + \sum_{h=1}^{k} p^{2h-h/k}G(p^h)^2\right),$$

where the product is over prime numbers. Thus for each positive number R, making use of (3.6) again, one obtains

$$\sum_{R<q\le 2R} q|G(q)|^2 \ll R^{1/k-1}\prod_{p\le 2R}(1+2/p) \ll R^{1/k-1}\log^2(2R),$$

which provides the first estimate of (3.4). The second estimate of (3.4) follows similarly.

In order to establish (3.5) we first observe that, in view of (3.4), the series $\sum_{q=1}^{\infty}\phi(q)|G(q)|^2$ is absolutely convergent. Moreover the multiplicative property of G implies that the latter series is equal to $\prod_p v_p$, where, for each prime number p,

$$v_p = 1 + \sum_{h=1}^{\infty}\phi(p^h)|G(p^h)|^2 = 1 + \sum_{h=1}^{k}\frac{p^{h-2k}(1-p^{-1})}{(1-p^{-k})^2} = (1-p^{-k})^{-1}.$$

The proof of the lemma is completed on recalling the Euler product for $\zeta(k)$.

The next lemma shows that, so long as Q is not too large, the behaviour of $f(\alpha)$ is dominated by $f^*(\alpha)$ on the major arcs.

Lemma 3.2. *Suppose that $1 \le Q \le \frac{1}{2}x^{1/2}$. Then for each positive number ε,*

$$\int_{\mathfrak{M}} |\Delta(\alpha)|^2 d\alpha \ll Q^{3-2/k} x^{2/k-1+\varepsilon}.$$

Proof. We base our treatment on a mean value estimate for $F(q,a;\delta)$, which we define for $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and δ satisfying $0 < \delta < x$, by

$$F(q,a;\delta) = \int_{|\beta| \le (2\delta)^{-1}} |\Delta(\beta + a/q)|^2 d\beta. \tag{3.7}$$

Defining the coefficients $u(n;q,a)$ by

$$u(n;q,a) = \begin{cases} \mu_k(n)e(na/q) - \zeta(k)^{-1}G(q), & \text{when } 1 \le n \le x, \\ 0, & \text{otherwise,} \end{cases} \tag{3.8}$$

we have

$$\Delta(\beta + a/q) = \sum_{n=1}^{\infty} u(n;q,a)e(n\beta).$$

In view of (3.7), it therefore follows from Lemma 1 of Gallagher (1970) that

$$F(q,a;\delta) \ll \delta^{-2} \int_{-\infty}^{\infty} \Bigg| \sum_{y < n \le y+\delta} u(n;q,a) \Bigg|^2 dy. \tag{3.9}$$

Observe next that for each positive number z, it follows from (2.6) that

$$\begin{aligned} \sum_{1 \le n \le z} \mu_k(n)e(na/q) &= \sum_{b=1}^{q} \sum_{\substack{1 \le n \le z \\ n \equiv b \pmod q}} \mu_k(n)e(ab/q) \\ &= \sum_{b=1}^{q} e(ab/q) \left(\frac{z}{\zeta(k)} g(q,(q,b)) + E(z;q,b) \right). \end{aligned} \tag{3.10}$$

Moreover, when $(a,q) = 1$ we deduce from (3.1) that

$$\sum_{b=1}^{q} e(ab/q)g(q,(q,b)) = \sum_{c=1}^{q} e(c/q)g(q,(q,c)) = G(q). \tag{3.11}$$

Thus, on writing

$$\Delta(z;q,a) = \sum_{b=1}^{q} e(ab/q)E(z;q,b), \tag{3.12}$$

we obtain from (3.8), (3.10) and (3.11) the conclusion that when $0 < z \le x$,

$$\sum_{1 \le n \le z} u(n;q,a) = \Delta(z;q,a) + O(|G(q)|). \tag{3.13}$$

Define

$$z(y) = \begin{cases} y, & \text{when } 0 \le y \le x, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$w(y) = \begin{cases} y+\delta, & \text{when } -\delta < y \le x-\delta, \\ x, & \text{when } x-\delta < y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

Then by (3.13) one has

$$\sum_{y<n\le y+\delta} u(n;q,a) = \Delta(w(y);q,a) - \Delta(z(y);q,a) + O(|G(q)|),$$

when $-\delta < y \le x$, and otherwise this sum is zero. Consequently, by (3.9),

$$\begin{aligned} F(q,a;\delta) &\ll \delta^{-2} \int_{-\delta}^{x} \left(|G(q)|^2 + |\Delta(z(y);q,a)|^2 + |\Delta(w(y);q,a)|^2 \right) dy \\ &\ll \delta^{-2} \left(x|G(q)|^2 + \delta|\Delta(x;q,a)|^2 + \int_0^x |\Delta(y;q,a)|^2 dy \right). \end{aligned} \tag{3.14}$$

We now return to our central theme. By the definition of $\mathfrak{M}$,

$$\begin{aligned} \int_{\mathfrak{M}} |\Delta(\alpha)|^2 d\alpha &= \sum_{1\le q\le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-Q/(qx)}^{Q/(qx)} |\Delta(\beta + a/q)|^2 d\beta \\ &\ll \log(2Q) \max_{1\le R\le Q} \Psi(x,Q,R), \end{aligned} \tag{3.15}$$

where

$$\Psi(x,Q,R) = \sum_{R<q\le 2R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} F(q,a;xR/(2Q)).$$

But from (3.12) one has

$$\sum_{a=1}^{q} |\Delta(z;q,a)|^2 = q \sum_{b=1}^{q} |E(z;q,b)|^2 ,$$

and on combining this observation with (3.14) and (2.7) we obtain the estimate

$$\Psi(x,Q,R) \ll \frac{Q^2}{xR^2} \sum_{R<q\le 2R} q|G(q)|^2 + \frac{Q}{x}\Upsilon(x,2R) + \frac{Q^2}{x^2R} \int_0^x \Upsilon(y,2R)dy.$$

Thus, by making use of Lemmata 2.2 and 3.1, we finally obtain

$$\begin{aligned} \max_{1\le R\le Q} \Psi(x,Q,R) &\ll x^{\varepsilon-1}Q^2 \max_{1\le R\le Q} \left(R^{1/k-3} + x^{2/k}Q^{-1}R^{2-2/k} + x^{2/k}R^{1-2/k} \right) \\ &\ll x^{\varepsilon-1}Q^2 + x^{2/k-1+\varepsilon}Q^{3-2/k}. \end{aligned}$$

The lemma is now immediate from (3.15).

4. Subtracting the major arcs

Having obtained a suitable estimate for $\int_{\mathfrak{M}} |\Delta(\alpha)|^2 d\alpha$, in the shape of Lemma 3.2, we may now complete the estimation of the major arcs in three stages. We begin with an asymptotic formula for the dominating contribution.

Lemma 4.1. *Suppose that $1 \le Q \le \frac{1}{2}x^{1/2}$. Then for each positive number ε,*

$$\int_{\mathfrak{M}} |f^*(\alpha)|^2 d\alpha = \frac{x}{\zeta(k)} + O(xQ^{1/k-1+\varepsilon}).$$

Proof. The definition of $\mathfrak{M}$, combined with (3.2), implies that

$$\int_{\mathfrak{M}} |f^*(\alpha)|^2 d\alpha = \zeta(k)^{-2} \sum_{1\le q\le Q} \phi(q)|G(q)|^2 \int_{-Q/(qx)}^{Q/(qx)} |I(\beta)|^2 d\beta. \tag{4.1}$$

On noting that by (3.1) one has

$$\int_{-Q/(qx)}^{Q/(qx)} |I(\beta)|^2 d\beta = \int_{-1/2}^{1/2} |I(\beta)|^2 d\beta + O\left(\int_{Q/(qx)}^{1/2} \|\beta\|^{-2} d\beta \right) = x + O(qx/Q),$$

it follows from (4.1) that

$$\int_{\mathfrak{M}} |f^*(\alpha)|^2 d\alpha - \zeta(k)^{-2} x \sum_{1\le q\le Q} \phi(q)|G(q)|^2 \ll \frac{x}{Q} \sum_{1\le q\le Q} q^2 |G(q)|^2.$$

The proof of the lemma is completed by making reference to Lemma 3.1.

Next we estimate a cross-term.

Lemma 4.2. *Suppose that $1 \le Q \le \frac{1}{2}x^{1/2}$. Then for each positive number ε,*

$$\int_{\mathfrak{M}} |f^*(\alpha)\Delta(\alpha)| d\alpha \ll x^{1/k+\varepsilon} Q^{1-1/(2k)}.$$

Proof. When $1 \le R \le Q$, define $\mathfrak{N}(q,a;R)$ by

$$\mathfrak{N}(q,a;R) = \left\{ \alpha \in [0,1) \ : \ |q\alpha - a| \le Rx^{-1} \right\},$$

and define $\mathfrak{N}(R)$ to be the union of the $\mathfrak{N}(q,a;R)$ with $0 \le a \le q \le R$ and $(a,q)=1$. Thus $\mathfrak{M} = \mathfrak{N}(Q)$. Define, further, $\mathfrak{P}(R) = \mathfrak{N}(2R) \backslash \mathfrak{N}(R)$. Then a dyadic dissection, followed by an application of Cauchy's inequality, yields

$$\int_{\mathfrak{M}} |f^*(\alpha)\Delta(\alpha)| d\alpha \ll \log(2Q) \max_{1\le R\le Q} \left(U_1(R)^{1/2} U_2(R)^{1/2} \right), \tag{4.2}$$

where

$$U_1(R) = \int_{\mathfrak{P}(R)} |f^*(\alpha)|^2 d\alpha \quad \text{and} \quad U_2(R) = \int_{\mathfrak{N}(2R)} |\Delta(\alpha)|^2 d\alpha.$$

But by using the definition of $\mathfrak{P}(R)$, we deduce from (3.2) that

$$U_1(R) \ll \sum_{R<q\le 2R} \phi(q)|G(q)|^2 \int_{-1/2}^{1/2} |I(\beta)|^2 d\beta + \sum_{1\le q\le R} \phi(q)|G(q)|^2 \int_{R/(qx)}^{1/2} |I(\beta)|^2 d\beta.$$

Consequently, on making use of Lemma 3.1, one obtains $U_1(R) \ll xR^{1/k-1+\varepsilon}$. Meanwhile Lemma 3.2 provides the estimate $U_2(R) \ll R^{3-2/k}x^{2/k-1+\varepsilon}$. Thus, on inserting the latter estimates into (4.2) we arrive at the conclusion

$$\int_{\mathfrak{M}} |f^*(\alpha)\Delta(\alpha)|d\alpha \ll x^{\varepsilon} \max_{1\le R\le Q} \left(x^{2/k}R^{2-1/k}\right)^{1/2},$$

and the lemma follows immediately.

In the final stage of our estimation we simply observe that

$$\int_{\mathfrak{M}} |f(\alpha)|^2 d\alpha = \int_{\mathfrak{M}} |f^*(\alpha) + \Delta(\alpha)|^2 d\alpha,$$

whence

$$\int_{\mathfrak{M}} |f(\alpha)|^2 d\alpha - \int_{\mathfrak{M}} |f^*(\alpha)|^2 d\alpha \ll \int_{\mathfrak{M}} |f^*(\alpha)\Delta(\alpha)|d\alpha + \int_{\mathfrak{M}} |\Delta(\alpha)|^2 d\alpha.$$

Thus, by Lemmata 3.2, 4.1 and 4.2, we obtain

$$\int_{\mathfrak{M}} |f(\alpha)|^2 d\alpha - \frac{x}{\zeta(k)} \ll xQ^{1/k-1+\varepsilon} + x^{1/k+\varepsilon}Q^{1-1/(2k)} + x^{2/k-1+\varepsilon}Q^{3-2/k}.$$

Having estimated the major arc contribution, we make use of (1.3) to conclude that

$$\begin{aligned}\int_{\mathfrak{m}} |f(\alpha)|^2 d\alpha &= \int_0^1 |f(\alpha)|^2 d\alpha - \int_{\mathfrak{M}} |f(\alpha)|^2 d\alpha \\ &\ll x^{1/k} + xQ^{1/k-1+\varepsilon} + x^{2/k-1+\varepsilon}Q^{3-2/k},\end{aligned}$$

and thus we have established Theorem 2.

5. The corollaries to Theorem 2

In this section we provide brief proofs of the corollaries to Theorem 2, beginning with Corollary 2.

The proof of Corollary 2 *to Theorem* 2*.* Recall the notation of Corollary 2, together with the definition of $\mathfrak{m}(Q;x)$ from the introduction. Since $[0,1)\setminus\mathfrak{m}(x^{2/5};x)$ has measure $O(x^{-1/5})$, and

$$\inf_{\alpha\in\mathcal{B}_\delta(x)} |f_2(\alpha;x)| \ge x^{2/5+\delta},$$

one has

$$\int_{\mathcal{B}_\delta(x)} d\alpha \ll x^{-1/5} + x^{-4/5-2\delta} \int_{\mathfrak{m}(x^{2/5};x)} |f_2(\alpha;x)|^2 d\alpha.$$

Then by Corollary 1 to Theorem 2 one has

$$\int_{\mathcal{B}_\delta(x)} d\alpha \ll x^{-1/5} + x^{\varepsilon-2\delta},$$

and the corollary follows.

As Corollary 3 is a well-known result, we content ourselves with only a brief sketch of the proof.

The proof of Corollary 3 *to Theorem* 2. Recall the notation of the introduction, together with that of §§3 and 4, and take $Q = x^\theta$ and $x = n$. Then

$$r_k(n) = \int_0^1 f_k(\alpha; x)^2 e(-\alpha n) d\alpha = \int_{\mathfrak{M}} f_k(\alpha; x)^2 e(-\alpha n) d\alpha + \int_{\mathfrak{m}} f_k(\alpha; x)^2 e(-\alpha n) d\alpha. \tag{5.1}$$

The second term on the right hand side of (5.1) may be estimated by use of the triangle inequality in combination with Theorem 2. We thus obtain

$$\left| \int_{\mathfrak{m}} f_k(\alpha; x)^2 e(-\alpha n) d\alpha \right| \le \int_{\mathfrak{m}(x^\theta; x)} |f_k(\alpha; x)|^2 d\alpha \ll x^{2\beta_k + \varepsilon}. \tag{5.2}$$

Meanwhile, by an argument akin to that used in §4, one obtains

$$\int_{\mathfrak{M}} f_k(\alpha; x)^2 e(-\alpha n) d\alpha - \mathcal{M} \ll x Q^{1/k-1+\varepsilon} + \int_{\mathfrak{M}} |\Delta(\alpha)|^2 d\alpha + \int_{\mathfrak{M}} |f^*(\alpha)\Delta(\alpha)| d\alpha,$$

where

$$\mathcal{M} = \zeta(k)^{-2} \sum_{1 \le q \le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} G(q)^2 e(-na/q) \int_{-1/2}^{1/2} I(\beta)^2 e(-\beta n) d\beta.$$

Thus Lemmata 3.2 and 4.2, together with some straightforward though tedious work on the singular series, provide the conclusion

$$\int_{\mathfrak{M}} f_k(\alpha; x)^2 e(-\alpha n) d\alpha = \mathfrak{S}_k(n) x + O(x^{2\beta_k + \varepsilon}). \tag{5.3}$$

The corollary follows on combining (5.1), (5.2) and (5.3).

Finally we investigate square-free numbers represented as the sum of three cubes.

The proof of Corollary 4 *to Theorem* 2. We provide only a sketch of the proof, the details involving standard techniques from the Hardy-Littlewood method. When P is a large real number, and η is a small positive number, denote by $S_\eta(P)$ the number of solutions of the equation

$$x_1^3 + y_1^3 + y_2^3 = x_2^3 + y_3^3 + y_4^3,$$

with $P/8 < x_1, x_2 \le P$ and $y_i \in \mathcal{A}(P, P^{3\eta})$ $(1 \le i \le 4)$. Also, when n is a positive integer, denote by $\rho_\eta(n)$ the number of representations of n in the form $n = x^3 + y^3 + z^3$ with $\frac{1}{4}n^{1/3} < x \le n^{1/3}$ and $y, z \in \mathcal{A}(n^{1/3}, n^\eta)$. Then plainly,

$$\sum_{P^3/2<n\le P^3} \rho_\eta(n)^2 \le S_\eta(P). \tag{5.4}$$

Moreover Theorem 1.2 of Wooley (1995) shows that when η is sufficiently small one has

$$S_\eta(P) \ll P^{13/4}. \tag{5.5}$$

We claim that for each large real number N one has

$$\sum_{N/2<n\le N} \mu^2(n)\rho_\eta(n) \gg N. \tag{5.6}$$

Temporarily assuming the truth of this claim, on applying Cauchy's inequality we deduce that

$$N^2 \ll \left| \sum_{N/2<n\le N} \mu^2(n)\rho_\eta(n)\right|^2 \le \left(\sum_{N/2<n\le N} \rho_\eta(n)^2\right)\left(\sum_{\substack{N/2<n\le N\\ \rho_\eta(n)>0}} \mu^2(n)\right),$$

whence by (5.4) and (5.5),

$$\mathcal{N}(N) \gg N^2/S_\eta(N^{1/3}) \gg N^{11/12}.$$

Thus, in order to complete the proof of the corollary it remains only to establish (5.6).

We apply the Hardy-Littlewood method. Write $P = (\frac{1}{2}N)^{1/3}$, and define the exponential sums

$$f(\alpha) = \sum_{N/2<n\le N} \mu^2(n)e(n\alpha),$$

$$g(\alpha) = \sum_{P/2<x\le P} e(\alpha x^3), \quad \text{and} \quad h(\alpha) = \sum_{x\in\mathcal{A}(P,P^{3\eta})} e(\alpha x^3).$$

Then

$$\sum_{N/2<n\le N} \mu^2(n)\rho_\eta(n) \ge \int_0^1 g(\alpha)h(\alpha)^2 f(-\alpha)d\alpha. \tag{5.7}$$

When Q is a positive real parameter, write $\mathfrak{m}(Q) = \mathfrak{m}(Q;N)$ and $\mathfrak{M}(Q) = [0,1) \setminus \mathfrak{m}(Q;N)$. We first take $Q = P^{2/3}$, and note that an application of Schwarz's inequality together with Theorem 2 and (5.5) leads to the bound

$$\int_{\mathfrak{m}(Q)} |f(\alpha)g(\alpha)h(\alpha)^2|d\alpha \le \left(\int_{\mathfrak{m}(Q)} |f(\alpha)|^2 d\alpha\right)^{1/2} \left(\int_0^1 |g(\alpha)h(\alpha)^2|^2 d\alpha\right)^{1/2}$$
$$\ll N^\varepsilon \left(NQ^{-1/2} + Q^2\right)^{1/2} (P^{13/4})^{1/2} = o(N).$$

Next, defining $f^*(\alpha)$ and $\Delta(\alpha)$ as in (3.2) and (3.3), mutatis mutandis, one has by a similar argument (now using Lemma 3.2),

$$\int_{\mathfrak{M}(Q)} |\Delta(\alpha)g(\alpha)h(\alpha)^2| d\alpha \ll N^{\varepsilon} \left(Q^2\right)^{1/2} \left(P^{13/4}\right)^{1/2} = o(N).$$

Consequently,

$$\int_0^1 f(-\alpha)g(\alpha)h(\alpha)^2 d\alpha = \int_{\mathfrak{M}(Q)} f^*(-\alpha)g(\alpha)h(\alpha)^2 d\alpha + o(N). \tag{5.8}$$

We now prune back to moduli bounded by X, where X is a sufficiently small power of $\log N$. When $0 \le a \le q \le X$, $(a,q) = 1$ and $|\alpha - a/q| \le q^{-1}XN^{-1}$, one has, by Theorem 4.1 and Lemma 4.6 of Vaughan (1981),

$$g(\alpha) \ll Pq^{-1/3}(1 + P^3|\alpha - a/q|)^{-1/3},$$

whence

$$\int_{\mathfrak{M}(Q)\setminus\mathfrak{M}(X)} |f^*(\alpha)g(\alpha)h(\alpha)^2| d\alpha \ll I_1 + I_2, \tag{5.9}$$

where

$$I_1 = \sum_{X<q\le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} PNq^{-1/3}|G(q)| \int_{-1/2}^{1/2} |h(\beta + a/q)|^2 (1 + N|\beta|)^{-4/3} d\beta \tag{5.10}$$

and

$$I_2 = \sum_{1\le q\le X} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} PNq^{-1/3}|G(q)| \int_{X/(qN)}^{1/2} |h(\beta + a/q)|^2 (1 + N|\beta|)^{-4/3} d\beta. \tag{5.11}$$

We treat I_1 by dropping the condition $(a,q) = 1$ from the second summation of (5.10), so that

$$I_1 \ll \sum_{X<q\le Q} Pq^{2/3}|G(q)|\mathcal{I}(P,q), \tag{5.12}$$

where $\mathcal{I}(P,q)$ denotes the number of solutions of the congruence $x^3 \equiv y^3 \pmod{q}$ with $1 \le x, y \le P$. Similarly, one has

$$I_2 \ll \sum_{1\le q\le X} PX^{-1/3}q|G(q)|\mathcal{I}(P,q). \tag{5.13}$$

In view of (3.6), one has $G(q) = 0$ unless q is cubefree. For those q which make a non-zero contribution to the sums in (5.12) or (5.13) we can therefore write $q = q_1 q_2^2$ with q_1, q_2 square-free, and satisfying $(q_1, q_2) = 1$. For a fixed choice of y, the number of solutions of the congruence $x^3 \equiv y^3 \pmod{q}$ may be estimated by using the Chinese Remainder Theorem. When p is a prime number exceeding 3, the congruence $x^3 \equiv y^3 \pmod p$ has at most 3 solutions, and the congruence $x^3 \equiv y^3 \pmod{p^2}$ has at most 3 solutions when $p \nmid y$, and $O(p)$ solutions when $p|y$. Thus, for a fixed choice of y, the number of solutions of the congruence $x^3 \equiv y^3 \pmod q$, with $1 \le x \le q$, is $O(q^\varepsilon(y, q_2))$. We therefore deduce that when $q < P$, one has $\mathcal{I}(P, q) \ll P^2 q^{\varepsilon - 1}$. On recalling (5.12) and (5.13), and applying the methods of the proof of Lemma 3.1, we therefore deduce that

$$I_1 + I_2 \ll P^3 X^{\varepsilon - 1/3}.$$

Consequently, by (5.9),

$$\int_{\mathfrak{M}(Q)\setminus\mathfrak{M}(X)} |f^*(\alpha)g(\alpha)h(\alpha)^2| d\alpha = o(N),$$

and by (5.7) and (5.8) the desired conclusion (5.6) will follow provided that one can establish the lower bound

$$\int_{\mathfrak{M}(X)} |f^*(\alpha)g(\alpha)h(\alpha)^2| d\alpha \gg N. \tag{5.14}$$

However, since X is a small power of $\log N$, our major arcs are now so thin that the process of replacing the exponential sums f, g and h by their major arc approximations is routine (see, for example, Vaughan (1989)), and thus we easily obtain an asymptotic formula for the left hand side of (5.14). The corollary follows easily.

6. Lower bounds for L^1-means

In this section we concern ourselves with lower bounds for L^1-means. We start by discussing the idea with which we opened this paper, namely that applications of the circle method can frequently be simplified provided that suitable L^1-means are small. In order to provide a more concrete formulation of this idea, consider a set of natural numbers $\mathcal{A}$. When N is a large real number, we write $\mathcal{A}(N)$ for the set $\mathcal{A} \cap [1, N]$. For the purposes of illustration, suppose that we wish to solve a diophantine problem involving kth powers of elements of $\mathcal{A}$, where k is a natural number. Then it is desirable to estimate the exponential sum $\sum_{x \in \mathcal{A}(N)} e(\alpha x^k)$. Suppose that $\mathcal{A}$ is reasonably dense, so that $\text{card}(\mathcal{A}(N)) \gg N^{1-\varepsilon}$, and that the L^1 mean of the exponential sum generated by $\mathcal{A}$ is small, so that

$$\int_0^1 \left| \sum_{x \in \mathcal{A}(N)} e(\alpha x) \right| d\alpha \ll N^\varepsilon. \tag{6.1}$$

Then the generating function $\sum_{x \in \mathcal{A}(N)} e(\alpha x^k)$ may be estimated by noting that

$$\begin{aligned}\sum_{x \in \mathcal{A}(N)} e(\alpha x^k) &= \sum_{1 \le y \le N} e(\alpha y^k) \int_0^1 \sum_{x \in \mathcal{A}(N)} e(\beta(y - x)) d\beta \\ &\le \sup_{\beta \in [0,1)} \left| \sum_{1 \le y \le N} e(\alpha y^k + \beta y) \right| \int_0^1 \left| \sum_{x \in \mathcal{A}(N)} e(\beta x) \right| d\beta.\end{aligned}$$

Then in view of our hypothesis on the L^1-mean, we obtain

$$\sum_{x\in\mathcal{A}(N)} e(\alpha x^k) \ll N^{\varepsilon} \sup_{\beta\in[0,1)} \left| \sum_{1\le y\le N} e(\alpha y^k + \beta y)\right|. \tag{6.2}$$

Moreover, since the latter exponential sum has variables running over a complete interval, it may be estimated for suitable α via Weyl differencing, or by applying Vinogradov's methods.

We now adapt the argument of Vaughan (1988) to establish Theorem 3.

The proof of Theorem 3. Since the intervals $(b/q-\varepsilon_q, b/q+\varepsilon_q)$ are disjoint for $1\le b\le q\le Q$ and $(b,q)=1$, one has

$$\int_0^1 |g(\alpha;\boldsymbol{\lambda})|\,d\alpha \ge \sum_{1\le q\le Q} \sum_{\substack{b=1\\(b,q)=1}}^{q} \int_{-\varepsilon_q}^{\varepsilon_q} |g(\beta+b/q;\boldsymbol{\lambda})|\,d\beta. \tag{6.3}$$

Moreover the function $e(z)$ is unimodular for real values of z, so that for each integer a and real number α, one has

$$|g(\alpha;\boldsymbol{\lambda})| = \left| \sum_{1\le n\le x} \lambda_n e\left((n-a)\alpha\right)\right|.$$

Consequently, on applying the triangle inequality to (6.3), and employing an elementary manipulation involving the Möbius function, one obtains

$$\int_0^1 |g(\alpha;\boldsymbol{\lambda})|\,d\alpha \ge \sum_{1\le q\le Q} \max_{1\le a\le q} \left|\mathcal{M}_{q,a}\right|, \tag{6.4}$$

where

$$\mathcal{M}_{q,a} = \sum_{b=1}^{q} \sum_{d|(b,q)} \mu(d) \sum_{1\le n\le x} \lambda_n e((n-a)b/q) \int_{-\varepsilon_q}^{\varepsilon_q} e((n-a)\beta)d\beta.$$

But

$$\sum_{b=1}^{q} \sum_{d|(b,q)} \mu(d) e\left((n-a)b/q\right) = \sum_{d|q} \mu(d) \sum_{\substack{b=1\\d|b}}^{q} e\left((n-a)b/q\right),$$

so that by orthogonality,

$$\mathcal{M}_{q,a} = \sum_{1\le n\le x} \lambda_n \frac{\sin(2\pi(n-a)\varepsilon_q)}{\pi(n-a)} \sum_{d|q} \mu(d)\Delta_{n,a}(q/d), \tag{6.5}$$

where $\Delta_{n,a}(r)$ is defined to be r when $n\equiv a \pmod r$, and to be zero otherwise. The theorem follows immediately from (6.4) and (6.5).

The proof of Corollary 1 to Theorem 3 is essentially trivial on expanding $\sin(bt)/t$ as a power series in t.

The proof of Corollary 1 *to Theorem* 3. Recall the notation of the statement of Theorem 3. Suppose that $\delta(x)$ satisfies the hypothesis of the statement of the corollary, and apply Theorem 3 with $\varepsilon_q = \delta(x)x^{-1}$. When $1 \le a \le q \le Q$ one has

$$S_{q,a} = \sum_{d|q} \mu(q/d) d \sum_{\substack{1\le n\le x\\ n\equiv a \pmod d}} \lambda_n \frac{2\delta(x)}{x}\left(1 + O((\delta(x))^2)\right).$$

Thus

$$\begin{aligned}\sum_{1\le q\le Q} \max_{1\le a\le q} &\left(|S_{q,a}| - \frac{2\delta(x)}{x} |E_{q,a}(x;\boldsymbol{\lambda})|\right) \\ &\ll \frac{\delta(x)^3}{x} \sum_{1\le q\le Q}\sum_{d|q} d \max_{1\le a\le q} \sum_{\substack{1\le n\le x\\ n\equiv a \pmod d}} |\lambda_n|, \\ &\ll \tilde{\lambda}(x)\delta(x)^3 \sum_{1\le q\le Q}\sum_{d|q}(1+d/x).\end{aligned}$$

The corollary follows from Theorem 3 on using an elementary divisor sum estimate.

We now illustrate the consequences of Theorem 3 with several examples. It is useful at this point to record the observation that when $a \le n \le x$, and $\varepsilon_q = c/x$ with $c \le 1/2$, one has

$$\frac{\sin(2\pi(n-a)\varepsilon_q)}{\pi(n-a)} \ge \frac{\sin(2\pi c)}{\pi x}.$$

Moreover the lower bound in the last inequality is maximised by setting $c = 1/4$.

Our first example provides a proof of the lower bound of (1.2).

Example 6.1. *One has*

$$\int_0^1 \left| \sum_{1\le n\le x} \mu_k(n) e(n\alpha)\right| d\alpha \gg x^{1/(2k)}. \tag{6.6}$$

Proof. We apply Theorem 3 with $\lambda_n = \mu_k(n)$, $Q = x^{1/2}$ and $\varepsilon_q = (4x)^{-1}$. Take $\mathcal{Q}$ to be the set of kth powers of squarefree numbers between 1 and Q, and observe that when q is squarefree one has

$$\left| \sum_{d|q^k} \mu(q^k/d) d \sum_{\substack{1\le n\le x\\ n\equiv 0 \pmod d}} \mu_k(n) \frac{\sin(2\pi n\varepsilon_q)}{\pi n}\right| \ge \frac{q^{k-1}}{\pi x} \sum_{\substack{1\le n\le x\\ n\equiv 0 \pmod{q^{k-1}}}} \mu_k(n).$$

Thus the conclusion of Theorem 3 yields

$$\int_0^1 |g(\alpha;\boldsymbol{\lambda})| d\alpha \ge \frac{1}{\pi x} \sum_{q^k\in\mathcal{Q}} q^{k-1} \sum_{\substack{1\le n\le x\\ n\equiv 0 \pmod{q^{k-1}}}} \mu_k(n).$$

On recalling the asymptotic formula (2.13), we therefore deduce that

$$\int_0^1 |g(\alpha;\boldsymbol{\lambda})|d\alpha \gg \sum_{q^k\in\mathcal{Q}} \frac{q^{k-1}}{x}\left(\frac{x\phi(q)}{q^k}+O(x^{1/k})\right) \gg \sum_{q^k\in\mathcal{Q}} \frac{\phi(q)}{q} + O(x^{\frac{1}{k}-\frac{1}{2}}). \quad (6.7)$$

But

$$\sum_{q^k\in\mathcal{Q}} \frac{\phi(q)}{q} = \sum_{1\le q\le x^{\frac{1}{2k}}} \mu^2(q)\frac{\phi(q)}{q},$$

and a simple elementary argument shows that the latter sum is $\gg x^{1/(2k)}$. The lower bound (6.6) now follows immediately from (6.7).

We next consider almost primes, establishing via a simple argument that the L^1-mean of the exponential sum over such numbers is large.

Example 6.2. *Let $\lambda_r(n)$ denote the characteristic function of the P_r-numbers, so that $\lambda_r(n)=1$ when the number of prime divisors of n is r, and $\lambda_r(n)=0$ otherwise. Then when r is a fixed natural number, one has*

$$\int_0^1 \left|\sum_{1\le n\le x} \lambda_r(n)e(n\alpha)\right| d\alpha \ge (1+o(1))\frac{(\log\log x)^{r-1}}{(r-1)!}\cdot\frac{x^{1/2}}{\pi(\log x)^2}. \quad (6.8)$$

Proof. We apply Theorem 3 with $\lambda_n=\lambda_r(n)$, $Q=x^{1/2}$ and $\varepsilon_q=(4x)^{-1}$. Take $\mathcal{Q}$ to be the set of prime numbers between 1 and Q, and observe that when q is a prime number one has

$$\left|\sum_{d|q}\mu(q/d)d \sum_{\substack{1\le n\le x\\ n\equiv 0 \pmod d}} \lambda_r(n)\frac{\sin(2\pi n\varepsilon_q)}{\pi n}\right| \ge \frac{1}{\pi x}\sum_{1\le n\le x}\lambda_r(n) - \frac{q}{x}\sum_{1\le n\le x/q}\lambda_{r-1}(n).$$

The lower bound (6.8) follows from the conclusion of Theorem 3 on recalling the well-known (see Theorem 437 of Hardy and Wright (1989), for example) estimate

$$\sum_{1\le n\le x}\lambda_s(n) \sim \frac{(\log\log x)^{s-1}}{(s-1)!}\cdot\frac{x}{\log x} \quad (s\ge 1).$$

We note that a more sophisticated argument should permit a slight sharpening of the above lower bound. Indeed, in the case $r=1$, corresponding to the primes, it is a simple matter to adapt our argument to obtain a lower bound of the form $cx^{1/2}/\log x$ for a suitable positive constant c. However, a bound of the latter strength is essentially immediate from Vaughan (1988).

The new iterative methods in additive number theory (see, for example, Vaughan & Wooley (1995)) employ exponential sums over smooth numbers, which is to say integers all of whose prime factors are small. In view of the discussion leading to (6.2) it is of some interest to determine bounds on the L^1-mean of exponential sums over such integers.

Example 6.3. *When η is a real number with $0<\eta<1/2$, and x is large, one has*

$$\int_0^1 \Bigg| \sum_{n\in\mathcal{A}(x,x^\eta)} e(n\alpha)\Bigg| d\alpha \gg_\eta \frac{x^{1/2}}{\log x}. \tag{6.9}$$

Proof. We apply Theorem 3 with $Q=x^{1/2}$, $\varepsilon_q=(4x)^{-1}$, and with $(\boldsymbol{\lambda})$ defined by $\lambda_n=1$ when $n\in\mathcal{A}(x,x^\eta)$, and $\lambda_n=0$ otherwise. Take $\mathcal{Q}$ to be the set of prime numbers between x^η and Q, and observe that when q is such a prime number one has

$$\Bigg|\sum_{d|q}\mu(q/d)d \sum_{\substack{1\le n\le x\\ n\equiv 0 \pmod d}} \lambda_n \frac{\sin(2\pi n\varepsilon_q)}{\pi n}\Bigg| \ge \frac{1}{\pi x}\sum_{n\in\mathcal{A}(x,x^\eta)} 1 \gg_\eta 1.$$

The lower bound (6.9) therefore follows immediately from the conclusion of Theorem 3.

A similar argument provides a lower bound of the same strength when the set $\mathcal{A}(x,x^\eta)$ is replaced by the set of integers up to x, all of whose prime factors exceed x^η. In this case one modifies the treatment by taking $\mathcal{Q}$ to be the set of squarefree integers in $\mathcal{A}(x^{1/2},x^\eta)$.

We conclude by remarking that Theorem 3 yields large L^1-means in circumstances where the sequence $(\boldsymbol{\lambda})$ is defined by some large sifting process. Thus further examples could be provided by taking λ_n to be the number of representations of n by a binary quadratic form, or indeed any norm form.

7. The proof of the upper bound in Theorem 1

The argument which leads to the upper bound for the L^1-mean given in (1.2) is elementary, making use of little more than a simple identity involving the Möbius function. Let x be a large real number, and let y be a parameter with $1\le y\le x^{1/k}$ to be chosen later. We note that

$$f_k(\alpha;x)=\sum_{1\le rs^k\le x}\mu(s)e(\alpha r s^k),$$

and write

$$f_k(\alpha;x)=S_1(\alpha)+S_2(\alpha), \tag{7.1}$$

where

$$S_1(\alpha)=\sum_{1\le s\le y}\mu(s)\sum_{1\le r\le x/s^k} e(\alpha r s^k), \tag{7.2}$$

and

$$S_2(\alpha)=\sum_{y<s\le x^{1/k}}\mu(s)\sum_{1\le r\le x/s^k} e(\alpha r s^k). \tag{7.3}$$

We first treat $S_1(\alpha)$, noting that by the triangle inequality,

$$\int_0^1 |S_1(\alpha)|d\alpha \le \sum_{1\le s\le y}\int_0^1 \Bigg|\sum_{1\le r\le x/s^k} e(\alpha r s^k)\Bigg| d\alpha.$$

Thus, by a change of variable,

$$\int_0^1 |S_1(\alpha)| d\alpha \le \sum_{1\le s\le y} \int_0^1 \left| \sum_{1\le r\le x/s^k} e(\alpha r) \right| d\alpha \ll \sum_{1\le s\le y} \int_0^1 \min\left\{ x/s^k, \|\alpha\|^{-1} \right\} d\alpha.$$

Consequently,

$$\int_0^1 |S_1(\alpha)| d\alpha \ll \sum_{1\le s\le y} \log(2x/s^k) \ll y\log(2x). \tag{7.4}$$

On the other hand, by Schwarz's inequality, one finds from (7.3) that

$$\left(\int_0^1 |S_2(\alpha)| d\alpha \right)^2 \le \int_0^1 |S_2(\alpha)|^2 d\alpha \le \sum_{\substack{1\le rs^k\le x\\ s>y}} \sum_{\substack{1\le uv^k\le x\\ v>y\\ rs^k=uv^k}} 1.$$

Thus, on using an elementary estimate for the divisor function, we deduce that

$$\int_0^1 |S_2(\alpha)| d\alpha \ll x^\varepsilon (xy^{1-k})^{1/2}. \tag{7.5}$$

On combining (7.1), (7.4) and (7.5) we deduce, again by the triangle inequality, that

$$\int_0^1 |f_k(\alpha; x)| d\alpha \ll x^\varepsilon (y + x^{1/2} y^{(1-k)/2}).$$

Hence, on taking $y = x^{1/(k+1)}$, we finally obtain the desired upper bound in (1.2).

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