

# THE SPECTRUM OF MULTIPLICATIVE FUNCTIONS

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*Dedicated to Richard Guy on his 80th birthday, for all the inspiring problems that he has posed*

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The first author is a Presidential Faculty Fellow. He is also supported, in part, by the National Science Foundation. The second author is supported by the American Institute of Mathematics (AIM), and, in part, by the National Science Foundation (DMS 97-29992).

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

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## 1. INTRODUCTION: DEFINITIONS AND PROPERTIES OF THE SPECTRUM

Let  $S$  be a subset of the unit disc  $\mathbb{U}$ , and let  $\mathbb{T}$  be the unit circle. Let  $\mathcal{F}(S)$  denote the class of completely (totally) multiplicative functions<sup>1</sup>  $f$  such that  $f(p) \in S$  for all primes  $p$ . Our main concern is:

*What numbers arise as mean-values of functions in  $\mathcal{F}(S)$ ?*

Precisely, we define

$$\Gamma_N(S) = \left\{ \frac{1}{N} \sum_{n \leq N} f(n) : f \in \mathcal{F}(S) \right\} \quad \text{and} \quad \Gamma(S) = \lim_{N \rightarrow \infty} \Gamma_N(S).$$

Here and henceforth, if we have a sequence of subsets  $J_N$  of the unit disc  $\mathbb{U} := \{|z| \leq 1\}$ , then by writing  $\lim_{N \rightarrow \infty} J_N = J$  we mean that  $z \in J$  if and only if there is a sequence of points  $z_N \in J_N$  with  $z_N \rightarrow z$  as  $N \rightarrow \infty$ . We call  $\Gamma(S)$  the *spectrum* of the set  $S$  and the object of this paper is to understand the spectrum. Although we can determine the spectrum explicitly only in one interesting case (where  $S = [-1, 1]$ ), we are able, in general, to qualitatively describe it and obtain some of its geometric structure. For example, qualitatively, the spectrum may be described in terms of Euler products and solutions to certain integral equations. Geometrically, we can always determine the boundary points of the spectrum (that is, the elements of  $\Gamma(S) \cap \mathbb{T}$ ) and show that the spectrum is connected. Moreover we can bound the spectrum, and make conjectures about some of its properties, though we have no precise idea of what it usually looks like.

We begin with a few immediate consequences of our definition:

- $\Gamma(\{1\}) = \{1\}$ .
- If  $S_1 \subset S_2$  then  $\Gamma(S_1) \subset \Gamma(S_2)$ .
- $\Gamma(S)$  is a closed subset of the unit disc  $\mathbb{U}$ .
- $\Gamma(S) = \Gamma(\overline{S})$  (where  $\overline{S}$  denotes the closure of  $S$ ).

*Henceforth, we shall assume that  $S$  is always closed.*

One of our main results, which formed the original motivation to study the questions discussed herein, is a precise description of the spectrum of  $[-1, 1]$ .

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<sup>1</sup>That is,  $f(mn) = f(m)f(n)$  for all positive integers  $m, n$

**Theorem 1.** *The spectrum of the interval  $[-1, 1]$  is the interval  $\Gamma([-1, 1]) = [\delta_1, 1]$  where*

$$\delta_1 = 1 - 2 \log(1 + \sqrt{e}) + 4 \int_1^{\sqrt{e}} \frac{\log t}{t+1} dt = -0.656999 \dots$$

Theorem 1 tells us that for any real-valued completely multiplicative function  $f$  with  $|f(n)| \leq 1$ ,

$$(1.1) \quad \sum_{n \leq x} f(n) \geq (\delta_1 + o(1))x.$$

In 1994, Roger Heath-Brown conjectured that there is some constant  $c > -1$  such that  $\sum_{n \leq x} f(n) \geq (c + o(1))x$ . Richard Hall [6] proved this conjecture, and, in turn, conjectured (as did Hugh L. Montgomery independently) the stronger estimate (1.1). Both Hall and Montgomery noticed that the estimate (1.1) is best possible by taking

$$(1.2) \quad f(q) = \begin{cases} 1 & \text{for primes } q \leq x^{1/(1+\sqrt{e})} \\ -1 & \text{for primes } x^{1/(1+\sqrt{e})} \leq q \leq x. \end{cases}$$

In this example, the reader can verify (or see [6]) that equality holds in (1.1). Our proof shows that this is essentially the only case when equality holds in (1.1):

**Corollary 1.** *Let  $x$  be sufficiently large, and let  $f$  be any real-valued completely multiplicative function with  $-1 \leq f(n) \leq 1$ . Then*

$$\sum_{n \leq x} f(n) \geq (\delta_1 + o(1))x.$$

*Equality holds above if and only if*

$$\sum_{p \leq x^{1/(1+\sqrt{e})}} \frac{1 - f(p)}{p} + \sum_{x^{1/(1+\sqrt{e})} \leq p \leq x} \frac{1 + f(p)}{p} = o(1).$$

By applying this Corollary to the completely multiplicative function  $f(n) = \left(\frac{n}{p}\right)$ , for some prime  $p$ , we deduce that the number of integers below  $x$  that are quadratic residues (mod  $p$ ) is

$$\frac{1}{2} \sum_{n \leq x} \left(1 + \left(\frac{n}{p}\right)\right) \geq \frac{1 + \delta_1}{2} x + o(x) = (\delta_0 + o(1))x,$$

say. In fact, the constant  $\delta_0 = 0.171500 \dots$ <sup>2</sup> More colloquially we have:

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<sup>2</sup>From the definition of  $\delta_1$  in Theorem 1, we can derive the following curious expression for  $\delta_0$ :

$$\delta_0 = 1 - \frac{\pi^2}{6} - \log(1 + \sqrt{e}) \log \frac{e}{1 + \sqrt{e}} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{(1 + \sqrt{e})^n}.$$

If  $x$  is sufficiently large then, for all primes  $p$ , more than 17.15% of the integers up to  $x$  are quadratic residues (mod  $p$ ).

The constant  $\delta_0$  here is best possible. To see this, we choose  $p$  such that  $\left(\frac{q}{p}\right)$  is given as in the Hall-Montgomery example (1.2); that infinitely many such primes exist follows from quadratic reciprocity and Dirichlet's theorem on primes in arithmetic progressions.

Naturally, one wonders if similar results hold for  $m$ -th power residues. We partially answer this question by demonstrating that for any prime  $\ell$ , the set of integers below  $x$  that are  $m$ -th power residues (mod  $\ell$ ) has positive density, and that its logarithmic density exceeds  $1/2^{m-1}$ .

**Theorem 2.** For integers  $m \geq 2$ , define

$$\gamma_m = \liminf_{x \rightarrow \infty} \inf_{\ell} \frac{1}{x} \sum_{\substack{n \leq x \\ n \equiv a^m \pmod{\ell}}} 1, \quad \text{and } \gamma'_m = \liminf_{x \rightarrow \infty} \inf_{\ell} \frac{1}{\log x} \sum_{\substack{n \leq x \\ n \equiv a^m \pmod{\ell}}} \frac{1}{n}.$$

Then  $\gamma_2 = \delta_0$ ,  $\gamma'_2 = 1/2$ , and for  $m \geq 3$ ,

$$0 < \gamma_m \leq \rho(m) \left( = \frac{1}{m^{m+o(m)}} \right) < \frac{1}{2^{m-1}} \leq \gamma'_m \leq \min_{\beta \geq 0} \frac{1}{e^\beta} \sum_{k=0}^{\infty} \frac{\beta^{km}}{(km)!} \left( \sim \frac{1}{e^{m/e}} \right).$$

Here  $\rho(u)$  is the Dickman-de Bruijn function, defined by  $\rho(u) = 1$  for  $0 \leq u \leq 1$ , and  $u\rho'(u) = -\rho(u-1)$  for all  $u \geq 1$ .

We do not know the exact values of  $\gamma_m$  and  $\gamma'_m$  for any  $m \geq 3$ . By calculating numerically the minimum over  $\beta$  in Theorem 2, we found that  $\gamma'_3 \leq 0.3245$ ,  $\gamma'_4 \leq 0.2187$ ,  $\gamma'_5 \leq 0.14792$ , and  $\gamma'_6 \leq 0.1003$ . Theorem 2 implies

For given integer  $m \geq 2$ , there exists a constant  $\pi_m > 0$  such that if  $x$  is sufficiently large then, for all primes  $p$ , more than  $\pi_m\%$  of the integers up to  $x$  are  $m$ th power residues (mod  $p$ ).

We now proceed to a more systematic treatment of the spectrum. For a given  $f \in \mathcal{F}(S)$ , the mean-value of  $f$  (that is,  $\lim_{x \rightarrow \infty} x^{-1} \sum_{n \leq x} f(n)$ ), if it exists, is obviously an element of the spectrum  $\Gamma(S)$ . We begin by trying to understand the subset of the spectrum consisting of such mean-values.

Let  $f$  be any multiplicative function with  $|f(n)| \leq 1$  for all  $n$ . Throughout this paper we define

$$\Theta(f, x) := \prod_{p \leq x} \left( 1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots \right) \left( 1 - \frac{1}{p} \right).$$

In [13], A. Wintner showed, by a simple convolution argument, that if  $\sum_p |1 - f(p)|/p$  converges then

$$(1.3) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n) = \Theta(f, \infty);$$

and so, if  $f \in \mathcal{F}(S)$  then  $\Theta(f, \infty) \in \Gamma(S)$ . As an application of Wintner's result, we can take  $f(p) = 1$  for all  $p > x$ , as long as  $1 \in S$ , so that  $\Theta(f, x) \in \Gamma(S)$  for all  $x$ . Thus, if  $1 \in S$ , we define the *Euler product spectrum* of  $S$  as

$$\Gamma_{\Theta}(S) = \lim_{x \rightarrow \infty} \{\Theta(f, x) : f \in \mathcal{F}(S)\},$$

which is a closed subset of  $\Gamma(S)$ . If  $1 \notin S$  then define  $\Gamma_{\Theta}(S) = \{0\}$ .

Proving an old conjecture of Erdős and Wintner, Wirsing [14] showed that every real multiplicative function with  $|f(n)| \leq 1$  has a mean-value. In fact, he proved that (1.3) always holds for such functions. Thus, when  $S \subset [-1, 1]$  Wirsing's Theorem gives that  $\Gamma_{\Theta}(S)$  is precisely the set of mean-values of elements in  $\mathcal{F}(S)$ . In view of Wintner's result, the critical point in Wirsing's Theorem is to show that if  $f$  is real valued and  $\sum_p (1 - f(p))/p$  diverges, then  $x^{-1} \sum_{n \leq x} f(n) \rightarrow 0$ .

The situation is more delicate for complex valued multiplicative functions. For example, the function  $f(n) = n^{i\alpha}$  ( $\alpha$  a non-zero real) does not have a mean-value; indeed  $\sum_{n \leq x} f(n) \sim x^{1+i\alpha}/(1+i\alpha)$ . Note that here  $\sum_p (1 - \operatorname{Re} p^{i\alpha})/p$  diverges but  $x^{-1} \sum_{n \leq x} n^{i\alpha}$  does not tend to 0. Halász [2] excluded this example by requiring that the set  $\{f(p)\}$  be everywhere dense on  $\mathbb{T}$ . In fact, he proved that if  $\sum_p (1 - \operatorname{Re} f(p)p^{-i\xi})/p$  diverges (which obviously does not hold for the troublesome example  $n^{i\alpha}$ ) for all real  $\xi$  then  $x^{-1} \sum_{n \leq x} f(n) \rightarrow 0$ ; and he quantified how fast this tends to 0.

**Lemma 1 (Halász).** *Let  $f$  be a multiplicative function with  $|f(n)| \leq 1$  for all  $n$ , and set*

$$M(f, x) = \min_{|t| \leq \log x} \sum_{p \leq x} \frac{1 - \operatorname{Re} f(p)p^{-it}}{p}.$$

Then

$$\sum_{n \leq x} f(n) \ll x e^{-M(f, x)/16}.$$

Halász comments that the factor 1/16 in the exponent can be replaced by the optimal constant 1. Over the years Halász' Theorem has been considerably refined ([5,7]), and recently Hall [5] found the following useful formulation.

**Lemma 1' (Hall).** *Let  $D$  be a convex subset of  $\mathbb{U}$  containing 0. If  $f \in \mathcal{F}(D)$  then*

$$\sum_{n \leq x} f(n) \ll x \exp\left(-\eta(D) \sum_{p \leq x} \frac{1 - \operatorname{Re} f(p)}{p}\right),$$

where  $\eta(D)$  is a constant determined by the geometry of  $D$  (see [5]). In particular, if  $\lambda(D)$  denotes the perimeter length of  $D$  then  $\eta(D) \geq (1 - \lambda(D)/2\pi)/2$ .

As a byproduct of our investigations here, we have been able to obtain explicit quantitative versions of Lemma 1 (with the strong exponent 1 there) and Lemma 1'. These will appear elsewhere.

Lemmas 1 and 1' are important tools in all our subsequent work here. We now note two immediate consequences of these results: If  $1 \notin S$  then (recalling that  $S$  is closed)

$$\sum_{p \leq x} \frac{1 - \operatorname{Re} f(p)}{p} \gg \sum_{p \leq x} \frac{1}{p} = \log \log x + O(1).$$

Further, we see easily that  $S$  can be contained in a convex region with perimeter length  $< 2\pi$ . By Lemma 1', it follows that  $\Gamma(S) = \{0\}$ . Thus we state

- If  $1 \notin S$  then  $\Gamma(S) = \{0\}$ .

Henceforth, we shall assume  $1 \in S$ .

Our second consequence characterizes the subsets  $S$  of  $\mathbb{U}$  with the property that (1.3) holds for all  $f \in \mathcal{F}(S)$ . Wirsing's result states that subsets of  $[-1, 1]$  have this property. To formulate our characterization fluidly, and for subsequent results, we introduce the notion of the *angle* of a set. For any  $V \subseteq \mathbb{U}$ , define

$$(1.4) \quad \operatorname{Ang}(V) := \sup_{\substack{v \in V \\ v \neq 1}} |\arg(1 - v)|.$$

Note that each such  $1 - v$  has positive real part, so  $0 \leq \operatorname{Ang}(V) \leq \pi/2$ . We adopt the convention that  $\operatorname{Ang}(\{1\}) = \operatorname{Ang}(\emptyset) = 0$ . Sometimes we will speak of the angle of a point  $z \in \mathbb{U}$  ( $z \neq 1$ ); by this we mean  $\operatorname{Ang}(z) = |\arg(1 - z)|$ .

**Corollary 2.** *Suppose  $S \subset \mathbb{U}$  and  $\operatorname{Ang}(S) < \pi/2$ . Then (1.3) holds for every  $f \in \mathcal{F}(S)$ ; that is, every  $f \in \mathcal{F}(S)$  has a mean-value. Thus,*

$$\Gamma_{\Theta}(S) = \left\{ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} f(n) : f \in \mathcal{F}(S) \right\} = \left\{ \Theta(f, \infty) : f \in \mathcal{F}(S) \right\}.$$

If  $S \subset [-1, 1]$  then  $\operatorname{Ang}(S) = 0$ , and so Corollary 2 generalizes Wirsing's result. If  $\alpha \neq 0$  is real then  $\operatorname{Ang}(\{p^{i\alpha}\}) = \pi/2$ , and thus Corollary 2 avoids the example  $f(n) = n^{i\alpha}$ . Corollary 2 follows from Wintner's result in the case that  $\sum_p |1 - f(p)|/p$  converges. If  $\operatorname{Ang}(S) < \pi/2$ , and  $f(p) \in S$  then  $|1 - f(p)| \asymp 1 - \operatorname{Re} f(p)$ . So the divergence of  $\sum_p |1 - f(p)|/p$  is equivalent to the divergence of  $\sum_p (1 - \operatorname{Re} f(p))/p$ , and so by Lemma 1' the mean-value of  $f$  is 0. This proves Corollary 2.

In general, the spectrum contains more elements than simply the Euler products. For example, the spectrum of Euler products for  $S = [-1, 1]$  is simply the interval  $[0, 1]$ . However, as Theorem 1 shows, the spectrum of  $S$  is more exotic. We now describe a family of integral equations whose solutions belong to the spectrum. In Theorem 3, we shall show that all points of the spectrum may be obtained by suitably combining an Euler product and a solution to one of these integral equations.

Recall that we assume  $S$  is closed and  $1 \in S$ . We define  $\Lambda(S)$  to be the set of values  $\sigma(u)$  obtained as follows. For a subset  $S$  of the unit disc we denote by  $S^*$  the convex hull of  $S$ . Let  $K(S)$  denote the class of measurable functions  $\chi : [0, \infty) \rightarrow S^*$  with  $\chi(t) = 1$  for

$0 \leq t \leq 1$ . We prove in Theorem 3.3 (below) that associated to each  $\chi$  there is a unique  $\sigma : [0, \infty) \rightarrow \mathbb{U}$  satisfying the following integral equation

$$(1.5) \quad u\sigma(u) = \sigma * \chi(u) = \int_0^u \sigma(u-t)\chi(t)dt \quad \text{for } u > 1,$$

with the initial condition  $\sigma(u) = 1$  for  $0 \leq u \leq 1$ .

Here, and throughout,  $f * g$  denotes the convolution of the two functions  $f$  and  $g$ : that is,  $f * g(x) = \int_0^x f(t)g(x-t)dt$ .

That the integral equation (1.5) is relevant to the study of multiplicative functions was already observed by Wirsing [14]. This connection may be seen from the following Proposition.

**Proposition 1.** *Let  $f$  be a multiplicative function with  $|f(n)| \leq 1$  for all  $n$  and  $f(n) = 1$  for  $n \leq y$ . Let  $\vartheta(x) = \sum_{p \leq x} \log p$  and define*

$$\chi(u) = \chi_f(u) = \frac{1}{\vartheta(y^u)} \sum_{p \leq y^u} f(p) \log p.$$

*Then  $\chi(t)$  is a measurable function taking values in the unit disc and with  $\chi(t) = 1$  for  $t \leq 1$ . Let  $\sigma(u)$  be the corresponding unique solution to (1.5). Then*

$$\frac{1}{y^u} \sum_{n \leq y^u} f(n) = \sigma(u) + O\left(\frac{u}{\log y}\right).$$

The converse to Proposition 1 is also true:

**Proposition 1 (Converse).** *Let  $S \subset \mathbb{U}$  and  $\chi \in K(S)$  be given. Given  $\epsilon > 0$  and  $u \geq 1$  there exist arbitrarily large  $y$  and  $f \in \mathcal{F}(S)$  with  $f(n) = 1$  for  $n \leq y$  and*

$$\left| \chi(t) - \frac{1}{\vartheta(y^t)} \sum_{p \leq y^t} f(p) \log p \right| \leq \epsilon \quad \text{for almost all } 0 \leq t \leq u.$$

*Consequently, if  $\sigma(u)$  is the solution to (1.5) for this  $\chi$  then*

$$\sigma(t) = \frac{1}{y^t} \sum_{n \leq y^t} f(n) + O(u^\epsilon - 1) + O\left(\frac{u}{\log y}\right) \quad \text{for all } t \leq u.$$

If  $J$  and  $K$  are two subsets of the unit disc, we define  $J \times K$  to be the set of elements  $z = jk$  where  $j \in J$  and  $k \in K$ .

**Theorem 3 (The Structure Theorem).** *For any closed subset  $S$  of  $\mathbb{U}$  with  $1 \in S$ ,  $\Gamma(S) = \Gamma_{\Theta}(S) \times \Lambda(S)$ .*

Researchers in the field have previously used results like Proposition 1 and Theorem 3 in special, usually extreme, cases (see [8, 10, 14], for instance), but this appears to be the first attempt to provide such a result in this generality. The idea of the proof of Theorem 3 is to decompose  $f \in \mathcal{F}(S)$  into two parts:  $f_s(p) = f(p)$  for  $p \leq y$  and  $f_s(p) = 1$  for  $p > y$ , and  $f_l(p) = 1$  for  $p \leq y$  and  $f_l(p) = f(p)$  for  $p > y$ . For appropriately chosen  $y$ , the average of  $f$  until  $x$  is approximated by the product of the averages of  $f_s$  and  $f_l$ . If  $y$  is small enough compared with  $x$ , then the average of  $f_s$  is approximated by  $\Theta(f_s, \infty) \in \Gamma_{\Theta}(S)$ . Proposition 1 shows that if  $y$  is not too small, the average of  $f_l$  is approximated by the solution to an integral equation. Combining these, one gets that  $\Gamma(S) \subset \Gamma_{\Theta}(S) \times \Lambda(S)$ . The proof that  $\Gamma_{\Theta}(S) \times \Lambda(S) \subset \Gamma(S)$  is similar, invoking the converse of Proposition 1.

As the case  $S = [-1, 1]$  illustrates,  $\Gamma_{\Theta}(S)$  represents the easy part of the spectrum while  $\Lambda(S)$  is more mysterious. Here Theorems 1 and 3 tell us that  $\Lambda(S) \subset [\delta_1, 1]$ . That is, given any  $\chi \in K([-1, 1])$  we have  $\sigma(u) \geq \delta_1$  for all  $u$  (where  $\sigma$  is the corresponding solution to (1.5)). An important example is the function  $\chi(t) = 1$  for  $t \leq 1$  and  $\chi(t) = -1$  for  $t > 1$ . Denote by  $\rho_-(u)$  the corresponding solution to (1.5). Then  $\rho_-(u)$  satisfies a differential-difference equation very similar to that satisfied by the Dickman-de Bruijn function. Namely,  $\rho_-(u) = 1$  for  $u \leq 1$  and for  $u > 1$ ,

$$u\rho'_-(u) = -2\rho_-(u-1).$$

It is not hard to verify that  $\rho_-(u)$  decreases for  $u$  in  $[1, 1 + \sqrt{e}]$  and increases for  $u > 1 + \sqrt{e}$ . The absolute minimum  $\rho_-(1 + \sqrt{e})$  is guaranteed by Theorem 1 to be  $\geq \delta_1$  and in fact  $\rho_-(1 + \sqrt{e}) = \delta_1$ . By continuity,  $\rho_-(u)$  takes on all values in the interval  $[\delta_1, 1]$  showing that  $\Lambda(S) \supset [\delta_1, 1]$ .

We now describe properties of  $\Gamma_{\Theta}(S)$ , which are also inherited by  $\Gamma(S)$ : In many cases, we get an explicit description of  $\Gamma_{\Theta}(S)$ . To state our results we introduce the set  $\mathcal{E}(S)$  defined as follows: If  $1 \in S \subset \mathbb{U}$  define

$$\mathcal{E}(S) = \{e^{-k(1-\alpha)} : k \geq 0, \alpha \text{ is in the convex hull of } S\},$$

so that  $\mathcal{E}(S)$  consists of various ‘‘spirals’’ connecting 1 to 0.

**Theorem 4.** (i) *For all closed subsets  $S$  of  $\mathbb{U}$  with  $1 \in S$ ,*

$$\mathcal{E}(S) \times [0, 1] \supset \Gamma_{\Theta}(S) = \Gamma_{\Theta}(S) \times \mathcal{E}(S) \supset \mathcal{E}(S).$$

*If  $z \in \Gamma_{\Theta}(S)$  then  $|z| \leq \exp(-|\arg(z)| \cot(\text{Ang}(S)))$ .*

(ii) *If the convex hull of  $S$  contains a real point other than 1, then*

$$\Gamma_{\Theta}(S) = \mathcal{E}(S) = \mathcal{E}(S) \times [0, 1].$$

*In particular,  $\Gamma_{\Theta}(S)$  is starlike; that is,  $\Gamma_{\Theta}(S)$  contains each line joining 0 to a point  $z \in \Gamma_{\Theta}(S)$ .*

We may describe the set  $\mathcal{E}(S)$  explicitly as follows: If  $S$  does not contain any element with positive imaginary part then define  $\mathcal{I}^+ = \emptyset$ . If  $S$  does contain elements with positive



imaginary part, then let  $z^+$  be an element of  $S$  with  $\text{Im}(z^+) > 0$  and such that  $\text{Ang}(z^+)$  is the largest among all  $z \in S$  with positive imaginary part. Define now  $\mathcal{I}^+$  to be the interior of the closed curve  $\{e^{-k(1-z^+)} : 0 \leq k \leq 2\pi/|\text{Im } z^+|\} \cup [e^{-2\pi(1-\text{Re } z^+)/|\text{Im } z^+|}, 1]$ . Similarly, define  $\mathcal{I}^-$  by focussing on elements of  $S$  with negative imaginary part. Then  $\mathcal{E}(S)$  is contained in  $\mathcal{I}^+ \cup \mathcal{I}^-$ ; and if the convex hull of  $S$  contains a real point other than 1, then  $\mathcal{E}(S) = \mathcal{I}^+ \cup \mathcal{I}^-$ .

It is easy to see that

- If  $\text{Ang}(S) = \pi/2$ , then  $\mathbb{U} = \overline{\mathcal{E}(S)}$ , so that  $\Gamma(S) = \mathbb{U}$ .

Thus the spectrum is of interest only when  $\text{Ang}(S) < \pi/2$ .

Combining Theorems 3 and 4 enables us to deduce some basic properties of the spectrum (see §6b for the proof of this Corollary).

**Corollary 3.** *Let  $S$  be a closed subset of  $\mathbb{U}$  with  $1 \in S$ .*

- Then  $\Gamma(S) = \Gamma(S) \times \mathcal{E}(S)$ . Consequently, the spectrum of  $S$  is connected. If the convex hull of  $S$  contains a real point other than 1, then the spectrum is starlike, and contains the shape  $\{z : |z| \leq \exp(-|\arg(z)| \cot(\text{Ang}(S)))\}$ .*
- If  $\alpha \in S$  then  $1 - (1 - \alpha) \log u \in \Lambda(S)$  for all  $1 \leq u \leq 2$ . If  $\pi/2 > \text{Ang}(S) > 0$  then  $\Gamma(S)$  contains elements not in  $\Gamma_\Theta(S)$ .*
- If  $1, e^{i\alpha}$  and  $e^{i\beta}$  are distinct elements of  $S$  then  $\mathcal{E}(S)$ , and so  $\Gamma(S)$ , contains the disc centered at the origin with radius  $\exp(-2\pi/(|\cot(\alpha/2) - \cot(\beta/2)|))$ .*

We have seen that the sets  $\Gamma_\Theta(S)$  and  $\Gamma(S)$  have the property that multiplying by  $\mathcal{E}(S)$  leaves them unchanged. It turns out that  $\Lambda(S)$  also has this property, leading to the following variant of Theorem 3, which reveals that  $\Lambda(S)$  typically contains all the information about the spectrum (see §6c for the proof of this Theorem).

**Theorem 3'.** *If  $S$  is a closed subset of  $\mathbb{U}$  with  $1 \in S$  then*

$$\Lambda(S) = \Lambda(S) \times \mathcal{E}(S),$$

and

$$\Lambda(S) \subset \Gamma(S) \subset \Lambda(S) \times [0, 1].$$

*If the convex hull of  $S$  contains a real point different from 1 then  $\Gamma(S) = \Lambda(S)$ .*

Next we bound the spectrum and determine  $\Gamma(S) \cap \mathbb{T}$ .

**Theorem 5.** *Suppose  $S$  is a closed subset of  $\mathbb{U}$  with  $1 \in S$ . The spectrum of  $S$  is  $\mathbb{U}$  if and only if  $\text{Ang}(S) = \pi/2$ . If  $\text{Ang}(S) = \theta < \pi/2$ , then there exists a positive constant  $A(\theta)$ , depending only on  $\theta$ , such that  $\Gamma(S)$  is contained in a disc centered at  $A(\theta)$  with radius  $1 - A(\theta)$ . In fact,  $A(\theta) = (28/411) \cos^2 \theta$  is permissible. Thus*

$$\Gamma(S) \cap \mathbb{T} = \begin{cases} \{1\} & \text{if } \text{Ang}(S) < \pi/2 \\ \mathbb{T} & \text{if } \text{Ang}(S) = \pi/2. \end{cases}$$

Applied to the set  $S = [-1, 1]$ , Theorem 5 shows that there exists  $c > -1$  such that  $\Gamma(S) \subset [c, 1]$ . Thus Theorem 5 generalises Hall's result on Heath-Brown's conjecture.

If  $z \in S$  is such that  $\text{Ang}(z) = \text{Ang}(S) = \theta$  then, taking  $k = \pi/|\text{Im } z|$ , we have  $-\exp(-\pi \cot \theta) = e^{-k(1-z)} \in \mathcal{E}(S) \subset \Gamma(S)$ . Therefore  $A(\theta) \leq (1 - \exp(-\pi \cot \theta))/2 \leq \frac{\pi}{2} \cos \theta$ .

By a simple calculation, we can show that  $\mathcal{E}(S)$ ,  $\Gamma_\Theta(S)$  and  $S$  all have the same angle. From Theorems 3 and 3' we see that  $\text{Ang}(\Gamma(S)) = \text{Ang}(\Lambda(S)) \geq \text{Ang}(S)$ . We believe that these angles are all equal:

**Conjecture 1.** *The angle of the set equals the angle of the spectrum. Thus*

$$\text{Ang}(\Gamma(S)) = \text{Ang}(\Lambda(S)) = \text{Ang}(\Gamma_\Theta(S)) = \text{Ang}(\mathcal{E}(S)) = \text{Ang}(S).$$

Given  $0 \leq \theta \leq \pi/2$  define  $H^\theta$  to be the subset of  $\mathbb{U}$  inside the lines  $\arg(1-z) = \pm\theta$ : thus,  $H^\theta$  is the set of all points  $z$  with  $\text{Ang}(z) \leq \theta$ . If Conjecture 1 holds then taking  $S = H^\theta$  there we deduce that  $\Gamma(H^\theta) \subset H^\theta$ . Conversely, if  $\Gamma(H^\theta) \subset H^\theta$  then for any  $S \subset \mathbb{U}$  with  $\text{Ang}(S) = \theta$  we must have  $S \subset H^\theta$  and so  $\Gamma(S) \subset \Gamma(H^\theta) \subset H^\theta$ . It follows at once that  $\text{Ang}(\Gamma(S)) = \text{Ang}(S)$ . Thus Conjecture 1 is equivalent to the following:

**Conjecture 1'.** *With  $H^\theta$  as defined above  $\Gamma(H^\theta) \subset H^\theta$ .*

We support Conjecture 1 by showing that  $\text{Ang}(S)$  and  $\text{Ang}(\Gamma(S))$  are comparable in the situations  $\text{Ang}(S) \rightarrow 0$  and  $\text{Ang}(S) \rightarrow \pi/2$ .

**Theorem 6.** *Suppose  $S \subset \mathbb{U}$  and  $\text{Ang}(S) = \theta = \pi/2 - \delta$ .*

- (i) *Then,  $\text{Ang}(\Gamma(S)) \ll \text{Ang}(S)$ .*
- (ii) *Further,*

$$\frac{\pi}{2} - \delta = \text{Ang}(S) \leq \text{Ang}(\Gamma(S)) \leq \frac{\pi}{2} - \frac{\sin \delta}{2}.$$

The first part of the Theorem says that  $\text{Ang}(S)$  and  $\text{Ang}(\Gamma(S))$  are comparable when  $\text{Ang}(S)$  is small. The second part of the Theorem is mainly interesting in the complementary case when  $\text{Ang}(S)$  is close to  $\pi/2$ . In fact, when  $\delta$  is small we see that we are away from the truth only by a factor of 2 (as  $\sin \delta \sim \delta$ ).

*Example.* Let  $k \geq 3$  and  $S_k$  denote the set of  $k$ -th roots of unity. If  $f \in \mathcal{F}(S_k)$  then  $f(n) \in S_k$  for all  $n$ . Hence  $\Gamma(S_k)$  is contained in the convex hull of  $S_k$ : that is, in the regular  $k$ -gon with vertices the  $k$ -th roots of unity. Notice that this implies  $\text{Ang}(\Gamma(S_k)) \leq \text{Ang}(S_k)$ , so that  $\text{Ang}(S_k) = \text{Ang}(\Gamma(S_k))$  by Theorem 6(ii), supporting Conjecture 1. Applying Corollary 3(iii) with the two points  $e^{\pm 2\pi i/k}$ , we conclude that  $\Gamma(S_k)$  is starlike and contains the disc centered at 0 with radius  $\exp(-\pi \tan(\pi/k))$ . Even in this simple case we have not been able to determine the spectrum  $\Gamma(S_k)$ , though we do know that  $2\pi^3/3k^2 + o(1/k^2) \leq \pi - \text{Area}(\Gamma(S_k)) \leq 2\pi^3/k + o(1/k)$ .

We define *the projection of* (a complex number)  $z$  *in the direction*  $e^{i\alpha}$  *to be*  $\text{Re}(e^{-i\alpha}z)$ . Theorem 1 may be re-interpreted as stating that if  $z \in \Gamma(\{\pm 1\})$  then the projection of  $z$  in the direction  $-1$  is  $\leq -\delta_1$ . Evidently if  $1 \in S$  then  $1 \in \Gamma(S)$  so there is always a  $z \in \Gamma(S)$  whose projection in the direction  $1$ , is  $1$ , and thus uninteresting to us. This motivates us to define the *maximal projection of the spectrum*  $\Gamma(S)$  *of a set*  $S \subset \mathbb{T}$  *as*

$$\max_{1 \neq \zeta \in S} \max_{z \in \Gamma(S)} \text{Re}(\zeta^{-1}z).$$

**Conjecture 2.** *Let  $S$  be a closed subset of  $\mathbb{T}$  with  $1 \in S$ . If  $\text{Ang}(S) = \theta$  then the maximal projection of  $\Gamma(S)$  is*

$$\max_{1 \neq \zeta \in S} \max_{z \in \Gamma(S)} \text{Re}(\zeta^{-1}z) = 1 - (1 + \delta_1) \cos^2 \theta.$$

One half of this conjecture is easy to establish: namely, the maximal projection is  $\geq 1 - (1 + \delta_1) \cos^2 \theta$ . To see this, let  $z = x^{-1} \sum_{n \leq x} f(n)$  where  $f$  is the completely multiplicative function defined by  $f(p) = 1$  for all  $p \leq x^{1/(1+\sqrt{e})}$ , and  $f(p) = \zeta$  for  $x^{1/(1+\sqrt{e})} \leq p \leq x$ , where  $\zeta \in \mathbb{T}$  and  $\text{Ang}(\zeta) = \theta$ . Then, a simple calculation (analogous to the calculation in the Hall-Montgomery example (1.2)) gives that the projection of  $z$  along  $\zeta$  is  $1 - (1 + \delta_1) \cos^2 \theta + o(1)$ .

**Theorem 7.**

- (i) *Conjecture 2 is true for the sets  $S = \{1, -1\}$  and  $S = \{1, -1, i, -i\}$ .*
- (ii) *For any closed subset  $S$  of  $\mathbb{T}$  with  $1 \in S$ , the maximal projection of  $\Gamma(S)$  is  $\leq 1 - (56/411) \cos^2 \theta$ , where  $\theta = \text{Ang}(S)$ .*

To facilitate comparison between Theorem 7 and Conjecture 2, we observe that  $1 + \delta_1 = 0.3430\dots$  whereas  $56/411 = 0.1362\dots$ . Thus Theorem 7 is not too far away from the (conjectured) truth.

Let  $S \subset \mathbb{T}$  and  $\theta = \text{Ang}(S)$  and define  $\alpha := e^{i(\pi-2\theta)}$ . Let  $S_\theta := \{1, \alpha, \bar{\alpha}\}$  and  $S^\theta$  be  $\{1\}$  together with the arc of  $\mathbb{T}$  anticlockwise from  $\alpha$  to  $\bar{\alpha}$ . Then  $S \subset S^\theta$ , and if  $S$  is symmetric about the real axis then  $S_\theta \subset S$ . Conjecture 2 is equivalent to the conjecture that  $\Gamma(S) \subset \Gamma(S^\theta)$  is contained inside the arc of the circle of radius  $r_\theta := 1 - (1 + \delta_1) \cos^2 \theta$ , centered at the origin, going anticlockwise from  $r_\theta \alpha$  to  $r_\theta \bar{\alpha}$ , and inside the tangent lines to the circle from these two points going to the right. We suspect that one should be able to restrict  $\Gamma(S^\theta)$  more than as in Conjectures 1 and 2, particularly on the left side ( $\text{Re}(z) < 0$ ) of the plane.

If  $S_\theta \subset S$  then  $\Gamma(S_\theta) \subset \Gamma(S)$ . Collecting several results above, we have seen that  $\Gamma(S_\theta)$  contains the interior of the shape given by the line joining 1 to  $1 - (1 - \alpha) \log 2$ , the contour  $c(u) = 1 - (1 - \alpha) \log u + ((1 - \alpha)^2/2) \int_{t=1}^{u-1} (\log(u-t)/t) dt$  for  $2 \leq u \leq 1 + \sqrt{e}$ , and the spiral  $c(1 + \sqrt{e})e^{-t(1-\alpha)}$ ,  $t \geq 0$  until it hits the real axis, along with their complex conjugates.

In §8 we investigate other notions of spectrum. For fixed  $\sigma > 0$ , the spectrum of

$$\lim_{x \rightarrow \infty} \left\{ \frac{\sum_{n \leq x} \frac{f(n)}{n^\sigma}}{\sum_{n \leq x} \frac{1}{n^\sigma}} : f \in \mathcal{F}(S) \right\}$$

is evidently determined by the Euler products if  $\sigma > 1$ , and turns out to be the same as  $\Gamma(S)$  for  $0 < \sigma < 1$ , as we show at the beginning of section 8. Thus the only new and interesting case is where  $\sigma = 1$ , which gives the *logarithmic spectrum*,  $\Gamma_0(S)$ . As might be expected, the logarithmic spectrum is easier to study than  $\Gamma(S)$ . In fact  $\Gamma_0(S)$  lies inside the convex hull of  $\Gamma(S)$ . Our next result allows us to bound  $\Gamma_0(S)$  independently of  $\Gamma(S)$ .

**Theorem 8.** *Suppose  $S$  is a closed subset of  $\mathbb{U}$  with  $1 \in S$ , and let  $\mathcal{R}$  denote the closure of the convex hull of the points  $\prod_{i=1}^n \frac{1+s_i}{2}$ , for all  $n \geq 1$ , and all choices of points  $s_1, \dots, s_n$  lying in the convex hull of  $S$ . Then  $\Gamma_0(S)$  is contained in  $\mathcal{R}$ .*

As a consequence of Theorem 8, we have  $\Gamma_0([-1, 1]) = [0, 1]$ , and also, lending credence to Conjecture 1, that  $\text{Ang}(S) = \text{Ang}(\Gamma_0(S))$ .

**Corollary 4.** *Let  $S$  be a closed subset of  $\mathbb{U}$  with  $1 \in S$ .*

- (i)  $\Gamma_0([-1, 1]) = [0, 1]$ .
- (ii) *If  $\alpha \in S$  then  $1 - (1 - \alpha)(\log u - 1 + \frac{1}{u}) \in \Gamma_0(S)$  for all  $1 \leq u \leq 2$ . If  $0 < \text{Ang}(S) < \pi/2$  then  $\Gamma_0(S)$  contains elements not in  $\Gamma_\Theta(S)$ .*
- (iii)  $\text{Ang}(S) = \text{Ang}(\Gamma_0(S))$ .
- (iv) *Suppose  $\text{Ang}(S) = \frac{\pi}{2} - \delta$  with  $\delta > 0$ . If  $z \in \Gamma_0(S)$  then  $|z| \leq (\cos \delta)^{|\arg(z)|/\delta}$  where we choose  $|\arg(z)| \in [0, \pi]$ .*

Most of the ideas above generalize to the spectrum of all multiplicative functions in  $\mathbb{U}$ ; that is where  $f(mn) = f(m)f(n)$  for all pairs of coprime integers  $m, n$ . Thus the mean-value of  $f$  depends now on the (independent values of)  $f(p^k)$  with  $k \geq 2$  as well as the  $f(p)$ . A priori it is not obvious what range we should allow for the  $f(p^k)$ ; it seems that the most useful choices are  $f(p^k) \in S = S_m$ , when  $S$  is the set of  $m$ th roots of unity and, otherwise,  $f(p^k) \in \mathbb{U}$  for all  $k \geq 2$ . We call this new spectrum  $\hat{\Gamma}(S)$ , and note that  $\Gamma(S) \subset \hat{\Gamma}(S)$ . Moreover we define  $\hat{\Gamma}_\Theta(S)$  to be the set of values  $\Theta(f, x)$  as before. Now Theorem 1, Corollary 1, Lemmas 1, Corollary 2, and Propositions 1 all hold, Theorem 3 with  $\hat{\Gamma}(S) = \hat{\Gamma}_\Theta(S) \times \Lambda(S)$ . The most significant change is that the analogue to Theorem 4 is not true since  $\hat{\Gamma}_\Theta(S)$  is not necessarily a subset of  $\mathcal{E}(S) \times [0, 1]$ . For example, if  $S = \{1, -1, i, -i\}$  take  $f$  for which  $f(p^k) = i$  for each  $k \geq 1$ , and  $f(q^k) = 0$  if  $q \neq p$ , so that  $z = \Theta(f, \infty) = 1 - 1/p + i/p$  does not satisfy  $|z| \leq e^{-|\arg(z)|}$ . Changes thus need to be made in subsequent results, which are easy but messy, and the theory necessarily loses some of its elegance since, now,  $\hat{\Gamma}(S)$  rarely equals  $\Lambda(S)$ . Note also that Conjecture 1 is untrue for  $\hat{\Gamma}(S)$  since if  $f(p) = \beta^2$  and  $f(p^k) = \beta$  for each  $k \geq 2$  where  $\beta = e^{i(\pi/2 - \text{Ang}(S))}$ , and  $f(q^k) = 0$  if  $q \neq p$ , then  $\text{Ang}(\Theta(f, \infty)) > \text{Ang}(S)$ .

Define for  $B > 0$

$$\alpha(B) = \limsup_{|D| \rightarrow \infty} \frac{1}{(\log |D|)^B} \sum_{n \leq (\log |D|)^B} \left( \frac{D}{n} \right)$$

and

$$\beta(B) = \liminf_{|D| \rightarrow \infty} \frac{1}{(\log |D|)^B} \sum_{n \leq (\log |D|)^B} \left( \frac{D}{n} \right),$$

where,  $D$  represents a fundamental discriminant. Plainly  $\alpha(B) = 1$  for  $B \leq 1$ , and in [1] we showed that  $\alpha(B) \geq \rho(B)$  where  $\rho$  is the Dickman-de Bruijn function. Further, we showed there that if the Generalized Riemann Hypothesis holds then  $\alpha(B) \leq \rho(B/2)$ . The exact value of  $\alpha(B)$  is not known for any  $B > 1$ , though we do conjecture that  $\alpha(B) = \rho(B)$  for all  $B > 0$ . Regarding  $\beta$ , we see from Theorem 1 that  $\beta(B) \geq \delta_1$  for all  $B$  and, in view

of the Hall-Montgomery example,  $\beta(B) = \delta_1$  for  $B \leq 1$ . Hybridizing this consequence of Theorem 1 and our result on  $\alpha(B)$ , Mark Watkins asked us whether  $\beta(B) < 0$  for all  $B$ . We see below that this is indeed so.

**Theorem 9.** *Given  $u \geq 1$ , let  $\mathcal{C}(u)$  denote the set of all measurable functions  $\chi$  such that  $\chi(t) = 1$  for  $t \leq 1$ ,  $\chi(t) \in [-1, 1]$  for  $1 \leq t \leq u$ , and  $\chi(t) = 0$  for  $t > u$ . Define*

$$\gamma(B) = \min_{u \geq 1} \min_{\chi \in \mathcal{C}(u)} \sigma(Bu),$$

for all  $B > 0$ , where  $\sigma$  refers to the solution to (1.5). Then  $\beta(B) \leq \gamma(B)$  for all  $B > 0$ , where  $-\rho(B) \leq \gamma(B) < 0$ .

Assuming the GRH, we can show that  $\beta(B) \geq \gamma(B/2)$ . In [1], we gave our reasons for believing that  $\alpha(B) = \rho(B)$ ; these also lead us to believe that  $\beta(B) = \gamma(B)$  for all  $B$ .

To help orient the reader we supply a brief overview of the following sections, and describe the logical dependencies among them. The reader interested in a proof of the structure theorem can skip §2 and proceed to §3a and §4. After this a perusal of §3b and §5 would lead to a proof of Theorem 1. The bulk of our general results on the spectrum are covered in §6 and §7; both these sections build upon the work of §3 and §4. Next §8 deals with other notions of spectrum, chiefly the logarithmic spectrum. Again the material of §3 and §4 is assumed here. Finally §2 and §9 may be read independently of the rest of the paper.

## 2. THE NATURAL AND LOGARITHMIC DENSITIES OF $m$ TH POWER RESIDUES UP TO $x$

### 2a. The proportion of $m$ th power residues up to $x$ .

As noted in the introduction, it is clear that  $\gamma_2 = \delta_0$ . Given a set of  $m$ -th roots of unity  $\alpha_p$  for each prime  $p \leq x$ , we see (by the Chebotarev density theorem) that there are infinitely many primes  $\ell \equiv 1 \pmod{m}$  such that there is a character  $\chi \pmod{\ell}$  of order  $m$  for which  $\chi(p) = \alpha_p$  for all  $p \leq x$ . Choose  $\alpha_p = 1$  for  $p \leq x^{1/m}$ , and  $\alpha_p = e^{2\pi i/m}$  for  $x^{1/m} < p \leq x$ . Then an integer  $n \leq x$  is an  $m$ -th power residue  $\pmod{\ell}$  if and only if all its prime divisors are  $\leq x^{1/m}$ . It is well-known that the number of such integers is  $(\rho(m) + o(1))x = m^{-m+o(m)}x$ . This gives the upper bound  $\gamma_m \leq \rho(m) = m^{-m+o(m)}$ .

We now show that  $\gamma_m > 0$  for  $m \geq 3$ . To this end, we require the following result of Hildebrand [9].

**Lemma 2.1 (Hildebrand).** *Fix  $\theta > 0$ . In the two limits below the sup and inf are taken over all completely multiplicative functions  $f$  with  $0 \leq f(n) \leq 1$ , such that  $\Theta(f, x) = e^{-\theta} + o(1)$ . We have*

$$\liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n) = \rho(e^\theta) \quad \text{and} \quad \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n) \leq e^{-\theta} \int_0^{e^\theta} \rho(t) dt.$$

The lower bound is attained when  $f(p) = 1$  for all  $p \leq x^{e^{-\theta}}$ , and  $f(p) = 0$  for all larger primes  $p$ .

The exact value of the lim sup above is still not known, though it must be at least the average value,  $\geq e^{-\theta}$ . Also  $\int_0^\infty \rho(t)dt = e^\gamma$ , and so the upper bound given above is not too far from the truth. In our application, it is the lim inf result that is useful.

**Proposition 2.2.** *Suppose  $m$  is a given positive integer and  $c \geq 0$  is a given constant. For any sufficiently large integer  $n$ , and prime  $\ell > n$ , with  $\ell \equiv 1 \pmod{m}$  suppose that for some divisor  $M$  of  $m$  one has*

$$\sum_{p \in P} \frac{1}{p} \leq c$$

where  $P$  is the set of primes  $\leq x$  that are not  $M$ th power residues  $\pmod{\ell}$ . Then either more than  $\frac{M}{2m} \sum_{n \leq x, (n, P)=1} 1$  integers up to  $x$ , that are coprime to  $P$ , are  $m$ th power of residues  $\pmod{\ell}$ ;  
Or there exists a divisor  $d > 1$  of  $m/M$  such that

$$\sum_{q \in Q} \frac{1}{q} \leq \kappa(c, m)$$

where  $Q$  is the set of primes  $\leq n$  that are not  $M$ dth power residues  $\pmod{\ell}$ . Here  $\kappa(c, m)$  is a constant that depends only on  $c$  and  $m$ .

*Proof.* Let  $G$  be a set of coset representatives for the characters  $\pmod{\ell}$  of order dividing  $m$  modulo the characters  $\pmod{\ell}$  of order dividing  $M$ . Note that if  $n$  is an  $M$ th power  $\pmod{\ell}$  then

$$\sum_{x \in G} \chi(n) = \begin{cases} |G| = m/M & \text{if } n \text{ is an } m\text{th power } \pmod{\ell} \\ 0 & \text{otherwise.} \end{cases}$$

So suppose for each  $\chi \in G$ , except the identity  $\chi_0$  one has

$$(2.1) \quad \left| \sum_{\substack{n \leq x \\ (n, P)=1}} \chi(n) \right| \leq \frac{1}{2|G|} \sum_{\substack{n \leq x \\ (n, P)=1}} 1$$

Then the number of  $m$ th powers  $\pmod{\ell}$  up to  $x$  is

$$\begin{aligned} &\geq \frac{1}{|G|} \sum_{\chi \in G} \sum_{\substack{n \leq x \\ (n, P)=1}} \chi(n) \geq \frac{1}{|G|} \left( \sum_{\substack{n \leq x \\ (n, P)=1}} 1 - (|G| - 1) \cdot \frac{1}{2|G|} \sum_{\substack{n \leq x \\ (n, P)=1}} 1 \right) \\ &\geq \frac{1}{2|G|} \sum_{\substack{n \leq x \\ (n, P)=1}} 1 = \frac{M}{2m} \sum_{\substack{n \leq x \\ (n, P)=1}} 1 \end{aligned}$$

On the other hand, if (2.1) does not hold for some  $\chi \in G$ ,  $\chi \neq \chi_0$ , then suppose  $\chi$  has order  $d > 1$  in  $G$ . (Thus  $\chi(p) \neq 1$  if and only if  $p \in Q$ .)

Then, by Lemma 1' (with  $D$  the convex hull formed by the  $m$ th roots of unity),

$$\frac{M}{2m} \sum_{\substack{n \leq x \\ (n, P)=1}} 1 \ll_m x \exp\left(-c_m \sum_{p \in Q} \frac{1}{p}\right).$$

By the first part of Lemma 2.1,  $\sum_{\substack{n \leq x \\ (n, P)=1}} 1 \gtrsim x \rho(c)$ . Thus  $\sum_{p \in Q} \frac{1}{p} \ll_{c, m} 1$  which gives the result.

*Proof of Theorem 2.* First we may assume that  $x$  is sufficiently large for the argument below to work. Second we may assume that  $m$  divides  $\ell - 1$  else we replace  $m$  by  $\gcd(\ell - 1, m)$ . Third we may assume that  $\ell > x$  else the proportion of such integers is certainly  $\gg 1/m$  from elementary considerations.

First take the Proposition with  $c_1 = 0$ ,  $P_1 = \emptyset$ ,  $M_1 = 1$ . Either the result follows immediately with  $\gamma_m \geq \frac{1}{2m}$  or there exists an integer  $d$ , as described. Let  $P_2 = Q$ ,  $M_2 = d$ , and  $c_2 = \kappa(0, m)$ . If so, apply the Proposition again; either we get  $\geq \frac{M_2}{2M} \sum_{\substack{n \leq x \\ (n, P_2)=1}} 1$  such

integers as desired, and this is  $\gtrsim \frac{M_2}{2m} \rho(c_2)x$  by Hildebrand's Lemma; or we get another integer  $d_2$  as described. If so apply the Proposition again and again with

$$P_{k+1} = Q_k, \quad M_{k+1} = d_k \quad \text{and} \quad c_{k+1} = k\kappa(c_k, m).$$

The process eventually terminates (since  $M_1 \mid M_2 \mid M_3 \mid \dots \mid m$  and each  $M_{k+1} > M_k$ ); when it does we get

$$\gtrsim \frac{M_k}{2m} \rho(c_k)x \quad \text{integers up to } x$$

which are  $m$ th power residues  $\pmod{\ell}$ . Thus  $\gamma_m$  exceeds the minimum of the  $\frac{M_k}{2m} \rho(c_k)$  over all possible such sequences  $M_1 \mid M_2 \mid \dots \mid m$  (of which there are evidently only finitely many).

## 2b. Logarithmic proportions of $m$ th power residues.

It is plain that  $\gamma'_m \leq 1/m$ , and so in particular  $\gamma'_2 \leq 1/2$ . Let  $\beta > 1$  be a parameter to be chosen shortly, and put  $\alpha_p = e^{2\pi i/m}$  if  $(\log \log x)^{1/\beta} \leq p \leq \log \log x$ , and  $\alpha_p = 1$  for all other primes  $p \leq x$ . Choose  $\ell \equiv 1 \pmod{m}$  such that there is a character  $\chi \pmod{\ell}$  of order  $m$  with  $\chi(p) = \alpha_p$  for all  $p \leq x$ . Let  $P$  denote the product of those primes  $p \leq x$  with  $\alpha_p \neq 1$ . We may write every  $n \leq x$  uniquely as  $NR$  where  $p \mid N \implies p \mid P$ , and  $p \mid R \implies p \nmid P$ . Note that  $\chi(n) = \chi(N) = 1$  if and only if the number of primes dividing  $N$  (counted with multiplicity) is a multiple of  $m$ . Thus

$$\begin{aligned} \gamma'_m &\leq \frac{1}{\log x} \sum_{\substack{n \leq x \\ \chi(n)=1}} \frac{1}{n} \leq \frac{1}{\log x} \sum_{\substack{R \leq x \\ (R, P)=1}} \frac{1}{R} \sum_{\substack{N \leq x, \chi(N)=1 \\ p \mid N \implies p \mid P}} \frac{1}{N} \\ &\leq \left( \frac{\varphi(P)}{P} + o(1) \right) \sum_{k=0}^{\infty} \frac{1}{(km)!} \left( \sum_{p \mid P} \frac{1}{p-1} \right)^{km}. \end{aligned}$$

Letting  $x \rightarrow \infty$ , and using the prime number theorem we obtain

$$\gamma'_m \leq \frac{1}{\beta} \sum_{k=0}^{\infty} \frac{(\log \beta)^{km}}{(km)!}.$$

Taking  $\beta = e^{m/e}$ , and using Stirling's formula, it follows that  $\gamma'_m \ll e^{-m/e}$ .

**Lemma 2.3.** *Let  $a_1, \dots, a_n$  and  $R_1, R_2, \dots, R_n, m \geq 2$  be integers. Then*

$$\#\{(r_1, \dots, r_n) \in \mathbb{Z}^n : \sum_{i=1}^n r_i a_i \equiv 0 \pmod{m}, \text{ with } 0 \leq r_i \leq R_i - 1\} \geq \frac{R_1 \dots R_n}{2^{m-1}}.$$

Note that Lemma 2.1 is 'best possible' in that if  $R_1 = \dots = R_{m-1} = 2$  and  $n = m - 1$  then the only solution has each  $r_i = 0$ , and thus we get equality above.

On the other hand, we naively expect the proportion typically to be close to  $1/m$  (rather than be as small as  $1/2^{m-1}$ ); and if this is so in context then we might expect to improve Corollary 2. below.

*Proof.* Given the  $a_i, R_i$  and  $m$  above we define  $p(\mathbf{a}, \mathbf{R})$ , the proportion of the sums that equal zero  $\pmod{m}$ , to be equal to

$$\frac{1}{R_1 \dots R_n} \#\{(r_1, \dots, r_n) \in \mathbb{Z}^n : \sum_{i=1}^n r_i a_i \equiv 0 \pmod{m}, \text{ with } 0 \leq r_i \leq R_i - 1\}.$$

The result that we wish to prove is that  $p(\mathbf{a}, \mathbf{R}) \geq 1/2^{m-1}$ . Let us suppose that we have a counterexample above with  $s(\mathbf{R}) := \sum_i (R_i - 2)$  minimal.

We will show that we must have each  $R_i = 2$ , else if  $R_n \geq 3$  then we will construct two new examples  $\mathbf{b}, \mathbf{B}$  and  $\mathbf{c}, \mathbf{C}$  with  $s(\mathbf{R}) > s(\mathbf{B}), s(\mathbf{C})$ , and with  $p(\mathbf{b}, \mathbf{B}) < 1/2^{m-1}$  or  $p(\mathbf{c}, \mathbf{C}) < 1/2^{m-1}$ , thus contradicting the minimality of the purported counterexample  $\mathbf{a}, \mathbf{R}$ . We thus have reduced proving Lemma 2.3 to the case where every  $R_i = 2$ , which we prove in Lemma 2.4 below.

Now we construct  $\mathbf{b}, \mathbf{B}$  and  $\mathbf{c}, \mathbf{C}$  as follows: Let  $b_i = c_i = a_i$  and  $B_i = C_i = R_i$  for  $1 \leq i \leq n-1$ . Let  $b_n = b_{n+1} = a_n$  with  $B_n = R_n - 1$  and  $B_{n+1} = 2$ ; and let  $c_n = (R_n - 1)a_n$  and  $C_n = 2$ . We see that  $s(\mathbf{B}) = s(\mathbf{R}) - 1$  and  $s(\mathbf{C}) = s(\mathbf{R}) - (R_n - 2) \leq s(\mathbf{R}) - 1$ .

Now for  $0 \leq j \leq R_n - 1$  we define

$$\tau_j = \#\{(r_1, \dots, r_{n-1}) \in \mathbb{Z}^n : ja_n + \sum_{i=1}^{n-1} r_i a_i \equiv 0 \pmod{m}, \text{ with } 0 \leq r_i \leq R_i - 1\}.$$

Thus with  $R := R_1 \dots R_{n-1}$  we have

$$p(\mathbf{a}, \mathbf{R}) = \frac{1}{R_n R} \{\tau_0 + \tau_1 + \dots + \tau_{R_n-1}\}; \quad p(\mathbf{c}, \mathbf{C}) = \frac{1}{2R} \{\tau_0 + \tau_{R_n-1}\};$$

$$\text{and } p(\mathbf{b}, \mathbf{B}) = \frac{1}{2(R_n - 1)R} \{\tau_0 + 2(\tau_1 + \dots + \tau_{R_n-2}) + \tau_{R_n-1}\}.$$

Therefore

$$\frac{1}{2^{m-1}} > p(\mathbf{a}, \mathbf{R}) = \frac{1}{R_n} \{(R_n - 1)p(\mathbf{b}, \mathbf{B}) + p(\mathbf{c}, \mathbf{C})\} \geq \min\{p(\mathbf{b}, \mathbf{B}), p(\mathbf{c}, \mathbf{C})\},$$

as required.



**Lemma 2.4.** *Let  $a_1, \dots, a_n$  and  $m \geq 2$  be integers. Then*

$$\#\{A \subseteq \{1, \dots, n\} : \sum_{i \in A} a_i \equiv 0 \pmod{m}\} \geq 2^{n-(m-1)}.$$

*Proof.* If  $n \leq m - 1$  the statement is trivial since we always can take  $A$  to be the empty set and thus get at least one such sum. We will assume henceforth that  $n \geq m$ .

Let  $A_0$  be the subsequence of  $a_i$ 's which are  $\equiv 0 \pmod{m}$ , and then let  $A_1 = \{1, \dots, n\} \setminus A_0$ . We shall define a sequence of subsets  $\{B_j\}_{j \geq 1}$  of  $A_1$ , with  $B_1 \subset B_2 \subset B_3 \subset \dots$  and each  $B_j$  having exactly  $j$  elements; and we will let  $C_j$  be the set of sums  $\pmod{m}$ , of the subsets of  $B_j$ .

We define  $B_1 = \{b_1\}$  where  $b_1$  is any element of  $A_1$ , so that  $C_1 = \{0, b_1\}$  has two elements. Given  $B_j$  (and thus  $C_j$ ) we attempt to select  $b_{j+1} \in A_1 \setminus B_j$ , so that  $C_{j+1}$  is larger than  $C_j$ . If this is possible we so construct  $B_{j+1}$  (that is, as  $B_j \cup \{b_{j+1}\}$ ) and move on to attempting the analogous construction with  $j + 1$ ; note that then  $C_{j+1}$  contains at least  $j + 2$  elements. If this construction is impossible, write  $j = k$ , and note that we must have  $b + C_k \subset C_k$  for every  $b \in A_1 \setminus B_k$ . But since  $0 \in C_k$  this would imply that  $b, 2b, 3b, \dots \in C_k$ . Indeed by repeatedly using the relation  $b + C_k \subset C_k$ , we see that the additive subgroup  $S$ , generated by the elements of  $A_1 \setminus B_k$ , must be a subset of  $C_k$ .

In fact there must be such a value of  $k$ , since if not then we would have  $m \geq |C_n| \geq n + 1$  which gives a contradiction. Note that  $m \geq |C_k| \geq k + 1$ .

Now select any subset  $R$  of  $A_0$ , and any subset  $T$  of  $A_1 \setminus B_k$ . Note that  $s := \sum_{a \in R} a + \sum_{a' \in T} a' \equiv \sum_{a' \in T} a' \pmod{m}$ , so that  $s \in S$ , and thus  $-s \in S \subset C_k$ . Therefore, by the definition of  $C_k$ , there exists a subset  $U$  of  $B_k$  with  $\sum_{a'' \in U} a'' \equiv -s \pmod{m}$ . Thus we have  $\sum_{a \in R \cup T \cup U} a \equiv 0 \pmod{m}$ , and so

$$\#\{A \subseteq \{1, \dots, n\} : \sum_{i \in A} a_i \equiv 0 \pmod{m}\} \geq \sum_{R \subset A_0} \sum_{T \subset A_1 \setminus B_k} 1 = 2^{n-k} \geq 2^{n-(m-1)}.$$

**Corollary 2.5.** *Let  $f$  be a completely multiplicative function where each  $f(p)$  is an  $m$ th root of unity. Then*

$$\frac{1}{\log x} \sum_{\substack{n \leq x \\ f(n)=1}} \frac{1}{n} \geq \frac{1}{2^{m-1}} + o(1).$$

*Proof.* If  $m = 1$  the result is trivial, so assume henceforth that  $m \geq 2$ , and such a function  $f$  is given. Given integer  $N$  we write  $N = p_1^{R_1-1} p_2^{R_2-1} \dots p_n^{R_n-1}$  where each  $R_i \geq 2$ . Moreover we can write  $f(p_j) = e^{2i\pi a_j/m}$  for each  $1 \leq j \leq n$ , where  $a_j$  is some integer. Thus the number of divisors of  $N$  for which  $f(d) = 1$ , is exactly the number of 1's that appear in the expansion

$$\sum_{d|N} f(d) = \prod_{j=1}^n \left( 1 + e^{2i\pi a_j/m} + e^{4i\pi a_j/m} + \dots + e^{2(R_j-1)i\pi a_j/m} \right),$$

which equals

$$\#\{(r_1, \dots, r_n) \in \mathbb{Z}^n : \sum_{i=1}^n r_i a_i \equiv 0 \pmod{m}, \text{ with } 0 \leq r_i \leq R_i - 1\}.$$

By Lemma 2.3, this is  $\geq R_1 \dots R_n / 2^{m-1} = d(N) / 2^{m-1}$ , where  $d(N)$  is the number of divisors of  $N$ . In other words

$$\sum_{\substack{d|N \\ f(d)=1}} 1 \geq d(N) / 2^{m-1}.$$

Since  $\sum_{N \leq x} \sum_{d|N} f(d) = \sum_{d \leq x} f(d) [x/d]$ , we deduce that

$$\sum_{\substack{d \leq x \\ f(d)=1}} \left[ \frac{x}{d} \right] = \sum_{N \leq x} \sum_{\substack{d|N \\ f(d)=1}} 1 \geq \sum_{N \leq x} \frac{d(N)}{2^{m-1}} = \frac{1}{2^{m-1}} \sum_{d \leq x} \left[ \frac{x}{d} \right],$$

from which we deduce the result.

In the proof of Corollary 2.5 we made extensive use of Lemma 2.3. However it may be that ‘typically’ Lemma 2.3 is not sharp. We now re-examine the lower bounds for  $\sum_{d|N, f(d)=1} 1$  given above. For simplicity, we will assume that  $m$  is prime and  $N$  is squarefree. Suppose that  $J$  is the set of integers for which  $a_j \not\equiv 0 \pmod{m}$  (where  $a_j$  is as defined above). Thus

$$\sum_{d|N, f(d)=1} 1 = \frac{1}{m} \sum_{\zeta^m=1} \prod_{j=1}^n (1 + \zeta^{a_j}).$$

We get a contribution of  $2^n$  from the  $\zeta = 1$  term. Otherwise, if  $\zeta = e^{2i\pi k/m}$  then  $|1 + \zeta^{a_j}| = 2|\cos(\pi a_j k/m)|$ . Therefore

$$\sum_{d|N, f(d)=1} 1 \geq \frac{2^n}{m} \left( 1 - \sum_{k=1}^{m-1} \prod_{j \in J} |\cos(\pi a_j k/m)| \right) \geq \frac{2^n}{m} \left( 1 - \sum_{k=1}^{m-1} |\cos^{|J|}(\pi k/m)| \right),$$

by an optimization argument. This is  $\gg 2^n/m$  if  $|J| \gg m^2$ ; thus, if a typical integer  $N \leq x$  has  $\gg m^2$  prime factors for which  $f(p) \neq 1$  then we might expect to improve considerably the lower bound in Corollary 2.5.

### 3. BASIC RESULTS ON INTEGRAL EQUATIONS

#### 3a. Existence and uniqueness of solutions and first estimates.

We begin with the following simple principle which we shall use repeatedly.

**Lemma 3.1.** *Let  $\alpha$  and  $\beta$  be two integrable functions from  $[0, \infty)$  to  $\mathbb{R}$ . Suppose that  $\alpha(u) \geq 0$  for all  $0 \leq u \leq 1$ , and that  $\beta_0 \geq \beta(u) \geq 0$  for all  $u$ . If  $u\alpha(u) \geq (\beta * \alpha)(u)$  then  $\alpha(u) \geq 0$  for all  $u$ . In particular, if  $u\alpha(u) \geq (1 * \alpha)(u)$  then  $\alpha(u) \geq 0$  for all  $u$ .*

*Proof.* It suffices to show  $\alpha(u) \geq 0$  for those points  $u \geq 1$  satisfying  $\alpha(u)/u^{\beta_0-1} \leq \alpha(t)/t^{\beta_0-1}$  for all  $t \leq u$ . For such a  $u$ ,

$$u\alpha(u) \geq \int_0^u \alpha(t)\beta(u-t)dt \geq \int_1^u \alpha(t)\beta(u-t)dt \geq \frac{\alpha(u)}{u^{\beta_0-1}} \int_1^u t^{\beta_0-1}\beta(u-t)dt.$$

If  $\alpha(u) < 0$  then we must have

$$u^{\beta_0} \leq \int_1^u t^{\beta_0-1}\beta(u-t)dt \leq \int_1^u \beta_0 t^{\beta_0-1}dt = u^{\beta_0} - 1,$$

which is a contradiction.

The condition that  $\beta$  is bounded may be relaxed. We need only that  $\beta$  is bounded on closed intervals. Thus, for example, the result holds for any continuous, non-negative function  $\beta$ .

Let  $\chi$  be an element of  $K(\mathbb{U})$ . Our first application of this Lemma is to show the existence and uniqueness of solutions to the integral equation (1.5). To this end, it is useful to define  $I_0(u) = I_0(u; \chi) = 1$ , and for  $k \geq 1$ ,

$$I_k(u) = I_k(u; \chi) = \int_{\substack{t_1, \dots, t_k \geq 1 \\ t_1 + \dots + t_k \leq u}} \frac{1 - \chi(t_1)}{t_1} \dots \frac{1 - \chi(t_k)}{t_k} dt_1 \dots dt_k.$$

Define for all  $k \geq 0$ ,

$$\sigma_k(u) = \sum_{j=0}^k \frac{(-1)^j}{j!} I_j(u; \chi), \quad \text{and} \quad \sigma_\infty(u) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} I_j(u; \chi).$$

**Lemma 3.2.** *For all  $j \geq 1$ ,*

$$(3.1) \quad uI_j(u) = (1 * I_j)(u) + j((1 - \chi) * I_{j-1})(u).$$

*Further  $u\sigma_k(u) = (1 * \sigma_k)(u) - ((1 - \chi) * \sigma_{k-1})(u)$  and  $u\sigma_\infty(u) = (\sigma_\infty * \chi)(u)$ .*

*Proof.* Observe that, for  $j \geq 1$ ,

$$\begin{aligned} (1 * I_j)(u) &= \int_1^u \int_{\substack{t_1, \dots, t_j \geq 1 \\ t_1 + \dots + t_j \leq t}} \frac{1 - \chi(t_1)}{t_1} \dots \frac{1 - \chi(t_j)}{t_j} dt_1 \dots dt_j dt \\ &= \int_{\substack{t_1, \dots, t_j \geq 1 \\ t_1 + \dots + t_j \leq u}} \frac{1 - \chi(t_1)}{t_1} \dots \frac{1 - \chi(t_j)}{t_j} (u - t_1 - \dots - t_j) dt_1 \dots dt_j \\ &= uI_j(u) - j \int_1^u (1 - \chi(t_1)) \int_{\substack{t_2, \dots, t_j \geq 1 \\ t_2 + \dots + t_j \leq u - t_1}} \frac{1 - \chi(t_2)}{t_2} \dots \frac{1 - \chi(t_j)}{t_j} dt_2 \dots dt_j \\ &= uI_j(u) - j((1 - \chi) * I_{j-1})(u), \end{aligned}$$

and (3.1) follows. Multiply both sides of (3.1) by  $(-1)^j/j!$  and sum from  $j = 1$  to  $k$ . This gives

$$\begin{aligned} u \sum_{j=1}^k \frac{(-1)^j}{j!} I_j(u) &= \sum_{j=1}^k \frac{(-1)^j}{j!} (1 * I_j)(u) + \sum_{j=1}^k \frac{(-1)^j}{(j-1)!} ((1-\chi) * I_{j-1})(u) \\ &= \sum_{j=1}^k \frac{(-1)^j}{j!} (1 * I_j)(u) - \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} ((1-\chi) * I_j)(u). \end{aligned}$$

Adding  $u$  to both sides we get  $u\sigma_k = 1 * \sigma_k - (1-\chi) * \sigma_{k-1}$ . Summing from  $j = 1$  to  $\infty$  (instead of 1 to  $k$ ) gives  $u\sigma_\infty = \sigma_\infty * \chi$ .

**Theorem 3.3.** *For a given  $\chi \in K(\mathbb{U})$ , there exists a unique solution to the integral equation (1.5). In fact,  $\sigma = \sigma_\infty$  is this unique solution, and satisfies  $|\sigma(u)| \leq 1$  for all  $u$ .*

*Proof.* By definition  $\sigma_\infty(u) = 1$  for  $0 \leq u \leq 1$ . Since  $u\sigma_\infty = \sigma_\infty * \chi$ , we see that  $\sigma_\infty$  is a solution to (1.5). We now show that it is unique. Let  $\sigma$  be another solution to (1.5) and put  $\alpha(u) = -|\sigma(u) - \sigma_\infty(u)|$ . Note that  $\alpha(u) = 0$  for  $0 \leq u \leq 1$  and that

$$u\alpha(u) = - \left| \int_0^u (\sigma(t) - \sigma_\infty(t)) \chi(u-t) dt \right| \geq - \int_0^u |\sigma(t) - \sigma_\infty(t)| dt = \int_0^u \alpha(t) dt.$$

Lemma 3.1 shows that  $\alpha(u) \geq 0$  always, whence  $\sigma = \sigma_\infty$ .

To show that the unique solution  $\sigma$  satisfies  $|\sigma(u)| \leq 1$  for all  $u$ , we take  $\alpha(u) = 1 - |\sigma(u)|$ . Again  $\alpha(u) = 0$  for  $0 \leq u \leq 1$ , and

$$u\alpha(u) = \int_0^u dt - \left| \int_0^u \sigma(t) \chi(u-t) dt \right| \geq \int_0^u (1 - |\sigma(t)|) dt = \int_0^u \alpha(t) dt.$$

Thus  $\alpha(u) \geq 0$  for all  $u$ , by Lemma 3.1, and the proof is complete.

**Lemma 3.4.** *Let  $\chi$  and  $\hat{\chi}$  be two elements of  $K(\mathbb{U})$ , and let  $\sigma$  and  $\hat{\sigma}$  be the corresponding solutions to (1.5). Then  $\sigma(u)$  equals*

$$(3.2) \quad \hat{\sigma}(u) + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int_{\substack{t_1, \dots, t_j \geq 1 \\ t_1 + \dots + t_j \leq u}} \frac{\hat{\chi}(t_1) - \chi(t_1)}{t_1} \dots \frac{\hat{\chi}(t_j) - \chi(t_j)}{t_j} \hat{\sigma}(u - t_1 - \dots - t_j) dt_1 \dots dt_j.$$

*Consequently, if  $|\chi(t) - \hat{\chi}(t)| \leq \chi_0$  for all  $t$  then  $|\sigma(u) - \hat{\sigma}(u)| \leq u^{\chi_0} - 1$ , for all  $u \geq 1$ .*

*Proof.* Let  $\hat{I}_k$  be the integral corresponding to  $\hat{\chi}$ . Writing  $1 - \chi$  as  $(1 - \hat{\chi}) + (\hat{\chi} - \chi)$  in the definition of  $I_k$  we deduce that

$$I_k(u) = \sum_{j=0}^k \binom{k}{j} \int_{\substack{t_1, \dots, t_j \geq 1 \\ t_1 + \dots + t_j \leq u}} \frac{\hat{\chi}(t_1) - \chi(t_1)}{t_1} \dots \frac{\hat{\chi}(t_j) - \chi(t_j)}{t_j} \hat{I}_k(u - t_1 - \dots - t_j) dt_1 \dots dt_j.$$

Multiply both sides by  $(-1)^k/k!$ , and sum from  $k = 0$  to  $\infty$ . This proves (3.2).

If  $|\chi(t) - \hat{\chi}(t)| \leq \chi_0$  then, from (3.2) and the fact that  $|\hat{\sigma}(t)| \leq 1$  always, we obtain for  $u \geq 1$

$$|\sigma(u) - \hat{\sigma}(u)| \leq \sum_{j=1}^{\infty} \frac{1}{j!} \left( \int_1^u \frac{\chi_0}{t} dt \right)^j = \sum_{j=1}^{\infty} \frac{(\chi_0 \log u)^j}{j!} = u^{\chi_0} - 1.$$

This completes the proof of Lemma 3.4.

**Lemma 3.5.** *Suppose  $\chi \in K(\mathbb{U})$  is given and let  $\sigma$  be the corresponding solution to (1.5). Then*

$$A(v) := \frac{1}{v} \int_0^v |\sigma(t)| dt$$

is a non-increasing function of  $v$ . Hence, for all  $u \geq v$ ,

$$|\sigma(u)| \leq A(v) = \frac{1}{v} \int_0^v |\sigma(t)| dt.$$

*Proof.* From (1.5), we have  $|\sigma(u)| \leq A(u)$  for all  $u$ . Differentiating the definition of  $A(v)$ , we have  $A'(v) = |\sigma(v)|/v - A(v)/v \leq 0$ , so  $A(v)$  is non-increasing and therefore  $|\sigma(u)| \leq A(u) \leq A(v)$  if  $u \geq v$ .

### 3b. Inclusion-Exclusion inequalities.

Our formula for  $\sigma_\infty (= \sigma)$  looks like an inclusion-exclusion type identity. For a real-valued function  $\chi$ , we now show how to obtain inclusion-exclusion inequalities for  $\sigma$ .

**Proposition 3.6.** *Suppose  $\chi \in K([-1, 1])$  is given. For all integers  $k \geq 0$ , and all  $u \geq 0$ ,  $(-1)^{k+1}(\sigma(u) - \sigma_k(u)) \geq 0$ . Thus  $\sigma_{2k+1}(u) \leq \sigma(u) \leq \sigma_{2k}(u)$ .*

*Proof.* From Lemma 3.2 we know that  $u\sigma_k = 1 * \sigma_k - (1 - \chi) * \sigma_{k-1}$ , and clearly  $u\sigma = 1 * \sigma - (1 - \chi) * \sigma$ . Subtracting these identities we get  $u(\sigma - \sigma_k) = 1 * (\sigma - \sigma_k) - (1 - \chi) * (\sigma - \sigma_{k-1})$ .

Put  $\alpha_k(u) = (-1)^{k+1}(\sigma(u) - \sigma_k(u))$  so that the above relation may be rewritten as

$$(3.3) \quad u\alpha_k = 1 * \alpha_k + (1 - \chi) * \alpha_{k-1}.$$

We will show that  $\alpha_k(u) \geq 0$  always by induction on  $k$ . Since  $\sigma_0 = 1$ , the case  $k = 0$  follows from Theorem 3.3. Suppose that  $\alpha_{k-1}$  has been shown to be non-negative. Since  $(1 - \chi)$  is always non-negative it follows from (3.3) that  $u\alpha_k(u) \geq (1 * \alpha_k)(u)$ . Clearly  $\alpha_k(u) = 0$  for  $0 \leq u \leq 1$ . Lemma 3.1 now shows that  $\alpha_k(u) \geq 0$  always, completing our proof.

We now develop some inclusion-exclusion type inequalities for the case when  $\chi \in K(\mathbb{U})$  is complex-valued. To state this, we make the following definitions: Put  $R_0(u) = C_0(u) = 1$  and for  $k \geq 1$  put

$$C_k(u) = C_k(u; \chi) = \int_{\substack{t_1, \dots, t_k \geq 1 \\ t_1 + \dots + t_k \leq u}} \frac{|\operatorname{Im} \chi(t_1)|}{t_1} \cdots \frac{|\operatorname{Im} \chi(t_k)|}{t_k} dt_1 \cdots dt_k,$$

and

$$R_k(u) = R_k(u; \chi) = \int_{\substack{t_1, \dots, t_k \geq 1 \\ t_1 + \dots + t_k \leq u}} \frac{1 - \operatorname{Re} \chi(t_1)}{t_1} \cdots \frac{1 - \operatorname{Re} \chi(t_k)}{t_k} dt_1 \cdots dt_k.$$

**Proposition 3.7.** *For all  $u$ ,  $|\operatorname{Im} \sigma(u)| \leq C_1(u)$ . Let  $\hat{\chi} = \operatorname{Re} \chi$  and let  $\hat{\sigma}$  denote the corresponding solution to (1.5). Then for all  $u$ ,  $|\operatorname{Re} \sigma(u) - \hat{\sigma}(u)| \leq C_2(u)/2$ . In particular  $1 - R_1(u) - C_2(u)/2 \leq \operatorname{Re} \sigma(u) \leq 1 - R_1(u) + (R_2(u) + C_2(u))/2$ .*

*Proof.* Observe that

$$(3.4) \quad u|\operatorname{Im} \sigma| = |\operatorname{Im} \sigma * \operatorname{Re} \chi + \operatorname{Re} \sigma * \operatorname{Im} \chi| \leq |\operatorname{Im} \sigma| * 1 + 1 * |\operatorname{Im} \chi|.$$

In the same way as we showed  $uI_k = 1 * I_k + k(1 - \chi) * I_{k-1}$  (see Lemma 3.2), it follows that

$$(3.5) \quad uC_k = 1 * C_k + k|\operatorname{Im} \chi| * C_{k-1}.$$

Define  $\alpha(u) = C_1(u) - |\operatorname{Im} \sigma(u)|$  so that  $\alpha(u) = 0$  for  $u \leq 1$ . Taking  $k = 1$  in (3.5) and subtracting (3.4), we get  $u\alpha(u) \geq 1 * \alpha$ . Lemma 3.1 shows that  $\alpha(u) \geq 0$  always.

Notice that

$$u(\operatorname{Re} \sigma(u) - \hat{\sigma}(u)) = \operatorname{Re} \sigma * \operatorname{Re} \chi - \operatorname{Im} \sigma * \operatorname{Im} \chi - \hat{\sigma} * \operatorname{Re} \chi$$

whence, using  $|\operatorname{Im} \sigma(t)| \leq C_1(t)$ ,

$$(3.6) \quad u|\operatorname{Re} \sigma(u) - \hat{\sigma}(u)| \leq |\operatorname{Re} \sigma - \hat{\sigma}| * 1 + |\operatorname{Im} \chi| * C_1.$$

Put  $\alpha(u) = C_2(u)/2 - |\operatorname{Re} \sigma(u) - \hat{\sigma}(u)|$  so that  $\alpha(u) = 0$  for  $0 \leq u \leq 1$ . Take  $k = 2$  in (3.5), divide by 2, and subtract (3.6). This gives  $u\alpha(u) \geq 1 * \alpha$  so that, by Lemma 3.1,  $\alpha(u) \geq 0$  always.

By Proposition 3.6 we see that  $1 - R_1(u) \leq \hat{\sigma}(u) \leq 1 - R_1(u) + R_2(u)/2$ . This gives the last assertion of the Proposition.

#### 4. PROOF OF THE STRUCTURE THEOREM

In this section we discuss the relation between the integral equation (1.5) and averages of multiplicative functions. In particular, we shall prove the Structure Theorem for the spectrum.

##### 4a. Variation of averages of multiplicative functions.

In this subsection we establish the following Proposition which seeks to show that the average value of a multiplicative function varies slowly.

**Proposition 4.1.** *Let  $f$  be a multiplicative function with  $|f(n)| \leq 1$  for all  $n$ . Let  $x$  be large, and suppose  $1 \leq y \leq x$ . Then*

$$\left| \frac{1}{x} \sum_{n \leq x} f(n) - \frac{1}{x/y} \sum_{n \leq x/y} f(n) \right| \ll \frac{\log 2y}{\log x} \exp \left( \sum_{p \leq x} \frac{|1 - f(p)|}{p} \right).$$

To prove this Proposition we require a consequence of Theorem 2 of Halberstam and Richert [4]. Suppose  $h$  is a non-negative multiplicative function with  $h(p^k) \leq 2\gamma^{k-1}$  for all prime powers  $p^k$ , for some  $\gamma$ ,  $0 < \gamma < 2$ . It follows from Theorem 2 of [4] that

$$(4.1) \quad \sum_{n \leq x} h(n) \leq \frac{2x}{\log x} \sum_{n \leq x} \frac{h(n)}{n} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\}.$$

Using partial summation we deduce from (4.1) that for  $1 \leq y \leq x^{1/2}$ ,

$$(4.2) \quad \begin{aligned} \sum_{x/y < n \leq x} \frac{h(n)}{n} &\leq \left\{ \frac{1}{\log x} + \log\left(\frac{\log x}{\log(x/y)}\right) \right\} \sum_{n \leq x} \frac{h(n)}{n} \left\{ 2 + O\left(\frac{1}{\log x}\right) \right\} \\ &\ll \frac{\log 2y}{\log x} \sum_{n \leq x} \frac{h(n)}{n}. \end{aligned}$$

Equipped with (4.1) and (4.2) we proceed to a proof of Proposition 4.1.

*Proof of Proposition 4.1.* Since the left side of the Proposition is trivially  $\ll 1$ , there is nothing to prove if  $y > \sqrt{x}$ . Suppose now that  $y < \sqrt{x}$ . Let  $g$  be the multiplicative function with  $g(p^k) = f(p^k) - f(p^{k-1})$  for each prime power. Then  $f(n) = \sum_{d|n} g(d)$ , and so

$$\left| \frac{1}{x} \sum_{n \leq x} f(n) - \sum_{d \leq x} \frac{g(d)}{d} \right| \leq \frac{1}{x} \sum_{d \leq x} |g(d)|.$$

Taking this statement for  $x$  and  $x/y$ , we get

$$\left| \frac{1}{x} \sum_{n \leq x} f(n) - \frac{1}{x/y} \sum_{n \leq x/y} f(n) \right| \leq \sum_{x/y \leq d \leq x} \frac{|g(d)|}{d} + \frac{1}{x} \sum_{d \leq x} |g(d)| + \frac{1}{x/y} \sum_{d \leq x/y} |g(d)|.$$

Since each  $|g(p^k)| \leq 2$ , it follows from (4.1) and (4.2) that the above is

$$\begin{aligned} &\ll \frac{\log 2y}{\log x} \sum_{n \leq x} \frac{|g(n)|}{n} \ll \frac{\log 2y}{\log x} \prod_{p \leq y} \left( 1 + \frac{|1 - f(p)|}{p} + \frac{2}{p^2} + \frac{2}{p^3} + \dots \right) \\ &\ll \frac{\log 2y}{\log x} \exp\left( \sum_{p \leq x} \frac{|1 - f(p)|}{p} \right). \end{aligned}$$

This proves the Proposition.

#### 4b. A useful identity.

**Lemma 4.2.** *Let  $f$  be a multiplicative function with  $|f(p^k)| \leq 1$ . Then*

$$\log x \int_0^1 \sum_{n \leq x^t} f(n) dt + O\left(\frac{x}{\log x}\right) = \int_0^1 \sum_{n \leq x^t} f(n) \sum_{m \leq x^{1-t}} \Lambda(m) f(m) dt.$$

*Proof.* Note that

$$\begin{aligned} \int_0^1 \sum_{n \leq x^t} f(n) \sum_{m \leq x^{1-t}} \Lambda(m) f(m) dt &= \sum_{nm \leq x} f(n) f(m) \Lambda(m) \frac{\log(x/nm)}{\log x} \\ &= \sum_{nm \leq x} f(mn) \Lambda(m) \frac{\log(x/mn)}{\log x} + O\left( \sum_{\substack{mn \leq x \\ (m,n) > 1}} \Lambda(m) \frac{\log(x/mn)}{\log x} \right), \end{aligned}$$

and writing  $r = nm$  this is

$$\begin{aligned} &= \sum_{r \leq x} f(r) \frac{\log(x/r)}{\log x} \sum_{m|r} \Lambda(m) + O\left( \frac{x}{\log x} \right) \\ &= \sum_{r \leq x} f(r) \log r \frac{\log(x/r)}{\log x} + O\left( \frac{x}{\log x} \right). \end{aligned}$$

Next observe that

$$\begin{aligned} \sum_{r \leq x} f(r) \log r \frac{\log(x/r)}{\log x} &= \int_0^1 \sum_{r \leq x^t} f(r) \log r dt \\ &= \log x \int_0^1 \sum_{r \leq x^t} f(r) dt + O\left( \log x \int_0^1 (1-t)x^t dt \right) \\ &= \log x \int_0^1 \sum_{r \leq x^t} f(r) dt + O\left( \frac{x}{\log x} \right). \end{aligned}$$

The two identities above establish the Lemma.

As a consequence of Lemma 4.2 we derive a convolution identity for the averages of  $f$  which will be very useful in our treatment of differential delay equations (see the proof of Proposition 1 below).

**Proposition 4.3.** *Let  $f$  be a multiplicative function with  $|f(p^k)| \leq 1$ . Then*

$$\sum_{n \leq x} f(n) + O\left( \frac{x}{\log x} \exp\left( \sum_{p \leq x} \frac{|1 - f(p)|}{p} \right) \right) = \int_0^1 \sum_{n \leq x^t} f(n) \sum_{p \leq x^{1-t}} f(p) \log p dt.$$

*Proof.* Applying Proposition 4.1 we find that

$$\sum_{n \leq x^t} f(n) = x^{t-1} \sum_{n \leq x} f(n) + O\left( (1-t) \exp\left( \sum_{p \leq x} \frac{|1 - f(p)|}{p} \right) \right).$$



Inserting this in the LHS of Lemma 4.2 we get

$$\sum_{n \leq x} f(n) + O\left(\frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|1 - f(p)|}{p}\right)\right) = \int_0^1 \sum_{n \leq x^t} f(n) \sum_{m \leq x^{1-t}} f(m) \Lambda(m) dt.$$

Since

$$\sum_{m \leq x^{1-t}} f(m) \Lambda(m) = \sum_{p \leq x^{1-t}} f(p) \log p + O(x^{(1-t)/2})$$

and

$$\int_0^1 \sum_{n \leq x^t} f(n) x^{(1-t)/2} \ll \sqrt{x} \int_0^1 x^{t/2} dt \ll \frac{x}{\log x},$$

we have proved the Proposition.

#### 4c. Removing the impact of the small primes.

The main result of this section is the following Proposition which separates the contribution of small primes.

**Proposition 4.4.** *Fix  $\pi/2 \geq \varphi > 0$ . Suppose that  $f \in \mathcal{F}(S)$  where  $S \subset \mathbb{U}$  with  $\text{Ang}(S) \leq \pi/2 - \varphi$ . For any  $\epsilon \geq \log 2 / \log x$ , let  $g$  be the completely multiplicative function with  $g(p) = 1$  if  $p \leq x^\epsilon$ , and  $g(p) = f(p)$  otherwise, so that  $g \in \mathcal{F}(S)$  also. Then*

$$\frac{1}{x} \sum_{n \leq x} f(n) = \Theta(f, x^\epsilon) \frac{1}{x} \sum_{m \leq x} g(m) + O_\varphi(\epsilon^\eta), \quad \text{where } \eta = \eta(\varphi) = \frac{\sin \varphi}{2\pi} \{\varphi - \sin \varphi\}.$$

We begin by deriving a weak version of Proposition 4.4 as a consequence of Proposition 4.1. Using this in conjunction with Lemma 1' we shall prove the stronger Proposition 4.4.

**Proposition 4.5.** *For any multiplicative function  $f$  with  $|f(p^k)| \leq 1$  for every prime power  $p^k$ , let  $s(f, x) := \sum_{p \leq x} |1 - f(p)|/p$ . For any  $1 > \epsilon \geq \log 2 / \log x$ , let  $g$  be the completely multiplicative function with  $g(p) = 1$  if  $p \leq x^\epsilon$ , and  $g(p) = f(p)$  otherwise. Then*

$$\frac{1}{x} \sum_{n \leq x} f(n) = \Theta(f, x^\epsilon) \frac{1}{x} \sum_{m \leq x} g(m) + O(\epsilon \exp(s(f, x))),$$

where the implicit constant is absolute.

*Proof.* Define the multiplicative function  $h$  by  $h(p^k) = f(p^k) - f(p^{k-1})$  if  $p \leq x^\epsilon$ , and  $h(p^k) = 0$  otherwise. Then  $f(n) = \sum_{m|n} h(n/m)g(m)$ , and so

$$(4.3) \quad \frac{1}{x} \sum_{n \leq x} f(n) = \sum_{n \leq x} \frac{h(n)}{n} \left( \frac{n}{x} \sum_{m \leq x/n} g(m) \right).$$

Now, Proposition 4.1 gives<sup>3</sup>

$$\frac{n}{x} \sum_{m \leq x/n} g(m) = \frac{1}{x} \sum_{m \leq x} g(m) + O\left(\frac{\log n}{\log x} \exp(s(g, x))\right).$$

Using this in (4.3) we obtain

$$(4.4) \quad \frac{1}{x} \sum_{n \leq x} f(n) = \sum_{n \geq 1} \frac{h(n)}{n} \frac{1}{x} \sum_{m \leq x} g(m) + O\left(\sum_{n=1}^{\infty} \frac{|h(n)|}{n} \frac{\log n}{\log x} \exp(s(g, x))\right).$$

Since  $\sum_{n \geq 1} h(n)/n = \Theta(f, x^\epsilon)$  the main term above corresponds to the main term of the Proposition. We now show how to handle the error term. Now  $h(n) = 0$  if  $n$  is divisible by a prime larger than  $x^\epsilon$  whence

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|h(n)|}{n} \log n &= \sum_{n=1}^{\infty} \frac{|h(n)|}{n} \sum_{m|n} \Lambda(m) = \sum_{p \leq x^\epsilon} \log p \sum_{\substack{n=1 \\ p|n}}^{\infty} \frac{|h(n)|}{n} + \sum_{\substack{p^k \\ k \geq 2}} \log p \sum_{\substack{n=1 \\ p^k|n}}^{\infty} \frac{|h(n)|}{n} \\ &\ll \sum_{p \leq x^\epsilon} \frac{\log p}{p} \sum_{n=1}^{\infty} \frac{|h(n)|}{n} + \sum_{\substack{p^k \\ k \geq 2}} \frac{\log p}{p^k} \sum_{n=1}^{\infty} \frac{|h(n)|}{n} \\ &\ll (\epsilon \log x + 1) \exp(s(f, x^\epsilon)). \end{aligned}$$

Inserting this in (4.4) we obtain that the error term there is

$$\ll (\epsilon + 1/\log x) \exp(s(f, x^\epsilon) + s(g, x)) \ll \epsilon \exp(s(f, x)).$$

*Proof of Proposition 4.4.* Suppose that  $z = e^{2it}$ . Then  $1 - \operatorname{Re}(z) = 2\sin^2 t$  and  $|1 - z| = 2|\sin t|$ . If we restrict  $t$  to the range  $\varphi \leq t \leq \pi/2$  then we get  $1 - \operatorname{Re}(z) = |1 - z| \sin t \geq |1 - z| \sin \varphi$ . Thus we obtain, in the notation of Proposition 4.5,

$$\sum_{p \leq x} \frac{1 - \operatorname{Re}(f(p))}{p} \geq (\sin \varphi) \sum_{p \leq x} \frac{|1 - f(p)|}{p} = s(f, x) \sin \varphi.$$

Now Lemma 1' implies that

$$(4.5) \quad \left| \sum_{n \leq x} f(n) \right| \ll_{\varphi} x \exp\left(-\frac{\sin \varphi}{\pi} (\varphi - \sin \varphi) s(f, x)\right),$$

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<sup>3</sup>Strictly speaking the error above must have  $\log 2n$  instead of  $\log n$ ; but there is no error in the case  $n = 1$  and when  $n \geq 2$ , clearly  $\log 2n \ll \log n$ .

and similarly,

$$\left| \sum_{n \leq x} g(n) \right| \ll_{\varphi} x \exp\left(-\frac{\sin \varphi}{\pi}(\varphi - \sin \varphi)s(g, x)\right).$$

Further,

$$|\Theta(f, x^{\epsilon})| \ll \exp\left(-\sum_{p \leq x^{\epsilon}} \frac{1 - \operatorname{Re} f(p)}{p}\right) \ll \exp(-\sin \varphi s(f, x^{\epsilon})),$$

whence

$$\begin{aligned} |\Theta(f, x^{\epsilon})| \left| \sum_{n \leq x} g(n) \right| &\ll_{\varphi} x \exp\left(-\frac{\sin \varphi}{\pi}(\varphi - \sin \varphi)(s(f, x^{\epsilon}) + s(g, x))\right) \\ &\ll_{\varphi} x \exp\left(-\frac{\sin \varphi}{\pi}(\varphi - \sin \varphi)s(f, x)\right). \end{aligned}$$

Together with (4.5), this proves the Proposition in the case  $s(f, x) \geq \log 1/\sqrt{\epsilon}$ . The case  $s(f, x) \leq \log 1/\sqrt{\epsilon}$  follows from Proposition 4.5.

#### 4d. Completing the proof of the Structure theorem.

We begin by proving Proposition 1 and its converse.

*Proof of Proposition 1.* Let  $s(u) = [y^u]^{-1} \sum_{n \leq y^u} f(n)$  so that  $s(u) = 1$  for  $u \leq 1$ . Proposition 4.3 tells us that

$$s(u) = \frac{1}{u} \int_0^u s(u-t) \frac{1}{y^t} \left( \sum_{p \leq y^t} f(p) \log p \right) dt + O\left(\frac{u}{\log y}\right).$$

By the prime number theorem  $\vartheta(y^t) = y^t + O(y^t/\log(ey^t))$  and so

$$s(u) = \frac{1}{u} \int_0^u s(u-t) \chi(t) dt + O\left(\frac{u}{\log y}\right).$$

Let  $C$  be the implied constant in the above estimate; that is, for all  $u \geq 1$ ,

$$(4.6) \quad \left| s(u) - \frac{1}{u} \int_0^u s(u-t) \chi(t) dt \right| \leq \frac{Cu}{\log y}.$$

We will demonstrate that  $|\sigma(u) - s(u)| \leq 2Cu/\log y$  which proves the Proposition. Put  $\alpha(u) = -|\sigma(u) - s(u)| + 2Cu/\log y$ . Plainly  $\alpha(u) \geq 0$  for  $u \leq 1$  and note that, using (4.6),

$$\begin{aligned} (1 * \alpha)(u) &= \frac{Cu^2}{\log y} - \int_0^u |\sigma(u-t) - s(u-t)| dt \\ &\leq \frac{Cu^2}{\log y} - \left| \int_0^u (\sigma(u-t) - s(u-t)) \chi(t) dt \right| \\ &\leq \frac{Cu^2}{\log y} - u|\sigma(u) - s(u)| + \frac{Cu^2}{\log y} = u\alpha(u). \end{aligned}$$

By Lemma 3.1,  $\alpha(u) \geq 0$  always, proving the Proposition.

*Proof of the converse to Proposition 1.* Let  $\chi$  be as in the statement of the converse to Proposition 1, and let  $\sigma$  denote the corresponding solution to (1.5).

Since  $\chi$  is measurable and  $\chi(t)$  belongs to the convex hull of  $S$ , we can find a step function  $\chi_1$  with the following properties:  $\chi_1(t) = 1$  for  $t \leq 1$ ,  $\chi_1(t)$  in the convex hull of  $S$  and  $|\chi(t) - \chi_1(t)| \leq \epsilon/2$  for almost all  $t \in [0, u]$ .<sup>4</sup> It is a simple exercise (left to the reader!) that  $\chi_1$  exists.

Next, we choose  $y$  large and find  $f \in \mathcal{F}(S)$  with  $f(p) = 1$  for  $p \leq y$  and such that if

$$\psi(t) = \frac{1}{\vartheta(y^t)} \sum_{p \leq y^t} f(p) \log p$$

then  $|\psi(t) - \chi_1(t)| \leq \epsilon/2$  for almost all  $t \in [0, u]$ . The existence of  $f$  is another straightforward exercise.

With this choice,  $|\chi(t) - \psi(t)| \leq \epsilon$  for almost all  $t \in [0, u]$ . Let  $\tilde{\sigma}$  denote the solution to  $u\tilde{\sigma}(u) = (\tilde{\sigma} * \psi)(u)$  with the initial condition  $\tilde{\sigma}(t) = 1$  for  $t \leq 1$ . By Proposition 1 we note that for  $t \leq u$ ,

$$\frac{1}{y^t} \sum_{n \leq y^t} f(n) = \tilde{\sigma}(t) + O\left(\frac{t}{\log y}\right).$$

From Lemma 3.4, we note that  $|\tilde{\sigma}(t) - \sigma(t)| \leq t^\epsilon - 1 \leq u^\epsilon - 1$ . This completes our proof.

We are now in a position to prove the Structure theorem.

*Proof of Theorem 3.* If  $\text{Ang}(S) = \pi/2$  then  $\Gamma(S) = \Gamma_\Theta(S) = \Lambda(S) = \mathbb{U}$ , and there is nothing to prove. So we suppose below that  $\text{Ang}(S) < \pi/2$ .

If  $z \in \Gamma(S)$  then there exist large  $x$  and  $f \in \mathcal{F}(S)$  for which  $\frac{1}{x} \sum_{n \leq x} f(n) = z + o(1)$ . Take  $y = \exp((\log x)^{\frac{2}{3}})$  and define  $g \in \mathcal{F}(S)$  by  $g(p) = 1$  for  $p \leq y$ , and  $g(p) = f(p)$  for  $p > y$ . By Proposition 4.4  $\frac{1}{x} \sum_{n \leq x} f(n) = \Theta(f, y) \frac{1}{x} \sum_{n \leq x} g(n) + o(1)$ . Take  $\chi(t) = 1$  for  $t \leq 1$ , and  $\chi(t) = \frac{1}{\vartheta(y^t)} \sum_{p \leq y^t} g(p) \log p$  for  $t > 1$ . Let  $\sigma$  denote the corresponding solution to (1.5). Proposition 1 tells us that  $\frac{1}{x} \sum_{n \leq x} g(n) = \sigma\left(\frac{\log x}{\log y}\right) + o(1)$ . It follows that

$$z = \frac{1}{x} \sum_{n \leq x} f(n) + o(1) = \Theta(f, y) \sigma\left(\frac{\log x}{\log y}\right) + o(1).$$

This shows that  $\Gamma(S) \subset \Gamma_\Theta(S) \times \Lambda(S)$ .

Suppose now that  $z_\theta \in \Gamma_\Theta(S)$ , and  $z_\sigma \in \Lambda(S)$  are given. Plainly for large  $y$  there exists  $g \in \mathcal{F}(S)$  with  $z_\theta = \Theta(g, y) + o(1)$ . Further suppose  $z_\sigma = \sigma(u)$  for some  $u$ , and  $\sigma$  a solution to (1.5) for some measurable function  $\chi$  with  $\chi(t) = 1$  for  $t \leq 1$ , and  $\chi(t)$  in the convex hull of  $S$  for all  $t$ . By Proposition 1 (Converse) we deduce that there exists  $h \in \mathcal{F}(S)$  with

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<sup>4</sup>That is, the inequality is violated only on a set of measure 0.

$h(p) = 1$  for  $p \leq y$  such that  $z_\sigma = \sigma(u) = \frac{1}{y^u} \sum_{n \leq y^u} h(n) + o(1)$ . Define  $f \in \mathcal{F}(S)$  by  $f(p) = g(p)$  if  $p \leq y$ , and  $f(p) = h(p)$  if  $p > y$ . By Proposition 4.4 it follows that

$$\frac{1}{y^u} \sum_{n \leq y^u} f(n) = \Theta(g, y) \frac{1}{y^u} \sum_{n \leq y^u} h(n) + o(1) = z_\theta z_\sigma + o(1).$$

Hence  $\Gamma_\Theta(S) \times \Lambda(S) \subset \Gamma(S)$ , proving Theorem 3.

## 5. DETERMINING THE SPECTRUM OF $[-1, 1]$ ; PROOF OF THEOREM 1

In this section we shall prove Theorem 1 and Corollary 1. In Theorem 3' we saw that  $\Gamma([-1, 1]) = \Lambda([-1, 1])$ , and we have already seen that  $\Lambda([-1, 1]) \supset [\delta_1, 1]$ . The following theorem shows that  $\Lambda([-1, 1]) \subset [\delta_1, 1]$ , and more.

**Theorem 5.1.** *Let  $\chi \in K([-1, 1])$  be given, and let  $\sigma$  denote the corresponding solution to (1.5). If  $\int_0^u \frac{1-\chi(t)}{t} dt < 1$  for all  $u$ , then  $\sigma(u)$  is always positive. On the other hand, if  $\int_0^{u_0} \frac{1-\chi(t)}{t} dt = 1$  for some real number  $u_0$ , then  $\sigma(u) \geq 0$  for all  $u \leq u_0$ , and  $|\sigma(u)| \leq |\delta_1|$  for all  $u \geq u_0$ . Moreover, if  $|\sigma(u) - \delta_1| \leq \epsilon$  then we must have  $u = (1 + 1/\sqrt{\epsilon})u_0 + O(\epsilon^{1/4})$  and*

$$\int_0^{u/(1+\sqrt{\epsilon})} \frac{1-\chi(t)}{t} dt + \int_{u/(1+\sqrt{\epsilon})}^u \frac{1+\chi(t)}{t} dt \ll \sqrt{\epsilon}.$$

Given Theorem 5.1 we now show how Corollary 1 may be deduced.

*Deduction of Corollary 1.* Given  $f \in \mathcal{F}([-1, 1])$ , choose  $y = \exp((\log x)^{\frac{2}{3}})$ , and define  $g \in \mathcal{F}([-1, 1])$  by  $g(p) = 1$  for  $p \leq y$ , and  $g(p) = f(p)$  for  $p > y$ . Define for  $t \geq 0$ ,

$$\chi(t) = \frac{1}{\vartheta(y^t)} \sum_{p \leq y^t} g(p) \log p,$$

and let  $\sigma$  denote the corresponding solution to (1.5).

By Proposition 4.4 (with  $S = [-1, 1]$ , and  $\varphi = \pi/2$ ) we have that

$$\begin{aligned} \frac{1}{x} \sum_{n \leq x} f(n) &= \Theta(f, y) \frac{1}{x} \sum_{n \leq x} g(n) + O\left(\left(\frac{\log y}{\log x}\right)^{(\pi/2-1)/(2\pi)}\right) \\ &= \Theta(f, y) \frac{1}{x} \sum_{n \leq x} g(n) + o(1). \end{aligned}$$

Appealing now to Proposition 1, this is

$$= \Theta(f, y) \left( \sigma\left(\frac{\log x}{\log y}\right) + O\left(\frac{\log x}{\log^2 y}\right) \right) + o(1) = \Theta(f, y) \sigma\left(\frac{\log x}{\log y}\right) + o(1).$$

Since  $\Theta(f, y) \in [0, 1]$ , it follows at once from Theorem 5.1, that  $\frac{1}{x} \sum_{n \leq x} f(n) \geq \delta_1 + o(1)$ . Further, if equality holds here then we must have  $\Theta(f, y) = 1 + o(1)$ , and  $\sigma\left(\frac{\log x}{\log y}\right) = \delta_1 + o(1)$ .

The conclusion of the corollary now follows upon using our knowledge of when equality in Theorem 5.1 can occur.

The remainder of this section will be concerned with the proof of Theorem 5.1. Recall from §3 the definitions of  $I_k(u; \chi)$ . By Proposition 3.6 with  $k = 0$  we have  $\sigma(u) \geq 1 - I_1(u; \chi)$ . Hence if  $I_1(u; \chi) = \int_0^u \frac{1-\chi(t)}{t} dt < 1$  for all  $u$  then  $\sigma(u) > 0$  always, which is the first case of our Theorem. So we may suppose that there is a number  $u_0$  such that  $I_1(u_0; \chi) = 1$ . Plainly  $\sigma(u) \geq 1 - I_1(u; \chi) \geq 1 - I_1(u_0; \chi) = 0$  if  $u \leq u_0$ . Hence it remains to be shown that  $|\sigma(u)| \leq |\delta_1|$  for all  $u \geq u_0$ , and to identify when  $\sigma(u)$  is “close” to  $\delta_1$ .

We begin by giving an outline of the underlying ideas of this proof. It is helpful first to gain an understanding of the extremal function  $\rho_-(t)$ , which we discussed briefly in the introduction. Recall that  $\rho_-(t) = 1$  for  $t \leq 1$ , and for  $t > 1$  is the unique continuous solution to the differential-difference equation  $t\rho'_-(t) = -2\rho_-(t-1)$ . Alternatively, in terms of integral equations, for  $v \geq 1$  we have

$$v\rho_-(v) = \int_{v-1}^v \rho_-(t) dt - \int_0^{v-1} \rho_-(t) dt.$$

By integrating  $\rho'_-(t)$  appropriately, and using the differential-difference relation, we obtain that

$$\rho_-(t) = 1 - 2 \log t, \quad \text{for } 1 \leq t \leq 2,$$

and that

$$\rho_-(t) = 1 - 2 \log t + \gamma(t), \quad \text{for } 2 \leq t \leq 3,$$

where we put  $\gamma(t) = 0$  for  $t \leq 2$ , and define for  $t \geq 2$

$$(5.1) \quad \gamma(t) = 4 \int_2^t \frac{\log(v-1)}{v} dv.$$

Notice that  $\rho_-(t) \geq 0$  for  $t \leq \sqrt{e}$ ,  $\rho_-(\sqrt{e}) = 0$ , and that  $\rho_-(t) \leq 0$  for  $\sqrt{e} \leq t \leq 3$ .<sup>5</sup> Hence note that

$$\begin{aligned} \delta_1 = \rho_-(1 + \sqrt{e}) &= \frac{1}{(1 + \sqrt{e})} \left( \int_{\sqrt{e}}^{1+\sqrt{e}} \rho_-(t) dt - \int_0^{\sqrt{e}} \rho_-(t) dt \right) \\ &= -\frac{1}{1 + \sqrt{e}} \int_0^{1+\sqrt{e}} |\rho_-(t)| dt, \end{aligned}$$

or alternatively,

$$(5.2) \quad |\delta_1| = |\rho_-(1 + \sqrt{e})| = \frac{1}{1 + \sqrt{e}} \int_0^{1+\sqrt{e}} |\rho_-(t)| dt.$$

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<sup>5</sup>In fact,  $\rho_-(t) \leq 0$  for all  $t \geq \sqrt{e}$  but we do not need this fact.

This identity lies at the heart of our proof.

Suppose for simplicity that  $u > u_0(1 + 1/\sqrt{e})$ ; we seek to show that  $|\sigma(u)| \leq |\delta_1|$ . By Lemma 3.5 we note that

$$\begin{aligned} |\sigma(u)| &\leq \frac{1}{u_0(1 + 1/\sqrt{e})} \int_0^{u_0(1+1/\sqrt{e})} |\sigma(t)| dt \\ &= \frac{1}{u_0(1 + 1/\sqrt{e})} \left( \int_0^{u_0} \sigma(t) dt + \int_{u_0}^{u_0(1+1/\sqrt{e})} |\sigma(t)| dt \right). \end{aligned}$$

Our idea is essentially to compare  $|\sigma(t)|$  with  $|\rho_-(t\sqrt{e}/u_0)|$ . We shall show that  $|\sigma(t)|$  is smaller on average than  $|\rho_-(t\sqrt{e}/u_0)|$ . From this and the above inequality it would follow that  $|\sigma(u)| \leq \frac{1}{(1+\sqrt{e})} \int_0^{1+\sqrt{e}} |\rho_-(t)| dt$ , and from (5.2) the result follows.

In order to carry this out, we introduce the parameters

$$(5.3) \quad \lambda = I_1(u_0(1 - 1/\sqrt{e}), \chi), \quad \text{and} \quad \tau = I_1(u_0/\sqrt{e}, \chi),$$

which satisfy  $0 \leq \lambda \leq \tau \leq 1$ . In §5b we present an argument which maximizes  $\int_0^{u_0} \sigma(t) dt$  under the constraint (5.3). We show there that

$$\frac{1}{u_0} \int_0^{u_0} \sigma(t) dt \leq 2 - \frac{2}{\sqrt{e}} - E_1(\lambda, \tau) = \frac{1}{\sqrt{e}} \int_0^{\sqrt{e}} \rho_-(t) dt - E_1(\lambda, \tau),$$

where  $E_1(\lambda, \tau)$  is an explicit non-negative function of  $\tau$  and  $\lambda$  (see Corollaries 5.6 and 5.7 below).

For  $u_0 \leq t \leq u_0(1 + 1/\sqrt{e})$ , we use the inclusion-exclusion inequalities of Proposition 3.6 to obtain estimates of the form

$$|\sigma(t)| \leq |\rho_-(\sqrt{e}t/u_0)| + E_2(\lambda, \tau, t/u_0),$$

for some non-negative function  $E_2(\lambda, \tau, t/u_0)$ . The key is to obtain very precise bounds for  $E_2(\lambda, \tau, t/u_0)$  such that

$$\frac{1}{u_0(1 + 1/\sqrt{e})} \int_{u_0}^{u_0(1+1/\sqrt{e})} E_2(\lambda, \tau, t/u_0) dt \leq \frac{E_1(\lambda, \tau)}{1 + 1/\sqrt{e}}.$$

In fact, we shall see that equality above holds only when  $\lambda = \tau = 0$ . Combining this with our bound on  $\int_0^{u_0} \sigma(t) dt$ , we shall have shown that  $|\sigma(t)|$  is smaller than  $|\rho_-(t\sqrt{e}/u_0)|$  on average; as desired.

### 5a. Preliminaries.

Throughout  $\lambda$  and  $\tau$  are as in (5.3). We shall find it useful to consider the function  $\hat{\chi}(t) = \chi(t)$  if  $t \leq u_0$  and  $\hat{\chi}(t) = 1$  for  $t > u_0$ . Let  $\hat{\sigma}$  denote the corresponding solution to (1.5). Below,  $\hat{I}_k(u)$  will denote  $I_k(u; \hat{\chi})$ . Note that  $\hat{I}_1(u) \leq 1$  for all  $u$ , and so by Proposition 3.6, it follows that  $\hat{\sigma}(u) \geq 1 - \hat{I}_1(u) \geq 0$  always.

**Lemma 5.2.** *In the range  $u_0 \leq u \leq 2u_0$  we have*

$$\max\left(-2\log\frac{u}{u_0}, -2\log\frac{u}{u_0} + \frac{\hat{I}_2(u)}{2} - \frac{\hat{I}_3(u)}{6}\right) \leq \sigma(u) \leq \frac{\hat{I}_2(u)}{2}.$$

*Proof.* By Lemma 3.4 we see that in this range

$$\sigma(u) = \hat{\sigma}(u) - \int_{u_0}^u \frac{1 - \chi(t)}{t} \hat{\sigma}(u - t) dt.$$

Since  $0 \leq 1 - \chi(t) \leq 2$ , and  $0 \leq \hat{\sigma}(u - t) \leq 1$ , it follows that

$$-2\log\frac{u}{u_0} + \hat{\sigma}(u) \leq \sigma(u) \leq \hat{\sigma}(u).$$

Moreover  $\hat{\chi}$  has been designed so that  $\hat{I}_1(u) = 1$  for all  $u \geq u_0$ . Therefore, by Proposition 3.6, we see that  $\hat{\sigma}(u) \leq 1 - \hat{I}_1(u) + \hat{I}_2(u)/2 = \hat{I}_2(u)/2$ , and also that  $\hat{\sigma}(u) \geq \max(0, 1 - \hat{I}_1(u) + \hat{I}_2(u)/2 - \hat{I}_3(u)/6) = \max(0, \hat{I}_2(u)/2 - \hat{I}_3(u)/6)$ . The Lemma follows.

In order to use Lemma 5.2 successfully, we require estimates for  $\hat{I}_2(u)$ , and  $\hat{I}_3(u)$ . We develop these in the next two Lemmas.

**Lemma 5.3.** *In the range  $u_0 \leq u \leq u_0(1 + 1/\sqrt{e})$  we have*

$$\begin{aligned} \hat{I}_2(u) &\leq \min\left(1, 2\gamma\left(\frac{u\sqrt{e}}{u_0}\right) + 2\lambda - \tau^2 + 2(\tau - \lambda)\hat{I}_1(u - u_0(1 - 1/\sqrt{e}))\right) \\ &\leq \min\left(1, 2\gamma\left(\frac{u\sqrt{e}}{u_0}\right) + 2\tau - \tau^2\right), \end{aligned}$$

and

$$\hat{I}_2(u) \geq 2\gamma\left(\frac{u\sqrt{e}}{u_0}\right).$$

*Proof.* Write  $\hat{\chi}_1(t) = (1 - \hat{\chi}(t))/t$ . We define  $\psi_0(t) = 0$  for  $t \leq u_0/\sqrt{e}$  and  $\psi_0(t) = \hat{\chi}_1(t)$  for  $t > u_0/\sqrt{e}$ . Define  $\psi_1(t) = 0$  if  $t \leq u_0/\sqrt{e}$  or if  $t > u_0$  and  $\psi_1(t) = 2/t$  if  $u_0/\sqrt{e} < t \leq u_0$ . Notice that  $\psi_0(t) \leq \psi_1(t)$  for all  $t \leq u_0$ , and so  $(1 * \psi_0 * \psi_0)(u) \leq (1 * \psi_1 * \psi_1)(u) = 2\gamma(u\sqrt{e}/u_0)$  for  $u \leq u_0(1 + 1/\sqrt{e})$ .

Since  $\hat{\chi}_1 \geq 0$  and  $\hat{I}_1 = 1 * \hat{\chi}_1 \leq 1$ , we get  $\hat{I}_2 = 1 * \hat{\chi}_1 * \hat{\chi}_1 \leq 1 * \hat{\chi}_1 \leq 1$ . Hence

$$\begin{aligned} \hat{I}_2 &= 1 * \hat{\chi}_1 * \hat{\chi}_1 = 1 * \psi_0 * \psi_0 + 1 * (\hat{\chi}_1 - \psi_0) * (\hat{\chi}_1 + \psi_0) \\ &\leq 2\gamma(u\sqrt{e}/u_0) + 1 * (\hat{\chi}_1 - \psi_0) * (\hat{\chi}_1 + \psi_0). \end{aligned}$$



Now, for  $u$  in this range,

$$\begin{aligned}
1 * (\hat{\chi}_1 - \psi_0) * (\hat{\chi}_1 + \psi_0) &= \int_1^{u_0/\sqrt{e}} \hat{\chi}_1(t_1) \left( \int_1^{u-t_1} (\hat{\chi}_1(t_2) + \psi_0(t_2)) dt_2 \right) dt_1 \\
&\leq \left( \int_1^{u_0(1-1/\sqrt{e})} \hat{\chi}_1(t_1) dt_1 \right) \left( \int_1^u (\hat{\chi}_1(t_2) + \psi_0(t_2)) dt_2 \right) \\
&\quad + \left( \int_{u_0(1-1/\sqrt{e})}^{u_0/\sqrt{e}} \hat{\chi}_1(t_1) dt_1 \right) \left( \int_1^{u-u_0(1-1/\sqrt{e})} (\hat{\chi}_1(t_2) + \psi_0(t_2)) dt_2 \right) \\
&= \lambda(1 + 1 - \tau) + (\tau - \lambda)(2\hat{I}_1(u - u_0(1 - 1/\sqrt{e})) - \tau).
\end{aligned}$$

This shows the middle upper bound of the Lemma, from which the last upper bound of the lemma follows as  $\lambda \leq \tau$  and  $\hat{I}_1(u - u_0(1 - 1/\sqrt{e})) \leq 1$ .

Observe that

$$\begin{aligned}
\hat{I}_2 &= 1 * \hat{\chi}_1 * \hat{\chi}_1 = 1 * \psi_1 * \psi_1 + 1 * (\hat{\chi}_1 - \psi_1) * (\hat{\chi}_1 + \psi_1) \\
&= 2\gamma(u\sqrt{e}/u_0) + 1 * (\hat{\chi}_1 - \psi_1) * (\hat{\chi}_1 + \psi_1).
\end{aligned}$$

We now show that  $1 * (\hat{\chi}_1 - \psi_1) \geq 0$  which would show the lower bound. If  $t > u_0$  then  $(1 * (\hat{\chi}_1 - \psi_1))(t) = \hat{I}_1(u_0) - \int_{u_0/\sqrt{e}}^{u_0} 2dv/v = 0$ . If  $t \leq u_0/\sqrt{e}$  then  $(1 * (\hat{\chi}_1 - \psi_1))(t) = (1 * \hat{\chi}_1)(t) \geq 0$ . Lastly if  $u_0/\sqrt{e} \leq t \leq u_0$ , then

$$1 * (\hat{\chi}_1 - \psi_1) = \int_0^t (\hat{\chi}_1(v) - \psi_1(v)) dv = \int_t^{u_0} \left( \frac{2}{v} - \frac{1 - \hat{\chi}(v)}{v} \right) dv \geq 0.$$

**Lemma 5.4.** *If  $u \leq u_0(1 + 1/\sqrt{e})$  then  $\hat{I}_3(u) \leq 3\lambda\hat{I}_2(u) + 3\tau^2$ .*

*Proof.* By definition

$$\hat{I}_3(u) = \int_{\substack{t_1, t_2, t_3 \geq 1 \\ t_1 + t_2 + t_3 \leq u}} \frac{1 - \hat{\chi}(t_1)}{t_1} \frac{1 - \hat{\chi}(t_2)}{t_2} \frac{1 - \hat{\chi}(t_3)}{t_3} dt_1 dt_2 dt_3.$$

Since  $u \leq u_0(1 + 1/\sqrt{e})$  it follows that either one of  $t_1, t_2, t_3$  is  $\leq u_0(1 - 1/\sqrt{e})$  or at least two of  $t_1, t_2, t_3$  must be  $\leq u_0/\sqrt{e}$ . The first case contributes  $\leq 3\lambda\hat{I}_2(u)$  and the second contributes  $\leq 3\tau^2$ .

**5b. Bounding  $\int_0^{u_0} |\sigma(u)| du$ .**

Note that if  $u \leq u_0$  then  $\sigma(u) = \hat{\sigma}(u) \geq 1 - \hat{I}_1(u) \geq 0$ . Since  $\sigma(u) \leq 1 - I_1(u) + I_2(u)/2$  by Proposition 3.6, we obtain

$$\frac{1}{u_0} \int_0^{u_0} |\sigma(u)| du = \frac{1}{u_0} \int_0^{u_0} \sigma(u) du \leq 1 - \frac{(1 * I_1)(u_0)}{u_0} + \frac{(1 * I_2)(u_0)}{2u_0}.$$

Observe that  $1 * I_2 = 1 * (1 * \chi_1 * \chi_1) = 1 * \chi_1 * 1 * \chi_1 = I_1 * I_1$ . Hence

$$(5.4) \quad \frac{1}{u_0} \int_0^{u_0} |\sigma(u)| du \leq \frac{1}{2} + \frac{1}{2u_0} ((1 - I_1) * (1 - I_1))(u_0).$$

If we had lower bounds for  $I_1(t)$  then we could use those in (5.4) to get an upper bound on  $\int_0^{u_0} |\sigma(u)| du$ .

**Lemma 5.5.** For  $0 \leq u \leq u_0$ ,  $I_1(u) \geq \psi_1(u)$  where

$$\psi_1(u) := \begin{cases} \max(0, \lambda + 2 \log((u/u_0)/(1 - 1/\sqrt{e}))) & \text{if } 0 \leq u \leq (1 - 1/\sqrt{e})u_0 \\ \max(\lambda, \tau + 2 \log(\sqrt{e}u/u_0)) & \text{if } (1 - 1/\sqrt{e})u_0 \leq u \leq u_0/\sqrt{e} \\ \max(\tau, 1 + 2 \log(u/u_0)) & \text{if } u_0/\sqrt{e} \leq u \leq u_0. \end{cases}$$

Note that if  $\tau \geq 2 \log 2 - 1$  then

$$\psi_1(u) \geq \psi_2(u) := \begin{cases} \max(0, 2 \log(2u/u_0)) & \text{if } 0 \leq u \leq e^{\tau/2}u_0/2 \\ \max(\tau, 1 + 2 \log(u/u_0)) & \text{if } e^{\tau/2}u_0/2 \leq u \leq u_0. \end{cases}$$

*Proof.* We denote each of the above ranges in the definition of  $\psi_1$  by  $[u_1, u_2]$ . Since  $I_1$  is non-decreasing, we know that that  $I_1(u) \geq I_1(u_1)$  (which gives the lower bounds  $0, \lambda, \tau$  respectively). Further  $I_1(u) \geq I_1(u_2) - \int_u^{u_2} 2dt/t = I_1(u_2) - 2 \log(u_2/u)$ , which gives the other lower bound for that range.

The bounds  $\psi_1(u) \geq \psi_2(u)$  follow from the definitions.

We could plug in the lower bound  $\psi_1$  in (5.4) to obtain an upper bound for  $\int_0^{u_0} |\sigma(u)| du$ . However the resulting expression is complicated and we prefer to obtain simpler, but still sufficiently strong, bounds. We first use Lemma 5.5 to deal with the simpler case when  $\tau \geq 2 \log 2 - 1$ .

**Corollary 5.6.** Suppose  $\tau \geq 2 \log 2 - 1$ , then

$$\frac{1}{u_0} \int_0^{u_0} |\sigma(u)| du \leq 1 + e^{\tau/2} \left(1 - \frac{2}{\sqrt{e}}\right).$$

*Proof.* Since  $I_1(t) \geq \psi_2(t)$ , and as  $\psi_2(t) = 0$  for  $t \leq u_0/2$  we see that

$$\frac{1}{u_0} \int_0^{u_0} |\sigma(u)| du \leq \frac{1}{2} + \frac{1}{2u_0} ((1 - \psi_2) * (1 - \psi_2))(u_0) = 1 - \frac{(1 * \psi_2)(u_0)}{2u_0}.$$

The corollary follows upon calculating  $(1 * \psi_2)(u_0)$ .

**Corollary 5.7.** We have

$$\frac{1}{u_0} \int_0^{u_0} |\sigma(u)| du \leq 2 - \frac{2}{\sqrt{e}} - \frac{\tau^2}{2\sqrt{e}} - \begin{cases} 0 & \text{if } \tau \leq 2 \log 2 - 1 \\ \lambda/12 & \text{if } \tau \leq 3/10. \end{cases}$$

*Proof.* Throughout the calculations in this proof we make use of the hypothesis that  $0 \leq \lambda \leq \tau \leq 2 \log 2 - 1$ . From (5.4), we know that the desired integral is  $\leq 1/2 + ((1 - \psi_1) * (1 - \psi_1)(u_0))/(2u_0)$ .

With some calculation we verify that

$$(5.5) \quad 1 - \frac{(1 * \psi_1)(u_0)}{u_0} = 2 \left( 2 - e^{-\lambda/2} - e^{-1/2} \left( e^{\tau/2} + e^{(\lambda-\tau)/2} - e^{-\lambda/2} \right) \right).$$

Since  $\tau \leq 2 \log 2 - 1$  we have  $\psi_1(u) = \lambda$  for  $u_0(1 - 1/\sqrt{e}) \leq u \leq u_0/2$ . Moreover, if  $u_0 - u > u_0/\sqrt{e}$  and  $\psi_1(u_0 - u) \neq \tau$  then, by definition, we must have  $u \leq u_0(1 - e^{(\tau-1)/2}) \leq u_0(1 - 1/\sqrt{e})e^{-\lambda/2}$ , so that  $\psi_1(u) = 0$ . Thus

$$\frac{\psi_1 * \psi_1(u_0)}{2u_0} = \frac{\tau}{u_0} \int_0^{u_0(1-1/\sqrt{e})} \psi_1(u) du + \frac{\lambda}{u_0} \int_{u_0(1-1/\sqrt{e})}^{u_0/2} \psi_1(u_0 - u) du.$$

With some calculation one can verify that this equals

$$(5.6) \quad \tau \left(1 - \frac{1}{\sqrt{e}}\right) (2e^{-\lambda/2} - 2 + \lambda) + \lambda^2 \left(\frac{1}{\sqrt{e}} - \frac{1}{2}\right) + \frac{\lambda}{\sqrt{e}} (2e^{(\lambda-\tau)/2} - 2 + (\tau - \lambda)).$$

Thus we know (5.4)  $\leq$  (5.5) + (5.6). We now obtain some simple upper bounds for the expressions in (5.5) and (5.6).

By observing that  $1 - e^{-\lambda/2} \leq \lambda/2 - \lambda^2/8 + \lambda^3/48$ , and that

$$e^{\tau/2} + e^{(\lambda-\tau)/2} - e^{-\lambda/2} \geq 1 + \lambda + \frac{\tau^2}{8} + \frac{\tau^3}{48} + \frac{(\lambda - \tau)^2}{8} + \frac{(\lambda - \tau)^3}{48} - \frac{\lambda^2}{8} \geq 1 + \lambda + \frac{\tau^2}{4} - \frac{\tau\lambda}{4},$$

we deduce the upper bound

$$(5.7) \quad (5.5) \leq 2 - \frac{2}{\sqrt{e}} - \frac{\tau^2}{2\sqrt{e}} - \lambda \left(\frac{2}{\sqrt{e}} - 1 - \frac{\tau}{2\sqrt{e}}\right) - \lambda^2 \left(\frac{1}{4} - \frac{\lambda}{24}\right).$$

Since  $2e^{-\xi/2} - 2 + \xi \leq \xi^2/4$  for all  $\xi \geq 0$ , we get

$$(5.6) \leq \frac{\tau\lambda^2}{4}(1 - 1/\sqrt{e}) + \lambda^2(1/\sqrt{e} - 1/2) + \frac{\lambda}{4\sqrt{e}}(\tau - \lambda)^2 \\ \leq \lambda \frac{\tau^2}{4\sqrt{e}} + \lambda^2 \left(\frac{1}{\sqrt{e}} - \frac{1}{2} + \frac{\lambda}{4\sqrt{e}}\right).$$

Combining this upper bound with (5.7), we get

$$(5.4) \leq 2 \left(1 - \frac{1}{\sqrt{e}}\right) - \frac{\tau^2}{2\sqrt{e}} - \lambda \left(\frac{2}{\sqrt{e}} - 1 - \frac{\tau}{2\sqrt{e}} - \frac{\tau^2}{4\sqrt{e}}\right) - \lambda^2 \left(\frac{3}{4} - \frac{1}{\sqrt{e}} - \frac{\lambda}{24} \left(1 + \frac{6}{\sqrt{e}}\right)\right).$$

We deduce our result by noting that

$$\frac{3}{4} - \frac{1}{\sqrt{e}} - \frac{\lambda}{24} \left(1 + \frac{6}{\sqrt{e}}\right) \geq \frac{3}{4} - \frac{1}{\sqrt{e}} - \frac{(2 \log 2 - 1)}{24} \left(1 + \frac{6}{\sqrt{e}}\right) > 0,$$

and

$$\frac{2}{\sqrt{e}} - 1 - \frac{\tau}{2\sqrt{e}} - \frac{\tau^2}{4\sqrt{e}} \geq \begin{cases} 0 & \text{if } \tau \leq 2 \log 2 - 1 \\ 1/12 & \text{if } \tau \leq 3/10. \end{cases}$$

**5c. Proof of Theorem 5.1 for large  $\tau(\geq 29/100)$ .**

Define  $\alpha = \exp(|\delta_1|/2 - 10^{-6})$ . By Lemmas 5.2 and 5.3 we know that for  $(1 + 1/\sqrt{e})u_0 \geq u \geq u_0$ ,  $|\sigma(u)| \leq \max(1/2, 2 \log(u/u_0))$ . It follows that Theorem 1 holds in the range  $0 \leq u \leq \alpha u_0$ . If  $u > \alpha u_0$  then by Lemma 3.5

$$(5.8) \quad |\sigma(u)| \leq \frac{1}{\alpha u_0} \int_0^{\alpha u_0} |\sigma(t)| dt$$

so it suffices to show that this integral is  $< |\delta_1| - 10^{-6}$  to complete the proof of Theorem 5.1 in this range of  $\tau$ .

We estimate this integral by bounding it in various ranges using several results from previous sections: First, we bound  $\int_0^{u_0} |\sigma(t)| dt$  by Corollary 5.6 when  $2 \log 2 - 1 \leq \tau \leq 1$ , and by the first part of Corollary 5.7 when  $29/100 \leq \tau \leq 2 \log 2 - 1$ .

Second, since  $|\sigma(u)| \leq \max(1/2, 2 \log(u/u_0))$  for  $2u_0/\sqrt{e} \leq u \leq u_0\alpha$ , we get the bound

$$\frac{1}{u_0} \int_{2u_0/\sqrt{e}}^{\alpha u_0} |\sigma(t)| dt \leq \int_{2/\sqrt{e}}^{e^{1/4}} \frac{dt}{2} + \int_{e^{1/4}}^{\alpha} 2 \log t dt.$$

Finally, we have  $|\sigma(u)| \leq \max(\tau - \tau^2/2, 2 \log(u/u_0))$  for  $u_0 \leq u \leq 2u_0/\sqrt{e}$ , by Lemmas 5.2 and 5.3 and since  $\gamma(u) = 0$  in this range. Thus if  $\exp(\tau/2 - \tau^2/4) \geq 2/\sqrt{e}$  (which happens when  $\tau \geq 0.5231 \dots$ ) then

$$\frac{1}{u_0} \int_{u_0}^{2u_0/\sqrt{e}} |\sigma(t)| dt \leq (2/\sqrt{e} - 1)(\tau - \tau^2/2)$$

whereas if  $\exp(\tau/2 - \tau^2/4) \leq 2/\sqrt{e}$  (which happens when  $\tau \leq 0.5231 \dots$ ) then

$$\int_{u_0}^{2u_0/\sqrt{e}} |\sigma(t)| dt \leq (\exp(\tau/2 - \tau^2/4) - 1)(\tau - \tau^2/2) + \int_{\exp(\tau/2 - \tau^2/4)}^{2/\sqrt{e}} \log t dt.$$

Combining the above upper bounds on the integrals in the appropriate ranges, we deduce, after several straightforward calculations, that the integral on the right side of (5.8) is indeed  $< |\delta_1| - 10^{-6}$  and so Theorem 5.1 follows (for  $\tau \geq 29/100$ ).

*Henceforth we suppose that  $\tau \leq 29/100$ .*

**5d. The range  $u_0 \leq u \leq (2/\sqrt{e})u_0$ .**

We suppose in this section that  $u_0 \leq u \leq (2/\sqrt{e})(u_0)$ . Note that  $\gamma(u\sqrt{e}/u_0) = 0$  in this range. Observe that

$$\begin{aligned} \hat{I}_1(u - u_0(1 - 1/\sqrt{e})) &= \tau + \int_{u_0/\sqrt{e}}^{u - u_0(1 - 1/\sqrt{e})} \hat{\chi}_1(t) dt \leq \tau + 2 \log(1 + \sqrt{e}(u/u_0 - 1)) \\ &\leq \tau + 2\sqrt{e} \log(u/u_0), \end{aligned}$$

where the last inequality follows because  $\log(1 + \sqrt{e}(e^x - 1)) \leq \sqrt{e}x$  for all  $x \geq 0$  (in fact, (right side)–(left side) is an increasing function of  $x$ ). Inserting this in the middle bound of Lemma 5.3, we see by Lemma 5.2 that

$$\sigma(u) \leq \frac{\hat{I}_2(u)}{2} \leq \lambda(1 - \tau) + \frac{\tau^2}{2} + 2\sqrt{e}(\tau - \lambda) \log \frac{u}{u_0}.$$

Now define

$$\nu = \frac{\lambda(1 - \tau) + \tau^2/2}{2(1 - \sqrt{e}(\tau - \lambda))}.$$

From our assumption that  $\tau \leq 29/100$ , it is easy to show that  $\nu \leq \tau/2 - \tau^2/4 < 0.15$ . Since  $\sigma(u) \geq -2 \log(u/u_0)$  by Lemma 5.2, we see by our upper bound above for  $\sigma(u)$  that that

$$\begin{aligned} |\sigma(u)| &\leq \max(2 \log(u/u_0), \lambda(1 - \tau) + \tau^2/2 + 2\sqrt{e}(\tau - \lambda) \log(u/u_0)) \\ &= \begin{cases} 2 \log(u/u_0) & \text{if } u \geq u_0 e^\nu \\ 2 \log(u/u_0) + 2(1 - \sqrt{e}(\tau - \lambda))(\nu - \log(u/u_0)) & \text{if } u \leq u_0 e^\nu. \end{cases} \end{aligned}$$

Using the above upper bounds, we deduce that

$$\begin{aligned} \frac{1}{u_0} \int_{u_0}^{2u_0/\sqrt{e}} |\sigma(u)| du &\leq \int_1^{2/\sqrt{e}} 2 \log t dt + 2(e^\nu - 1 - \nu)(1 - \sqrt{e}(\tau - \lambda)) \\ &\leq \int_1^{2/\sqrt{e}} 2 \log t dt + \frac{4}{15} \frac{(\lambda(1 - \tau) + \tau^2/2)^2}{1 - \sqrt{e}(\tau - \lambda)}, \end{aligned}$$

by the definition of  $\nu$  and since  $e^x - 1 - x \leq 8x^2/15$  if  $0 \leq x \leq 0.15$ .

We simplify this a little by observing that, since  $\nu \leq \tau/2 - \tau^2/4$ ,

$$\begin{aligned} \frac{(\lambda(1 - \tau) + \tau^2/2)^2}{1 - \sqrt{e}(\tau - \lambda)} &= \lambda(1 - \tau) \frac{\lambda(1 - \tau)}{1 - \sqrt{e}(\tau - \lambda)} + \frac{\tau^2}{2} \frac{\lambda(1 - \tau)}{1 - \sqrt{e}(\tau - \lambda)} + \tau^2 \nu \\ &\leq \lambda\tau(1 - \tau)^2 + \frac{\tau^2}{2}\tau(1 - \tau) + \tau^2 \left( \frac{\tau}{2} - \frac{\tau^2}{4} \right). \end{aligned}$$

Inserting this in the previous estimate, and using  $\tau \leq 29/100$  we obtain

$$(5.9) \quad \frac{1}{u_0} \int_{u_0}^{2u_0/\sqrt{e}} |\sigma(u)| du \leq \int_1^{2/\sqrt{e}} 2 \log t dt + \frac{\tau^2}{16} + \frac{\lambda}{25} = \int_1^{2/\sqrt{e}} |\rho_-(t\sqrt{e})| dt + \frac{\tau^2}{16} + \frac{\lambda}{25}.$$

**5e. The range  $2u_0/\sqrt{e} \leq u \leq (1 + 1/\sqrt{e})u_0$ .**

From Lemmas 5.2, 5.3 and 5.4 we see that in this range

$$\sigma(u) \leq \frac{\hat{I}_2(u)}{2} \leq \gamma\left(\frac{u\sqrt{e}}{u_0}\right) + \tau - \frac{\tau^2}{2},$$

and

$$\sigma(u) \geq -2 \log \frac{u}{u_0} + \frac{\hat{I}_2(u)}{2} - \frac{\hat{I}_3(u)}{6} \geq -2 \log \frac{u}{u_0} + (1 - \lambda)\gamma\left(\frac{u\sqrt{e}}{u_0}\right) - \frac{\tau^2}{2}.$$

Hence

$$(5.10) \quad |\sigma(u)| \leq \max\left(2 \log \frac{u}{u_0} - (1 - \lambda)\gamma\left(\frac{u\sqrt{e}}{u_0}\right) + \frac{\tau^2}{2}, \gamma\left(\frac{u\sqrt{e}}{u_0}\right) + \tau - \frac{\tau^2}{2}\right).$$

**Lemma 5.8.** *The function  $2 \log(t) - (1 - \lambda)\gamma(t\sqrt{e}) + \tau^2/2$  is increasing in the range  $2/\sqrt{e} \leq t < (1 + 1/\sqrt{e})$ , and it is  $> \tau - \tau^2/2 + \gamma(t\sqrt{e})$ .*

*Proof.* For  $2/\sqrt{e} \leq t < (1 + 1/\sqrt{e})$  we have

$$\frac{d}{dt} (2 \log(t) - (1 - \lambda)\gamma(t\sqrt{e}) + \tau^2/2) = \frac{1}{t} (2 - 4(1 - \lambda) \log(t\sqrt{e} - 1)) > 0,$$

which gives the first statement. Now,

$$\frac{d}{dt} (\log(t) - \gamma(t\sqrt{e})) = \frac{1}{t} (1 - 4 \log(t\sqrt{e} - 1)),$$

which is positive in  $(2/\sqrt{e}, e^{-1/2} + e^{-1/4})$  and negative in  $(e^{-1/2} + e^{-1/4}, (1 + 1/\sqrt{e}))$ . So the minimum of  $\log(t) - \gamma(t\sqrt{e})$  is attained at one of the end points  $2/\sqrt{e}$  or  $(1 + 1/\sqrt{e})$ . The values taken by  $\log(t) - \gamma(t\sqrt{e})$  at these two points are  $0.19\dots$  and  $0.1829\dots$ , respectively, which are both larger than  $1/8 \geq (\tau - \tau^2)/2$ . Thus  $\log t - \gamma(t\sqrt{e}) \geq (\tau - \tau^2)/2$  throughout our range. Doubling this and adding  $\tau^2/2 + \gamma(t\sqrt{e})$  to both sides implies the second assertion of the lemma, since  $\lambda\gamma(t\sqrt{e}) \geq 0$

By (5.10) and Lemma 5.8 we see that for any  $2u_0/\sqrt{e} \leq u \leq (1 + 1/\sqrt{e})u_0$  we have

$$\frac{1}{u_0} \int_{2u_0/\sqrt{e}}^u |\sigma(t)| dt \leq \int_{2/\sqrt{e}}^{u/u_0} \left( 2 \log t - (1 - \lambda)\gamma(t\sqrt{e}) + \frac{\tau^2}{2} \right) dt$$

and that

$$\begin{aligned} \left( 1 + \frac{1}{\sqrt{e}} - \frac{u}{u_0} \right) |\sigma(u)| &\leq \left( 1 + \frac{1}{\sqrt{e}} - \frac{u}{u_0} \right) \left( 2 \log \frac{u}{u_0} - (1 - \lambda)\gamma(u\sqrt{e}) + \frac{\tau^2}{2} \right) \\ &\leq \int_{u/u_0}^{1+1/\sqrt{e}} \left( 2 \log t - (1 - \lambda)\gamma(t\sqrt{e}) + \frac{\tau^2}{2} \right) dt. \end{aligned}$$

Adding these two inequalities, and noting that  $\int_{2/\sqrt{e}}^{1+1/\sqrt{e}} \gamma(t\sqrt{e}) dt = 0.0416\dots < 1/24$  we arrive at

$$\begin{aligned} \frac{1}{u_0} \int_{2u_0/\sqrt{e}}^u |\sigma(t)| dt + \left( 1 + \frac{1}{\sqrt{e}} - \frac{u}{u_0} \right) |\sigma(u)| \\ &\leq \int_{2/\sqrt{e}}^{1+1/\sqrt{e}} (2 \log t - \gamma(t\sqrt{e})) dt + \frac{\lambda}{24} + \frac{\tau^2}{2} \left( 1 - \frac{1}{\sqrt{e}} \right) \\ (5.11) \quad &= \int_{2/\sqrt{e}}^{1+1/\sqrt{e}} |\rho_-(t\sqrt{e})| dt + \frac{\lambda}{24} + \frac{\tau^2}{2} \left( 1 - \frac{1}{\sqrt{e}} \right). \end{aligned}$$

**5f. Completion of the proof of Theorem 5.1.**

Recall that  $\alpha = \exp(|\delta_1|/2 - 10^{-6})$  and that by Lemmas 5.2 and 5.3 we have  $|\sigma(u)| \leq \max(1/2, 2 \log(u/u_0))$  when  $u_0 \leq u \leq (1 + \sqrt{e})u_0$ . Moreover  $\tau \leq 29/100$ . Thus Theorem 5.1 holds in the range  $u \leq \alpha u_0$ ; and so, below, we suppose that  $u > \alpha u_0 > 2u_0/\sqrt{e}$ . Put  $v = \min(u_0(1 + 1/\sqrt{e}), u)$  so that by Lemma 3.5

$$\begin{aligned} \left(1 + \frac{1}{\sqrt{e}}\right) |\sigma(u)| &= \frac{v}{u_0} |\sigma(u)| + \left(1 + \frac{1}{\sqrt{e}} - \frac{v}{u_0}\right) |\sigma(v)| \\ &\leq \frac{1}{u_0} \int_0^v |\sigma(t)| dt + \left(1 + \frac{1}{\sqrt{e}} - \frac{v}{u_0}\right) |\sigma(v)|. \end{aligned}$$

Using the second part of Corollary 5.7 together with (5.9) and (5.11), and recalling that  $2 - 2/\sqrt{e} = \int_0^1 |\rho_-(t\sqrt{e})| dt$ , we see that this is

$$\leq \int_0^{1+1/\sqrt{e}} |\rho_-(t\sqrt{e})| dt - \tau^2 \left( \frac{1}{\sqrt{e}} - \frac{9}{16} \right) - \frac{\lambda}{600} \leq \left(1 + \frac{1}{\sqrt{e}}\right) |\delta_1| - \frac{\tau^2}{25} - \frac{\lambda}{600},$$

because of the identity (5.2). It follows from this that  $|\sigma(u)| \leq |\delta_1|$  for all  $u \geq u_0$ . Further,  $|\sigma(u) - \delta_1| \leq \epsilon$  implies that  $\lambda \leq \tau \ll \sqrt{e}$ . Since  $I_1(u_0; \chi) = 1$ , we must have

$$(5.12) \quad \int_0^{u_0/\sqrt{e}} \frac{1 - \chi(t)}{t} dt + \int_{u_0/\sqrt{e}}^{u_0} \frac{1 + \chi(t)}{t} dt = 2\tau \ll \sqrt{e}.$$

We now try to pinpoint further the case when  $|\sigma(u) - \delta_1| \leq \epsilon$ . Put  $\chi_-(t) = 1$  for  $t \leq u_0/\sqrt{e}$ , and  $\chi_-(t) = -1$  for  $t > u_0/\sqrt{e}$ . Note that the corresponding solution to (1.5) is  $\rho_-(t\sqrt{e}/u_0)$ . Using Lemma 3.4 it follows that

$$\sigma(u) = \rho_-\left(\frac{u\sqrt{e}}{u_0}\right) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} D_k,$$

say, where

$$D_k = \int_{\substack{t_1, \dots, t_k \geq 1 \\ t_1 + \dots + t_k \leq u}} \frac{\chi_-(t_1) - \chi(t_1)}{t_1} \dots \frac{\chi_-(t_k) - \chi(t_k)}{t_k} \rho_-\left(\frac{(u - t_1 - \dots - t_k)\sqrt{e}}{u_0}\right) dt_1 \dots dt_k.$$

Suppose first that  $u_0 < u \leq 3u_0/\sqrt{e}$ . Notice that when  $k \geq 2$ , at least one of the  $t_i$  must be  $\leq u_0$ . It follows that for  $k \geq 2$ ,

$$|D_k| \leq k \left( \int_0^{u_0} \frac{|\chi_-(t) - \chi(t)|}{t} dt \right) \left( \int_0^{3u_0/\sqrt{e}} \frac{|\chi_-(t) - \chi(t)|}{t} dt \right)^{k-1}.$$

By (5.12), we see that the first factor above is  $\leq C\sqrt{\epsilon}$  for some constant  $C$ ; and clearly the second factor is  $\leq (C\sqrt{\epsilon} + 2 \int_{u_0}^{3u_0/\sqrt{\epsilon}} \frac{dt}{t})^{k-1} \leq 2^{k-1}$ , if  $\epsilon$  is small enough. So, for  $k \geq 2$ ,  $|D_k| \leq 2^{k-1} C k \sqrt{\epsilon}$  whence it follows by (5.12) that

$$\begin{aligned} \sigma(u) &= \rho_- \left( \frac{u\sqrt{\epsilon}}{\epsilon} \right) - \int_0^u \frac{\chi_-(t) - \chi(t)}{t} \rho_- \left( \frac{(u-t)\sqrt{\epsilon}}{u_0} \right) dt + O(\sqrt{\epsilon}) \\ &= \rho_- \left( \frac{u\sqrt{\epsilon}}{u_0} \right) + \int_{u_0}^u \frac{1 + \chi(t)}{t} \rho_- \left( \frac{(u-t)\sqrt{\epsilon}}{u_0} \right) dt + O(\sqrt{\epsilon}). \end{aligned}$$

Notice that  $\rho_-$  is a non-increasing function in the range  $[0, 3 - \sqrt{\epsilon}]$ . Thus the  $\rho_-((u-t)\sqrt{\epsilon}/u_0)$  term in the right side of the equation above lies between  $\rho_-(0) = 1$  and  $\rho_-(3 - \sqrt{\epsilon}) = 0.3978\dots$ . Hence when  $u_0 \leq u \leq 3u_0/\sqrt{\epsilon}$ , we conclude that

$$(5.13) \quad \frac{1}{3} \int_{u_0}^u \frac{1 + \chi(t)}{t} dt + O(\sqrt{\epsilon}) \leq \sigma(u) - \rho_- \left( \frac{u\sqrt{\epsilon}}{u_0} \right) \leq \int_{u_0}^u \frac{1 + \chi(t)}{t} dt + O(\sqrt{\epsilon}).$$

In the range  $t \in [0, 3]$  we know that  $\rho_-$  has its minimum of  $\delta_1$  at  $t = (1 + \sqrt{\epsilon})$ , and further it is easy to check that  $(\rho_-(t) - \delta_1) \gg (t - (1 + \sqrt{\epsilon}))^2$ . From this and (5.13) it follows that if  $|\sigma(u) - \delta_1| \leq \epsilon$  for  $u$  in the range  $[u_0, 3u_0/\sqrt{\epsilon}]$  then we must have  $u/u_0 = (1 + 1/\sqrt{\epsilon}) + O(\epsilon^{\frac{1}{4}})$ , and that  $\int_{u_0}^u \frac{1 + \chi(t)}{t} dt = O(\sqrt{\epsilon})$ . This proves Theorem 5.1 in this range of  $u$ .

Now suppose that  $u > 3u_0/\sqrt{\epsilon}$ . By Lemma 3.5, we note that

$$|\sigma(u)| \leq \frac{1}{3u_0/\sqrt{\epsilon}} \int_0^{3u_0/\sqrt{\epsilon}} |\sigma(t)| dt.$$

Now by (5.13) and a simple computation, we see that for  $t \leq 3u_0/\sqrt{\epsilon}$ ,

$$\rho_- \left( \frac{t\sqrt{\epsilon}}{u_0} \right) + O(\sqrt{\epsilon}) \leq \sigma(t) \leq \rho_- \left( \frac{t\sqrt{\epsilon}}{u_0} \right) + 2 \log \frac{u}{u_0} + O(\sqrt{\epsilon}) \leq -\rho_- \left( \frac{t\sqrt{\epsilon}}{u_0} \right) + O(\sqrt{\epsilon})$$

whence  $|\sigma(t)| \leq |\rho_-(t\sqrt{\epsilon}/u_0)| + O(\sqrt{\epsilon})$ . Inserting this into our bound for  $|\sigma(u)|$ , we get

$$|\sigma(u)| \leq \frac{\sqrt{\epsilon}}{3u_0} \int_0^{3u_0/\sqrt{\epsilon}} |\rho_-(t\sqrt{\epsilon}/u_0)| dt + O(\sqrt{\epsilon}) \leq 0.61 + O(\sqrt{\epsilon}).$$

Thus if  $\epsilon$  is small enough, then  $|\sigma(u) - \delta_1| \leq \epsilon$  is impossible for  $u > 3u_0/\sqrt{\epsilon}$ . This completes our proof of Theorem 5.1.

## 6. THE EULER PRODUCT SPECTRUM

### 6a. Proof of Theorem 4.

Suppose  $z \in \Gamma_{\Theta}(S)$  so that there exists  $f \in \mathcal{F}(S)$  with  $\Theta(f, \infty) = z$ . Suppose  $\alpha \in S$  and define  $g = g_y \in \mathcal{F}(S)$  by  $g(p) = f(p)$  for  $p \leq y$ ,  $g(p) = \alpha$  for  $y < p \leq y^{e^k}$  and



$g(p) = 1$  for  $p > y^{e^k}$ ; here  $k \geq 0$  is some real number. Then by the prime number theorem  $\Theta(g, \infty) = \Theta(f, y)e^{-k(1-\alpha)} + O(1/\log y)$ . Letting  $y \rightarrow \infty$  we have shown that  $e^{-k(1-\alpha)}z \in \Gamma_\Theta(S)$  for all  $\alpha \in S$  and all  $k \geq 0$ .

Now suppose  $\alpha = \sum_{j=1}^l k_j \alpha_j$  belongs to the convex hull of  $S$ ; where  $\alpha_j \in S$  and  $k_j \geq 0$  with  $\sum_{j=1}^l k_j = 1$ . If  $z \in \Gamma_\Theta(S)$  we see, by using the result of the preceding paragraph  $l$  times, that for all  $k \geq 0$ ,

$$e^{-k(k_1(1-\alpha_1)+\dots+k_l(1-\alpha_l))}z = e^{-k(1-\alpha)}z \in \Gamma_\Theta(S).$$

This shows that  $\Gamma_\Theta(S) \supset \mathcal{E}(S) \times \Gamma_\Theta(S)$ . Since  $1 \in \Gamma_\Theta(S)$  and  $1 \in \mathcal{E}(S)$  we have  $\Gamma_\Theta(S) = \mathcal{E}(S) \times \Gamma_\Theta(S) \supset \mathcal{E}(S)$ .

To demonstrate that  $\Gamma_\Theta(S) \subset \mathcal{E}(S) \times [0, 1]$ , we require the following technical lemma.

**Lemma 6.1.** *Let  $p > 1$  and  $\alpha = a + ib \in \mathbb{U}$  with  $\alpha \neq \pm 1$ . Let*

$$s := -\frac{1}{b} \arg(p - \alpha) = \frac{1}{|b|} \arctan\left(\frac{|b|}{p-a}\right), \text{ and put } r := \left(\frac{p-1}{p-\alpha}\right) e^{(1-\alpha)s}.$$

*Then  $r$  is a real number in the range  $0 \leq r \leq 1$ .*

The upper bound for  $r$  is tight and is attained in the situation  $\alpha \rightarrow 1$ . We can also show that  $r \geq e^{-2/(p+1)}((p-1)/(p+1))$  which is attained when  $\alpha \rightarrow -1$ ; but this is not necessary for our applications.

*Proof.* Since  $-bs = \arg(p - \alpha)$  we see that  $r$  is a non-negative real number. Since  $\arctan(t) \leq t$  for all  $t \in [0, \infty)$  ( $\arctan$  is to lie between 0 and  $\pi/2$  here) we get

$$|e^{(1-\alpha)s}| = e^{(1-a)s} = \exp\left(\frac{1-a}{|b|} \arctan\left(\frac{|b|}{p-a}\right)\right) \leq \exp\left(\frac{1-a}{p-a}\right).$$

Next, as  $e^{-t} \geq 1 - t$ , we get  $|(p-1)/(p-\alpha)| \leq (p-1)/(p-a) \leq \exp(-(1-a)/(p-a))$ . Hence

$$r = |r| = \frac{p-1}{|p-\alpha|} |e^{(1-\alpha)s}| \leq \exp\left(-\frac{1-a}{p-a} + \frac{1-a}{p-a}\right) = 1,$$

as desired.

For  $f \in \mathcal{F}(S)$ , let  $k_p = 0$  if  $f(p)$  is real, and  $k_p = -\arg(p-f(p))/\text{Im } f(p) \geq 0$  otherwise. By Lemma 6.1, we may conclude that

$$\frac{p-1}{p-f(p)} = r_p e^{-k_p(1-f(p))}$$

for a real number  $0 \leq r_p \leq 1$ . Taking the product over all primes we get

$$\Theta(f, \infty) = \left(\prod_p r_p\right) \exp\left(-\sum_p k_p(1-f(p))\right).$$

This shows that  $\Gamma_\Theta(S) \subset \mathcal{E}(S) \times [0, 1]$ .

For every  $1 \neq \alpha \in \mathbb{U}$  we note that  $(1 - \operatorname{Re} \alpha)/|\operatorname{Im} \alpha| = \cot(|\arg(1 - \alpha)|)$ . Hence for all  $1 \neq \alpha$  in the convex hull of  $S$  we have  $(1 - \operatorname{Re} \alpha)/|\operatorname{Im} \alpha| \leq \cot(\operatorname{Ang}(S))$ . This shows that if  $z \in \mathcal{E}(S)$  then  $|z| \leq \exp(-|\arg z| \cot(\operatorname{Ang}(S)))$ . Since  $\Gamma_\Theta(S) \subset \mathcal{E}(S) \times [0, 1]$  the same upper bound holds for all  $z \in \Gamma_\Theta(S)$ .

Now suppose that  $1 \neq \beta$  is a real number in the convex hull of  $S$ . Then

$$\mathcal{E}(S) \supset \mathcal{E}(S) \times \{e^{-k(1-\beta)} : k \geq 0\} = \mathcal{E}(S) \times [0, 1].$$

Hence  $\mathcal{E}(S) = \mathcal{E}(S) \times [0, 1]$  in this case, which completes the proof of Theorem 4.

**6b. Proof of Corollary 3.** By Theorems 3 and 4,

$$\Gamma(S) = \Gamma_\Theta(S) \times \Lambda(S) = \mathcal{E}(S) \times \Gamma_\Theta(S) \times \Lambda(S) = \mathcal{E}(S) \times \Gamma(S).$$

If  $S = \{1\}$  then  $\Gamma(S) = \{1\}$  and  $\operatorname{Ang}(S) = 0$  so that (ii) is immediate in this case. Suppose then that  $1 \neq \alpha \in S$ . If  $z \in \Gamma(S)$  then we know that  $e^{-k(1-\alpha)}z \in \Gamma(S)$  for all  $k \geq 0$ . Letting  $k$  vary from 0 to  $\infty$  we get a spiral connecting  $z$  to 0. This shows that  $\Gamma(S)$  is connected. If the convex hull of  $S$  contains a real point other than 1 then we know from Theorem 4 that  $\Gamma_\Theta(S) = [0, 1] \times \Gamma_\Theta(S)$  is starlike. Hence  $\Gamma(S) = \Gamma_\Theta(S) \times \Lambda(S) = [0, 1] \times \Gamma_\Theta(S) \times \Lambda(S) = [0, 1] \times \Gamma(S)$  is starlike as well. This completes the proof of part (i).

Suppose  $\chi(t) = 1$  for  $t \leq 1$  and  $\chi(t) = \alpha$  for  $t > 1$ , and let  $\sigma$  denote the corresponding solution to (1.5). Then by Theorem 3.3, we see that

$$\sigma(u) = 1 - \int_1^u \frac{1 - \chi(t)}{t} dt = 1 - (1 - \alpha) \log u, \quad \text{for } 1 \leq u \leq 2.$$

If  $\alpha$  is in the convex hull of  $S$ , this shows that  $1 - (1 - \alpha) \log u \in \Lambda(S)$  for  $1 \leq u \leq 2$ . Suppose now that  $\pi/2 > \operatorname{Ang}(S) > 0$  and that  $\pm 1 \neq \zeta \in S$  with  $\operatorname{Ang}(\zeta) = \operatorname{Ang}(S) = \theta$ , say. Suppose  $1 < u \leq 2$  and let  $z = 1 - (1 - \zeta) \log u$  so that  $z \in \Lambda(S)$ . If  $|\arg z| = \nu$  then a simple geometric consideration shows that  $|z| = \sin \theta / \sin(\theta + \nu)$ . On the other hand, if  $z \in \Gamma_\Theta(S)$  then by Theorem 4

$$|z| \leq \exp(-\nu \cot \theta) \leq \frac{1}{1 + \nu \cot \theta} < \frac{1}{\cos \nu + \sin \nu \cot \theta} = \frac{\sin \theta}{\sin(\theta + \nu)},$$

which is a contradiction. This proves part (ii).

If  $1, e^{i\alpha}$  and  $e^{i\beta}$  are distinct elements of  $S$  then for all  $k, l \geq 0$  we know that  $e^{-k(1-e^{i\alpha})-l(1-e^{i\beta})} \in \mathcal{E}(S)$ . Now let us fix the real part of  $k(1 - e^{i\alpha}) + l(1 - e^{i\beta})$ ; that is let us fix  $2k \sin^2(\alpha/2) + 2l \sin^2(\beta/2) = r$ , say. Then as  $k$  varies from 0 to  $r/(2 \sin^2(\alpha/2))$  we see that the imaginary part of  $k(1 - e^{i\alpha}) + l(1 - e^{i\beta})$ , which is  $-k \sin \alpha - l \sin \beta$ , varies continuously from  $-r \cot(\beta/2)$  to  $-r \cot(\alpha/2)$ . If the variation in the imaginary part is larger than  $2\pi$  in magnitude then  $\mathcal{E}(S)$  clearly contains the circle with center 0 and radius  $e^{-r}$ ; this happens provided  $r \geq 2\pi/|\cot(\alpha/2) - \cot(\beta/2)|$ . Hence we have proved (iii).

**6c. Proof of Theorem 3'.**

Suppose  $z_\sigma \in \Lambda(S)$ . We shall show that for any  $k \geq 0$  and any  $\alpha \in S$ ,  $e^{-k(1-\alpha)}z_\sigma \in \Lambda(S)$ . Using this repeatedly (as in §6a) it would follow that  $z_\sigma e^{-k(1-\alpha)} \in \Lambda(S)$  for any  $k \geq 0$  and  $\alpha$  in the convex hull of  $S$ . This means that  $\Lambda(S) \supset \Lambda(S) \times \mathcal{E}(S)$ , and since  $1 \in \mathcal{E}(S)$  it would follow that  $\Lambda(S) = \Lambda(S) \times \mathcal{E}(S)$ .

Since  $z_\sigma \in \Lambda(S)$ , we know that there is a measurable function  $\chi \in K(S)$ , and  $u \geq 1$  such that  $z_\sigma = \sigma(u)$ , where  $\sigma$  is the corresponding solution to (1.5). By Proposition 1 (Converse), for large  $x$  we may find  $f \in \mathcal{F}(S)$  with  $f(p) = 1$  for  $p \leq x^{\frac{1}{u}}$  such that

$$\frac{1}{x} \sum_{n \leq x} f(n) = \sigma(u) + o(1) = z_\sigma + o(1).$$

Now put  $y = \exp((\log x)^{\frac{2}{3}})$ , and define  $g \in \mathcal{F}(S)$  by  $g(p) = 1$  for  $p \leq y$  or  $p > y^{e^k}$  and  $g(p) = \alpha$  for  $y < p \leq y^{e^k}$ . Hence  $\Theta(g, x^{\frac{1}{u}}) = e^{-k(1-\alpha)} + o(1)$ . Consider  $h \in \mathcal{F}(S)$  defined by  $h(p) = g(p)$  if  $p \leq x^{\frac{1}{u}}$ , and  $h(p) = f(p)$  for  $p > x^{\frac{1}{u}}$ . By Proposition 1, we see easily that  $\frac{1}{x} \sum_{n \leq x} h(n) + o(1)$  belongs to  $\Lambda(S)$ . On the other hand, appealing to Proposition 4.5 we obtain

$$\begin{aligned} \frac{1}{x} \sum_{n \leq x} h(n) &= \Theta(g, x^{\frac{1}{u}}) \frac{1}{x} \sum_{n \leq x} f(n) + O\left(\frac{\log y}{\log x} \exp\left(\sum_{y < p \leq y^{e^k}} \frac{2}{p} + \sum_{x^{\frac{1}{u}} < p \leq x} \frac{2}{p}\right)\right) \\ &= e^{-k(1-\alpha)} z_\sigma + o(1). \end{aligned}$$

Letting  $x \rightarrow \infty$ , we conclude that  $e^{-k(1-\alpha)}z_\sigma \in \Lambda(S)$ , as desired.

Plainly  $\Lambda(S) \subset \Gamma(S)$ . Further, by the result just established and Theorems 3 and 4,

$$\Lambda(S) \times [0, 1] = \Lambda(S) \times \mathcal{E}(S) \times [0, 1] \supset \Lambda(S) \times \Gamma_\Theta(S) = \Gamma(S).$$

If the convex hull of  $S$  contains a real point other than 1 then by Theorem 4,  $\mathcal{E}(S) = \mathcal{E}(S) \times [0, 1] = \Gamma_\Theta(S)$  and hence  $\Lambda(S) = \Lambda(S) \times [0, 1] = \Gamma(S)$ .

## 7. ANGLES AND PROJECTIONS OF THE SPECTRUM

**7a. Proof that  $\text{Ang}(\Gamma(S)) \ll \text{Ang}(S)$ .**

Let  $\text{Ang}(S) = \theta$  and we seek to show that  $\text{Ang}(\Lambda(S)) = \text{Ang}(\Gamma(S)) \ll \theta$ . Suppose  $\chi \in K(S)$  is given, and let  $\sigma$  denote the corresponding solution to (1.5). We need to show that  $\text{Ang}(\sigma(u)) \ll \theta$  for all  $u$ ; or, in other words,  $|\text{Im } \sigma(u)| \ll \theta(1 - \text{Re } \sigma(u))$ .

We may suppose that  $\theta$  is sufficiently small, else the result is trivial. We let  $\hat{\chi} = \text{Re } \chi$  and let  $\hat{\sigma}$  denote the corresponding solution to (1.5). Recall that, in the notation of §3b,  $R_1(u) = R_1(u; \chi) = \int_1^u \frac{1 - \text{Re } \chi(t)}{t} dt = \int_1^u \frac{1 - \hat{\chi}(t)}{t} dt$ . By applying Lemma 1', taking  $D$  there to be the region  $\{z \in \mathbb{U} : \text{Ang}(z) \leq \pi/4\}$ , we see that for all  $v$ ,

$$(7.1) \quad |\sigma(v)|, |\hat{\sigma}(v)| \leq c_1 \exp(-c_2 R_1(v))$$

where  $c_1$  and  $c_2$  are absolute positive constants.

By simple trigonometry, for any  $z \in \mathbb{U}$ ,  $\text{Ang}(z) \leq \arcsin |z|$ . Hence we may assume that  $|\sigma(u)| \gg \sin \theta \gg \theta$ , whence  $R_1(u) \ll \log(1/\theta)$ , by (7.1).

By Lemma 3.4

$$\sigma(u) = \hat{\sigma}(u) - \int_1^u \frac{\hat{\chi}(t) - \chi(t)}{t} \hat{\sigma}(u-t) dt + O\left(\sum_{j=2}^{\infty} \frac{1}{j!} \left(\int_1^u \frac{|\hat{\chi}(t) - \chi(t)|}{t} dt\right)^j\right).$$

Since  $\chi$  is in the convex hull of  $S$  and  $\text{Ang}(S) = \theta$ ,  $|\text{Im } \chi(t)| = |\hat{\chi}(t) - \chi(t)| \leq \tan \theta (1 - \text{Re } \chi(t))$ . Hence  $\int_1^u \frac{|\hat{\chi}(t) - \chi(t)|}{t} dt \leq \tan \theta R_1(u)$ . Using this above, and as  $R_1(u) \ll \log(1/\theta)$ , we get

$$(7.2) \quad \sigma(u) = \hat{\sigma}(u) + i \int_1^u \frac{\text{Im } \chi(t)}{t} \hat{\sigma}(u-t) dt + O(\theta^2 R_1(u)^2).$$

If  $R_1(u) \geq 1$  then by Theorem 5.1,  $|\hat{\sigma}(u)| \leq |\delta_1|$ . If  $R_1(u) \leq 1$  then by the inclusion-exclusion inequalities of Proposition 3.6,  $\hat{\sigma}(u) \leq \hat{\sigma}_2(u) \leq 1 - R_1(u) + R_1(u)^2/2 \leq 1 - R_1(u)/2$ . Hence, using (7.2) and as  $R_1(u) \ll \log(1/\theta)$ ,

$$1 - \text{Re } \sigma(u) = 1 - \hat{\sigma}(u) + O(\theta^2 R_1(u)^2) \geq R_1(u) + O(\theta^2 R_1(u)^2) \gg \min(R_1(u), 1),$$

since  $\theta$  is sufficiently small.

Taking imaginary parts in (7.2), and recalling  $|\text{Im } \chi(t)| \ll \theta(1 - \text{Re } \chi(t))$  and (7.1), we see that

$$\begin{aligned} |\text{Im } \sigma(u)| &\leq \int_1^u \frac{|\text{Im } \chi(t)|}{t} |\hat{\sigma}(u-t)| dt + O(\theta^2 R_1(u)^2) \\ &\ll \theta \int_{u/2}^u \frac{1 - \text{Re } \chi(t)}{t} dt + \theta \int_1^{u/2} \frac{1 - \text{Re } \chi(t)}{t} e^{-c_2 R_1(u-t)} dt + \theta^2 R_1(u) \\ &\ll \theta(R_1(u) - R_1(u/2)) + \theta R_1(u/2) \exp(-c_2 R_1(u/2)) + \theta^2 R_1(u). \end{aligned}$$

Since  $R_1(u) - R_1(u/2) \leq 2 \log 2 \ll 1$ , and  $R_1(u) \ll \log(1/\theta)$ , the above shows that  $|\text{Im } \sigma(u)| \ll \theta \min(R_1(u), 1)$ . Combining this with our lower bound for  $1 - \text{Re } \sigma(u)$ , gives  $\text{Ang}(\sigma(u)) \ll \tan(\text{Ang}(\sigma(u))) = |\text{Im } \sigma(u)| / (1 - \text{Re } \sigma(u)) \ll \theta$ , completing the proof.

### 7b. The maximal projection of $S = \{\pm 1, \pm i\}$ .

In this section we shall prove Theorem 7(i). The result for  $S = \{1, -1\}$  follows from Theorem 5.1, and so we may restrict ourselves to the case  $S = \{\pm 1, \pm i\}$ . By Theorem 3' we see that  $\Gamma(S) = \Lambda(S)$  so we shall work here with  $\Lambda(S)$ . Let  $\chi \in K(\{\pm 1, \pm i\})$  be given, and let  $\sigma$  be the corresponding solution to (1.5). We shall show that for all  $u$ ,  $\text{Re } \sigma(u) \geq -(1 + |\delta_1|)/2$  and  $|\text{Im } \sigma(u)| \leq (1 + |\delta_1|)/2$ , so that the maximal projection of  $\{\pm 1, \pm i\}$  is  $(1 + |\delta_1|)/2$  as conjectured; that is, Theorem 7(i).

**Lemma 7.1.** *Let  $\chi'$  be any real-valued measurable function satisfying*

$$|\operatorname{Re} \chi(t)| \leq \frac{1 + \chi'(t)}{2} \quad \text{and} \quad |\operatorname{Im} \chi(t)| \leq \frac{1 - \chi'(t)}{2}.$$

for all  $t$ . Let  $\sigma'$  be the corresponding solution to (1.5). Then, for all  $u$ ,

$$|\operatorname{Re} \sigma(u)| \leq \frac{1 + \sigma'(u)}{2} \quad \text{and} \quad |\operatorname{Im} \sigma(u)| \leq \frac{1 - \sigma'(u)}{2}.$$

*Proof.* Let  $\beta(u) := (1 + \sigma'(u))/2 - |\operatorname{Re} \sigma(u)|$  and  $\gamma(u) := (1 - \sigma'(u))/2 - |\operatorname{Im} \sigma(u)|$ . Since

$$\begin{aligned} u|\operatorname{Re} \sigma(u)| &\leq |\operatorname{Re} \chi| * |\operatorname{Re} \sigma| + |\operatorname{Im} \chi| * |\operatorname{Im} \sigma| \\ &\leq \frac{1 + \chi'}{2} * |\operatorname{Re} \sigma| + \frac{1 - \chi'}{2} * |\operatorname{Im} \sigma|, \end{aligned}$$

we deduce that  $u\beta(u) \geq (1 + \chi')/2 * \beta + (1 - \chi')/2 * \gamma$ . Similarly, by bounding  $|\operatorname{Im} \sigma(u)|$  we get  $u\gamma(u) \geq (1 - \chi')/2 * \beta + (1 + \chi')/2 * \gamma$ . Taking  $\alpha(u) = \min\{\beta(u), \gamma(u)\}$  we have  $\alpha(u) = 0$  for  $0 \leq u \leq 1$ , and we deduce from the above that  $u\alpha(u) \geq 1 * \alpha$ . Therefore  $\alpha(u) \geq 0$  for all  $u$ , by Lemma 3.1.

*Proof of Theorem 7(i).* We wish to show that  $|\operatorname{Im} \sigma(u)|$  and  $-\operatorname{Re} \sigma(u)$  are both  $\leq (1 - \delta_1)/2$ . Note that  $\chi'$  exists, as in Lemma 7.1 since the convex hull of  $S$  is described by the conditions  $|\operatorname{Re} \chi(t)| + |\operatorname{Im} \chi(t)| \leq 1$ . By Theorem 5.1, we know that  $\sigma'(u) \geq \delta_1$  always. Hence by Lemma 7.1,  $|\operatorname{Im} \sigma(u)| \leq (1 - \delta_1)/2 = (1 + |\delta_1|)/2$ . Further, if  $I_1(u; \chi') \geq 1$  then  $|\sigma'(u)| \leq |\delta_1|$  by Theorem 5.1, so that  $|\operatorname{Re} \sigma(u)| \leq (1 + |\delta_1|)/2$ .

We now handle the case when  $I_1(u, \chi') \leq 1$ . Put  $\hat{\chi} = \operatorname{Re} \chi$  and let  $\hat{\sigma}$  be the corresponding real-valued solution to (1.5). By Proposition 3.7 and Theorem 5.1,

$$\operatorname{Re} \sigma(u) \geq \hat{\sigma}(u) - \frac{C_2(u)}{2} \geq \delta_1 - \frac{C_2(u)}{2} \geq \delta_1 - \frac{C_1(u)^2}{2}.$$

Now

$$C_1(u) = \int_1^u \frac{|\operatorname{Im} \chi(t)|}{t} dt \leq \int_1^u \frac{(1 - \chi'(u))}{2t} dt \leq \frac{1}{2},$$

and so  $\operatorname{Re} \sigma(u) \geq \delta_1 - 1/8 > -(1 + |\delta_1|)/2$ , which completes our proof.

### 7c. Towards the proofs of Theorems 5, 6(ii), and 7(ii).

In the following subsections, we suppose that  $S$  is a given subset of  $\mathbb{U}$  with  $\operatorname{Ang}(S) = \theta < \pi/2$ . Suppose that  $\chi \in K(S)$  is given, and that  $\sigma(u)$  is the corresponding solution to (1.5). Define

$$P(u) = \int_0^u \min(2, (1 - \operatorname{Re} \chi(t)) \sec^2 \theta) \frac{dt}{t}.$$

Let  $u_0$  be such that  $P(u_0) + P(u_0/2) = 1$ ; if no such point exists, set  $u_0 = \infty$ .

**Lemma 7.2.** *With these notations  $P(u) \cos^2 \theta \leq R_1(u) \leq P(u)$ , where  $R_i, C_i$  are as in section 3b. Further*

$$(7.3) \quad R_1(u)^2 + C_1(u)^2 \leq R_1(u)P(u),$$

$$(7.4) \quad R_2(u) + C_2(u) \leq \min(R_1(u)P(u), 2R_1(u)\sqrt{P(u/2)P(u)}),$$

$$(7.5) \quad |\operatorname{Im} \sigma(u)| \leq \sqrt{R_1(u)(P(u) - R_1(u))},$$

and

$$(7.6) \quad R_1(u)(1 - P(u)/2) \leq 1 - \operatorname{Re} \sigma(u) \leq R_1(u)(1 + P(u)/2).$$

*Proof.* It is clear from the definitions that  $P(u) \cos^2 \theta \leq R_1(u) \leq P(u)$ . Since  $\chi(t)$  lies in the convex hull of  $S$ , and  $\operatorname{Ang}(S) = \theta$ , we have

$$|\operatorname{Im} \chi(t)| \leq \min(\sqrt{1 - (\operatorname{Re} \chi(t))^2}, (1 - \operatorname{Re} \chi(t)) \tan \theta).$$

Using Cauchy's inequality we obtain

$$C_1(u)^2 \leq R_1(u) \int_1^u \min(1 + \operatorname{Re} \chi(t), (1 - \operatorname{Re} \chi(t)) \tan^2 \theta) \frac{dt}{t}.$$

Adding  $R_1(u)^2$  to the above, we obtain (7.3). By Proposition 3.7,  $|\operatorname{Im} \sigma(u)| \leq C_1(u)$ , and so we deduce (7.5).

Plainly  $R_2(u) \leq R_1(u)^2$ , and  $C_2(u) \leq C_1(u)^2$ . So the first bound in (7.4) follows from (7.3). Further, from the definition of  $R_2$ , we have  $R_2(u) \leq 2R_1(u/2)R_1(u) - R_1(u/2)^2 \leq 2R_1(u/2)R_1(u)$ , and similarly  $C_2(u) \leq 2C_1(u/2)C_1(u)$ . By Cauchy's inequality, and (7.3),

$$\begin{aligned} R_2(u) + C_2(u) &\leq 2(R_1(u/2)^2 + C_1(u/2)^2)^{1/2}(R_1(u)^2 + C_1(u)^2)^{1/2} \\ &\leq 2\sqrt{R_1(u/2)P(u/2)R_1(u)P(u)}, \end{aligned}$$

and the second bound of (7.4) follows as  $R_1(u/2) \leq R_1(u)$ .

By Proposition 3.7 we know that

$$R_1(u) - \frac{R_2(u) + C_2(u)}{2} \leq 1 - \operatorname{Re} \sigma(u) \leq R_1(u) + \frac{R_2(u) + C_2(u)}{2},$$

and using the first bound of (7.4), we obtain (7.6).

We next prove a technical Lemma which will be useful in the proof of Lemma 7.4.

**Lemma 7.3.** *If  $a, b \geq c > 0$  and  $0 \leq x, y \leq 1$  then*

$$2ax + 2by - (\sqrt{ax} + \sqrt{by})^2 \geq c(x + y)(2 - x - y).$$

*Proof.* Without loss of generality assume  $a \geq b$ . We shall prove that result for  $c = b$ , and then the more general statement follows. First note that  $(\sqrt{a} + \sqrt{b})(2 - x) \geq 2\sqrt{b} \geq 2\sqrt{by}$ . Multiplying this through by  $(\sqrt{a} - \sqrt{b})x$  and adding  $b(2y - y^2 + 2x - x^2)$  to both sides, we get  $a(2x - x^2) + b(2y - y^2) \geq 2xy\sqrt{ab} + b(2(x + y) - (x + y)^2)$  after some re-arranging. This directly implies the result.

**Lemma 7.4.** *Suppose that  $u \geq u_0$ . Then*

$$|\sigma(u)|^2 \leq 1 - \frac{\cos^2 \theta}{u_0} \int_0^{u_0/2} (P(t) + P(u_0 - t))(2 - P(t) - P(u_0 - t))dt.$$

*Proof.* If  $t \leq u_0$  then  $R_1(t) \leq P(t) \leq 1$  by Lemma 7.2, and so by Proposition 3.7,

$$\begin{aligned} |\operatorname{Re} \sigma(t)| &\leq \max\left(1 - R_1(t) + \frac{R_2(t) + C_2(t)}{2}, -1 + R_1(t) + \frac{C_2(t)}{2}\right) \\ &= 1 - R_1(t) + \frac{R_2(t) + C_2(t)}{2}. \end{aligned}$$

Using (7.5) we deduce

$$\begin{aligned} |\sigma(t)|^2 &\leq 1 - 2R_1(t) + R_1(t)P(t) + R_2(t) + C_2(t) \\ &\quad + (R_2(t) + C_2(t))\left(\frac{R_2(t) + C_2(t)}{4} - R_1(t)\right). \end{aligned}$$

By (7.4),  $R_2(t) + C_2(t) \leq R_1(t)P(t) \leq R_1(t)$  and so for  $t \leq u_0$ , we have shown

$$(7.7) \quad |\sigma(t)|^2 \leq 1 - 2R_1(t) + R_1(t)P(t) + R_2(t) + C_2(t).$$

By Lemma 3.5, Cauchy's inequality, and the above bound we obtain for  $u \geq u_0$

$$\begin{aligned} |\sigma(u)|^2 &\leq \left(\frac{1}{u_0} \int_0^{u_0} |\sigma(t)| dt\right)^2 \leq \frac{1}{u_0} \int_0^{u_0} |\sigma(t)|^2 dt \\ (7.8) \quad &\leq \frac{1}{u_0} \int_0^{u_0} (1 - 2R_1(t) + R_1(t)P(t) + R_2(t) + C_2(t))dt. \end{aligned}$$

Denote  $\chi_1(t) = (1 - \operatorname{Re} \chi(t))/t$ , so that  $R_1 = 1 * \chi_1$  and  $R_2 = 1 * \chi_1 * \chi_1$ . It follows that  $1 * R_2 = 1 * \chi_1 * 1 * \chi_1 = R_1 * R_1$ . In like manner,  $1 * C_2 = C_1 * C_1$ . Using this, Cauchy's inequality, and (7.3), we obtain

$$\begin{aligned} \int_0^{u_0} (R_2(t) + C_2(t))dt &= (R_1 * R_1)(u_0) + (C_1 * C_1)(u_0) \\ &\leq (\sqrt{R_1^2 + C_1^2} * \sqrt{R_1^2 + C_1^2})(u_0) \leq (\sqrt{R_1 P} * \sqrt{R_1 P})(u_0) \\ &= 2 \int_0^{u_0/2} \sqrt{R_1(t)P(t)R_1(u_0 - t)P(u_0 - t)}dt. \end{aligned}$$

Using this in (7.8) we deduce that  $|\sigma(u)|^2 \leq 1 - J$  where

$$J = \frac{1}{u_0} \int_0^{u_0/2} \left( 2R_1(t) + 2R_1(u_0 - t) - \left( \sqrt{R_1(t)P(t)} + \sqrt{R_1(u_0 - t)P(u_0 - t)} \right)^2 \right) dt.$$

For  $0 \leq t \leq u_0/2$ , take  $a = R_1(t)/P(t)$ ,  $b = R_1(u_0 - t)/P(u_0 - t)$ , so that  $a$  and  $b$  are  $\geq \cos^2 \theta$  by Lemma 7.2. Take  $x = P(t)$  and  $y = P(u_0 - t)$ , so that both  $x$  and  $y$  are  $\leq 1$ . Applying Lemma 7.3, the integrand in the definition of  $J$  is  $\geq \cos^2 \theta (P(t) + P(u_0 - t))(2 - P(t) - P(u_0 - t))$ ; which proves the Lemma.

Using Lemma 7.4 we can get an explicit bound on  $|\sigma(u)|$  when  $u \geq u_0$ .

**Proposition 7.5.** *If  $u \geq u_0$  then  $|\sigma(u)| \leq 1 - (56/411) \cos^2 \theta$ .*

*Proof.* Put  $\alpha = P(u_0/2)$  so that  $0 \leq \alpha \leq 1/2$ , and  $P(u_0) = 1 - \alpha$ . For  $0 \leq t \leq u_0/2$ , note that

$$P(t) \geq P(u_0/2) - \int_t^{u_0/2} 2 \frac{dv}{v} = \alpha - 2 \log(u_0/(2t)),$$

and also  $P(t) \geq 0$ . Similarly

$$P(u_0 - t) \geq P(u_0) - \int_{u_0-t}^{u_0} 2 \frac{dv}{v} = 1 - \alpha - 2 \log(u_0/(u_0 - t)),$$

and also  $P(u_0 - t) \geq P(u_0/2) = \alpha$ . Thus if we put

$$m(t) = \begin{cases} 1 - \alpha + 2 \log(1 - t/u_0) & \text{for } t/u_0 \leq 1 - e^\alpha/\sqrt{e} \\ \alpha & \text{for } 1 - e^\alpha/\sqrt{e} \leq t/u_0 \leq 1/(2e^{\alpha/2}) \\ 2\alpha + 2 \log(2t/u_0) & \text{for } 1/(2e^{\alpha/2}) \leq t/u_0 \leq 1/2, \end{cases}$$

then  $P(t) + P(u_0 - t) \geq m(t) \geq 0$  for each  $0 \leq t \leq u_0/2$ .

Note that  $P(t) + P(u_0 - t) \leq P(u_0/2) + P(u_0) = 1$ , and that the function  $y(2 - y)$  is increasing in the range  $0 \leq y \leq 1$ . Hence

$$\begin{aligned} \frac{1}{u_0} \int_0^{u_0/2} (P(t) + P(u_0 - t))(2 - P(t) - P(u_0 - t)) dt &\geq \frac{1}{u_0} \int_0^{u_0/2} m(t)(2 - m(t)) dt \\ &= (12 - 4\alpha) \frac{e^\alpha}{\sqrt{e}} - 13 - 2\alpha^2 + (6 - 2\alpha)e^{-\alpha/2}, \end{aligned}$$

after some calculations. This function of  $\alpha$  attains a unique minimum in the range  $(0, 1/2)$ , at  $\alpha_0 = 0.08055\dots$ , at which point its value is  $\geq 0.272516916\dots \geq 112/411$ . Inserting this into Lemma 7.4, and taking square roots of both sides we obtain the result.

For convenience, in the next three subsections we put  $\lambda = \lambda_\theta = (28/411) \cos^2 \theta$ .



**7d. Proof of Theorem 5.**

For all  $u$ , we seek to show that the distance of  $\sigma(u)$  from  $\lambda$  is  $\leq 1 - \lambda$ . Suppose  $u \geq u_0$ . By the triangle inequality the distance of  $\sigma(u)$  from  $\lambda$  is  $\leq \lambda$  plus the distance from  $\sigma(u)$  to 0. By Proposition 7.5, the latter distance is  $\leq 1 - 2\lambda$ , so that our claim holds in this case.

Suppose  $u \leq u_0$ . Observe that  $\min\{t, 2\sqrt{t(1-t)}\} \leq 2 - t - (2+t)(28/411)$  for all  $0 \leq t \leq 1$ . Taking  $t = P(u)$ , multiplying through by  $R_1(u)$  and observing that  $P(u/2) \leq 1 - t$ , we obtain  $R_2(u) + C_2(u) \leq R_1(u)(2 - P(u)) - 2\lambda(1 - \operatorname{Re} \sigma(u))$ , from (7.4) and the second inequality in (7.6). By (7.7) we deduce that  $2\lambda(1 - \operatorname{Re} \sigma(u)) \leq 1 - |\sigma(u)|^2$  and so, re-arranging,  $(\operatorname{Re} \sigma(u) - \lambda)^2 + (\operatorname{Im} \sigma(u))^2 \leq (1 - \lambda)^2$ .

It follows that  $\Lambda(S)$  is contained in the circle centered at  $\lambda$  with radius  $1 - \lambda$ , and Theorem 5 follows since  $\Gamma(S) \subset [0, 1] \times \Lambda(S)$ .

**7e. Proof of Theorem 6(ii).**

We shall show that  $\operatorname{Ang}(\sigma(u)) \leq \frac{\pi}{2} - \frac{\sin \delta}{2}$ . Suppose first that  $u \geq u_0$ . Note that  $\operatorname{Ang}(\sigma(u)) \leq \arcsin(|\sigma(u)|) \leq \arcsin(1 - 2\lambda)$ , by Proposition 7.5. Now  $\arcsin(1 - 2\lambda) \leq \pi/2 - \sqrt{4\lambda}$ , and our claim follows in this case since  $\cos \theta = \sin \delta$ , and  $\sqrt{112/411} > 1/2$ .

Thus we may suppose  $u < u_0$ . By definition,  $\operatorname{Ang}(\sigma(u)) = \arctan(|\operatorname{Im} \sigma(u)|/(1 - \operatorname{Re} \sigma(u)))$ . By (7.5) and (7.6),

$$\frac{|\operatorname{Im} \sigma(u)|}{(1 - \operatorname{Re} \sigma(u))} \leq \frac{\sqrt{R_1(u)(P(u) - R_1(u))}}{R_1(u)(1 - P(u)/2)} \leq 2\sqrt{P(u)/R_1(u) - 1} \leq 2 \tan \theta,$$

since  $P(u) \leq 1$  as  $u < u_0$ , and  $P(u)/R_1(u) \leq \sec^2 \theta$  by Lemma 7.2.

For  $0 \leq x < 1$  we have  $(1+x)/(1-x) \geq 4x/(1-x^2)$ . Taking  $x = \tan(\theta/2)$  we deduce that  $2 \tan \theta \leq \tan(\pi/4 + \theta/2)$ . Thus  $\operatorname{Ang}(\sigma(u)) \leq \arctan(2 \tan \theta) \leq \pi/4 + \theta/2 = \pi/2 - \delta/2 \leq \frac{\pi}{2} - \frac{\sin \delta}{2}$ , as desired. We have shown that  $\operatorname{Ang}(\Lambda(S)) \leq \frac{\pi}{2} - \frac{\sin \delta}{2}$ , and Theorem 6(ii) follows.

**7f. Proof of Theorem 7(ii).**

We show that the projection of  $\sigma(u)$  on  $S$  is  $\leq 1 - 2\lambda$ . From this it follows that the maximal projection of  $\Lambda(S)$  (and hence of  $\Gamma(S)$  by Theorem 3') is  $\leq 1 - 2\lambda$ , proving Theorem 7(ii). If  $u \geq u_0$  then the projection of  $\sigma(u)$  on  $S$  is  $\leq |\sigma(u)| \leq 1 - 2\lambda$ , by Proposition 7.5, and our claim follows. Thus we may suppose that  $u < u_0$ . Since  $\operatorname{Ang}(S) = \theta = \pi/2 - \delta$ , we need to show that  $\operatorname{Re}(e^{-i\gamma}\sigma(u)) \leq 1 - 2\lambda$  for  $2\delta \leq |\gamma| \leq \pi$  (taking the projection along  $\zeta = e^{i\gamma}$ ).

Recall that by (7.7),  $|\sigma(u)|^2 \leq 1 - 2R_1(u) + R_1(u)P(u) + R_2(u) + C_2(u)$ . Using (7.4) together with the bound  $P(u/2) \leq 1 - P(u)$ , we deduce that, since  $R_1(u) \geq P(u) \cos^2 \theta$  by Lemma 7.2,

$$\begin{aligned} |\sigma(u)|^2 &\leq 1 - 2R_1(u)(1 - P(u)/2 - \min\{P(u)/2, \sqrt{P(u)(1 - P(u))}\}) \\ &\leq 1 - 2 \cos^2 \theta P(u)(1 - P(u)/2 - \min\{P(u)/2, \sqrt{P(u)(1 - P(u))}\}) \\ &\leq 1 - 4\lambda \end{aligned}$$

in the range  $1/6 \leq P(u) \leq 1$ , as may be verified using Maple. Thus in this range of  $P(u)$ , the projection of  $\sigma(u)$  on  $S$  is  $|\sigma(u)| \leq 1 - 2\lambda$ , as desired.

Now suppose  $P(u) \leq 1/6$ . From the above argument we know that  $|\sigma(u)|^2 \leq 1 - 2R_1(u)(1 - P(u)) \leq 1 - 5R_1(u)/3$ , so that  $|\sigma(u)| \leq 1 - 5R_1(u)/6 \leq 1 - 2\lambda$  if  $R_1(u) > 12\lambda/5 = (112/685)\cos^2\theta$ .

So we are left with the case  $P(u) \leq 1/6$ , and  $R_1(u) \leq \cos^2\theta/6 \leq 1/6$ . By (7.5),  $|\operatorname{Im} \sigma(u)| \leq \sqrt{R_1(u)P(u)} \leq (1/6)\cos\theta$ , and by (7.6),  $\operatorname{Re} \sigma(u) \geq 1 - R_1(u)(1 + P(u)/2) \geq 1 - (1/6)(13/12) = 59/72$ . Hence  $\tan(|\arg \sigma(u)|) = |\operatorname{Im} \sigma(u)|/\operatorname{Re} \sigma(u) < \cos\theta \leq \cot\theta = \tan\delta$ . Thus  $|\arg \sigma(u)| \leq \delta$ , and so if  $2\delta \leq |\gamma| \leq \pi$ ,  $|\arg(e^{-i\gamma}\sigma(u))| > \delta$ . So the projection of  $\sigma(u)$  on  $e^{i\gamma}$  is  $\leq \cos\delta = \sin\theta \leq 1 - (1/2)\cos^2\theta$ . This completes the proof of Theorem 7(ii).

## 8. GENERALIZED NOTIONS OF THE SPECTRUM: THE LOGARITHMIC SPECTRUM

We may generalize the notion of spectrum by considering the values

$$\left( \sum_{n \leq N} \kappa(n) \right)^{-1} \sum_{n \leq N} f(n)\kappa(n)$$

for  $f \in \mathcal{F}(S)$  as  $N \rightarrow \infty$ , where  $\kappa(n)$  is a given positive valued function (we considered the case  $\kappa = 1$  above). In this setting one quickly becomes curious about the weights  $\kappa(n) = 1/n^\sigma$  for a given real number  $\sigma \geq 0$ . If  $\sigma > 1$  then the sum converges absolutely and so we obtain the set of Euler products  $\zeta(\sigma)^{-1} \prod_p (1 - f(p)/p^\sigma)^{-1}$ . If  $\sigma < 1$  then the new spectrum is exactly the same as  $\Gamma(S)$ , since if  $f \in \mathcal{F}(S)$  is completely multiplicative then, for any given  $\sigma < 1$ , we have

$$(8.1) \quad \left( \sum_{n \leq x} \frac{1}{n^\sigma} \right)^{-1} \sum_{n \leq x} \frac{f(n)}{n^\sigma} = \frac{1}{x} \sum_{n \leq x} f(n) + o(1).$$

To see this note that if  $\sum_{p \leq x} (1 - \operatorname{Re} f(p))/p \rightarrow \infty$  then both sides of the equation are  $o(1)$  by Lemma 1' and partial summation. Thus we may assume that  $\sum_{p \leq x} |1 - f(p)|/p \asymp_S \sum_{p \leq x} (1 - \operatorname{Re} f(p))/p \ll 1$ . Let  $g(p^k) = f(p^k) - f(p^{k-1})$  for each prime power. By (4.1) we have  $\sum_{n \leq t} |g(n)| \ll (t/\log t) \exp(\sum_{p \leq t} |g(p)|/p) \ll t/\log t$  when  $t \leq x$ ; and so  $\sum_{d \leq x} |g(d)|/d^\sigma \ll x^{1-\sigma}/(1-\sigma) \log x$  by partial summation. Therefore, since  $\sum_{n \leq t} n^{-\sigma} = t^{1-\sigma}/(1-\sigma) + O(1)$ , we obtain

$$\begin{aligned} \sum_{n \leq x} \frac{f(n)}{n^\sigma} &= \sum_{d \leq x} \frac{g(d)}{d^\sigma} \sum_{n \leq x/d} \frac{1}{n^\sigma} = \sum_{d \leq x} \frac{g(d)}{d^\sigma} \left( \frac{1}{1-\sigma} \left( \frac{x}{d} \right)^{1-\sigma} + O(1) \right) \\ &= \frac{x^{1-\sigma}}{1-\sigma} \sum_{d \leq x} \frac{g(d)}{d} + O\left( \sum_{d \leq x} \frac{|g(d)|}{d^\sigma} \right) \\ &= \left( \sum_{n \leq x} \frac{1}{n^\sigma} \right) \left\{ \sum_{d \leq x} \frac{g(d)}{d} + O\left( \frac{1}{\log x} \right) \right\}. \end{aligned}$$

Comparing the formula at  $\sigma$  with the formula at  $\sigma = 0$  gives (8.1).

This leaves us with the case  $\sigma = 1$ , that is  $\kappa(n) = 1/n$ , which gives rise to the logarithmic spectrum  $\Gamma_0(S)$  mentioned in the introduction. We now proceed to a study of this spectrum, beginning with some general results on logarithmic means. Elsewhere we will apply these methods to obtain upper bounds on  $L(1, \chi)$ .<sup>6</sup>

One may also consider other other choices of  $\kappa(n)$ ; for example,  $\kappa(n) = d_k(n)$ , the  $k$ th divisor function. It would be interesting to determine this spectrum when  $S = \{\pm 1\}$ .

### 8a. Generalities on logarithmic means.

**Proposition 8.1.** *Let  $f$  be a multiplicative function with  $|f(n)| \leq 1$  for all  $n$ , and put  $g(n) = \sum_{d|n} f(d)$ . Then*

$$\frac{1}{\log x} \left| \sum_{n \leq x} \frac{f(n)}{n} \right| \leq 2e^{2\gamma} \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{|g(p)|}{p} + \frac{|g(p^2)|}{p^2} + \dots\right) + O\left(\frac{1}{\log x}\right).$$

*Proof.* Since

$$\sum_{n \leq x} g(n) = \sum_{n \leq x} \sum_{d|n} f(d) = \sum_{d \leq x} f(d) \left(\frac{x}{d} + O(1)\right) = x \sum_{d \leq x} \frac{f(d)}{d} + O(x),$$

we see that

$$\frac{1}{\log x} \left| \sum_{n \leq x} \frac{f(n)}{n} \right| \leq \frac{1}{x \log x} \sum_{n \leq x} |g(n)| + O\left(\frac{1}{\log x}\right).$$

Note that  $|g(n)|$  is a non-negative multiplicative function with  $|g(n)| \leq d(n)$  for all  $n$ . Hence by Theorem 2 of Halberstam and Richert [4] (see (4.1) above) we obtain

$$\begin{aligned} \frac{1}{x \log x} \sum_{n \leq x} |g(n)| &\leq \frac{2}{x \log^2 x} \sum_{n \leq x} \frac{|g(n)|}{n} + O\left(\frac{1}{\log x}\right) \\ &\leq \frac{2}{x \log^2 x} \prod_{p \leq x} \left(1 + \frac{|g(p)|}{p} + \frac{|g(p^2)|}{p^2} + \dots\right) + O\left(\frac{1}{\log x}\right). \end{aligned}$$

The result follows from Mertens' theorem.

Since  $2 - |1 + z| \leq (1 - \operatorname{Re} z)/2$  whenever  $|z| \leq 1$ , the right side of the equation in Proposition 8.1 is

$$\ll \exp\left(-\sum_{p \leq x} \frac{2 - |g(p)|}{p}\right) \leq \exp\left(-\frac{1}{2} \sum_{p \leq x} \frac{1 - \operatorname{Re} f(p)}{p}\right).$$

---

<sup>6</sup>In the spirit of P.J. Stephens [12] who showed that  $|L(1, \chi_d)| \leq \frac{1}{4}(2 - \frac{2}{\sqrt{e}} + o(1)) \log |d|$  where  $\chi_d$  is a quadratic character with conductor  $|d|$ . We establish similar results for higher order characters.

More precisely one obtains

$$\frac{1}{\log x} \left| \sum_{n \leq x} \frac{f(n)}{n} \right| \leq \frac{26e^{2\gamma}}{\pi^2} \exp \left( -\frac{1}{2} \sum_{p \leq x} \frac{1 - \operatorname{Re} f(p)}{p} \right),$$

a weak, but relatively easy and effective, analogue of Lemma 1' for logarithmic means. Moreover this has the advantage that  $f$  need not be restricted to a subset of  $\mathbb{U}$  since the case  $f(n) = n^{it}$  does not impede us here (since  $\sum_{n \leq x} n^{i\alpha-1} \ll 1$ ).

Next we derive analogues of Propositions 4.1, 4.4, and 4.5. As the above example indicates, the situation here is much simpler. For example, the analogue of Proposition 4.1 is the trivial estimate

$$\frac{1}{\log x} \sum_{n \leq x} \frac{f(n)}{n} - \frac{1}{\log(x/y)} \sum_{n \leq x/y} \frac{f(n)}{n} \ll \frac{\log 2y}{\log x},$$

which is valid for all functions  $f$  with  $|f(n)| \leq 1$ , and all  $1 \leq y \leq \sqrt{x}$ . Using this estimate (in place of Proposition 4.1) and arguing exactly as in the proof of Proposition 4.5 we arrive at the following Proposition (see also Lemma 5 of Hildebrand [11]).

**Proposition 8.2.** *Let  $f$  be any multiplicative function with  $|f(n)| \leq 1$ . Let  $g$  be the completely multiplicative function defined by  $g(p) = 1$  for  $p \leq y$  and  $g(p) = f(p)$  for  $p > y$ . Then*

$$\frac{1}{\log x} \sum_{n \leq x} \frac{f(n)}{n} = \Theta(f, y) \frac{1}{\log x} \sum_{n \leq x} \frac{g(n)}{n} + O\left(\frac{\log y}{\log x} \exp(s(f, y))\right),$$

where  $s(f, y) = \sum_{p \leq y} |1 - f(p)|/p$ . The remainder term above is  $\ll (\log y)^3 / \log x$ .

We omit the proof of Proposition 8.2 since it is almost identical to that of Proposition 4.5.

Observe that

$$\frac{1}{\log y^u} \sum_{n \leq y^u} \frac{f(n)}{n} = \frac{1}{u} \int_0^u \frac{1}{[y^t]} \sum_{n \leq y^t} f(n) dt + O\left(\frac{1}{u \log y}\right),$$

which implies that  $\Gamma_0(S)$  lies inside the convex hull of  $\Gamma(S)$ . From this equation, we deduce the following analogues of Proposition 1 and its converse.

**Proposition 8.3.** *Let  $f$  and  $\chi$  be as in Proposition 1. Then*

$$\frac{1}{\log y^u} \sum_{n \leq y^u} \frac{f(n)}{n} = \frac{1}{u} \int_0^u \sigma(t) dt + O\left(\frac{u}{\log y}\right).$$

**Proposition 8.3 (Converse).** *Let  $f$  and  $\chi$  be as in the converse of Proposition 1. Then for all  $1/\log y \leq t \leq u$*

$$\frac{1}{t} \int_0^t \sigma(v)dv = \frac{1}{\log y^t} \sum_{n \leq y^t} \frac{f(n)}{n} + O(u^\epsilon - 1) + O\left(\frac{u}{\log y}\right).$$

Let  $S$  be a closed subset of  $\mathbb{U}$  with  $1 \in S$ . We define  $\Lambda_0(S)$  to be the set of values  $\frac{1}{u} \int_0^u \sigma(t)dt$  obtained as follows: Let  $\chi$  be any element of  $K(S)$ , and let  $\sigma$  denote the corresponding solution to (1.5). Then  $\Lambda_0(S)$  is the set of all values  $\frac{1}{u} \int_0^u \sigma(t)dt = \frac{1}{u}(1 * \sigma)(u)$  for all  $u > 0$ , and all such choices of  $\chi$ . Note that  $\Lambda_0(S)$  is in the convex hull of  $\Lambda(S)$ .

Combining Proposition 8.2, with Proposition 8.3 and its Converse, we obtain the following Structure Theorem for the logarithmic spectrum.

**Theorem 8.4.** *Let  $S$  be a closed subset of  $\mathbb{U}$  with  $1 \in S$ . Then  $\Gamma_0(S) = \Gamma_\Theta(S) \times \Lambda_0(S)$ . Further  $\Lambda_0(S) = \Lambda_0(S) \times \mathcal{E}(S)$ , and so*

$$\Lambda_0(S) \subset \Gamma_0(S) \subset \Lambda_0(S) \times [0, 1].$$

Theorem 8.4 is proved exactly in the same way as Theorems 3 and 3'; so we omit its proof. We end this subsection by making the following useful observation:<sup>7</sup>

$$\begin{aligned} u(1 * \sigma)(u) &= \int_0^u (u - t)\sigma(t)dt + \int_0^u t\sigma(t)dt = (1 * (1 * \sigma))(u) + (1 * (t\sigma(t)))(u) \\ (8.2) \quad &= (1 * 1 * \sigma)(u) + (1 * \chi * \sigma)(u) = ((1 * \sigma) * (1 + \chi))(u). \end{aligned}$$

**8b. Bounding  $\Gamma_0(S)$ : Proof of Theorem 8.**

If  $S = \{1\}$  then  $\Gamma_0(S) = \mathcal{R} = \{1\}$ , and there is nothing to prove. Suppose that  $S$  contains an element  $\alpha \neq 1$ . Then  $(\frac{1+\alpha}{2})^n \in \mathcal{R}$  for all  $n \geq 1$ . As  $n \rightarrow \infty$  this sequence of points converges to 0, and since  $\mathcal{R}$  is closed, we deduce that  $0 \in \mathcal{R}$ . By convexity it follows that  $\mathcal{R} = \mathcal{R} \times [0, 1]$ . Hence, by Theorem 8.4, we need only show that  $\Lambda_0(S) \subset \mathcal{R}$  in order to establish Theorem 8.

We define, for any complex number  $z$ , its  $\mathcal{R}$ -norm  $\|z\|_{\mathcal{R}} := \min_{r \in \mathcal{R}} |z - r|$ ; that is  $\|z\|_{\mathcal{R}}$  is the shortest distance from  $z$  to  $\mathcal{R}$ . We first make a couple of general observations about this norm:

Let  $X$  be a measurable subset of the real line, and suppose  $f$  is a non-negative measurable function with  $\int_X f(x)dx = 1$ . Then for any measurable function  $g$ ,<sup>8</sup>

$$(8.3) \quad \left\| \int_X f(x)g(x) dx \right\|_{\mathcal{R}} \leq \int_X f(x)\|g(x)\|_{\mathcal{R}} dx.$$

<sup>7</sup>More generally,  $u(\sigma_1 * \sigma_2)(u) = ((\chi_1 + \chi_2) * (\sigma_1 * \sigma_2))(u)$ .

<sup>8</sup>An analogous convexity result holds for sums: If  $a_i \geq 0$  with  $\sum a_i = 1$ , then  $\|\sum a_i z_i\|_{\mathcal{R}} \leq \sum a_i \|z_i\|_{\mathcal{R}}$ .

To see (8.3), suppose  $r(x)$  is a point in  $\mathcal{R}$  closest to  $g(x)$ . Then  $\int_X f(x)r(x)dx$  is a convex combination of the points  $r(x)$ , and so is an element of  $\mathcal{R}$ . Therefore

$$\begin{aligned} \left\| \int_X f(x)g(x)dx \right\|_{\mathcal{R}} &\leq \left| \int_X f(x)g(x)dx - \int_X f(x)r(x)dx \right| \leq \int_X f(x)|g(x) - r(x)|dx \\ &= \int_X f(x)\|g(x)\|_{\mathcal{R}}dx, \end{aligned}$$

which proves (8.3).

Let  $s$  be any point in the convex hull of  $S$  and let  $r$  be a point in  $\mathcal{R}$  closest to given  $z$ . By the definition of  $\mathcal{R}$ , we know that  $r\frac{1+s}{2}$  is also a point in  $\mathcal{R}$ , and so

$$(8.4) \quad \left\| z\frac{1+s}{2} \right\|_{\mathcal{R}} \leq \left| z\frac{1+s}{2} - r\frac{1+s}{2} \right| = \left| \frac{1+s}{2} \right| \|z\|_{\mathcal{R}} \leq \|z\|_{\mathcal{R}}.$$

Suppose  $\chi \in K(S)$  is given and let  $\sigma$  be the corresponding solution to (1.5). We shall show that  $\frac{1}{u}(1 * \sigma)(u) \in \mathcal{R}$  for all  $u$ . This proves that  $\Lambda_0(S)$  (and so  $\Gamma_0(S)$ ) is contained in  $\mathcal{R}$ . Define  $\alpha(u) = -u\left\|\frac{1}{u}(1 * \sigma)(u)\right\|_{\mathcal{R}}$ . Plainly  $\alpha(u) = 0$  for  $u \leq 1$ , and we shall show below that it is always non-negative so that  $\alpha(u) = 0$  for all  $u$ , which proves that  $\frac{1}{u}(1 * \sigma)(u) \in \mathcal{R}$ .

By (8.2) we see that

$$\left\| \frac{1}{u}(1 * \sigma)(u) \right\|_{\mathcal{R}} = \left\| \frac{1}{u^2} \int_0^u 2v \left( \frac{1}{v}(1 * \sigma)(v) \right) \frac{1 + \chi(u-v)}{2} dv \right\|_{\mathcal{R}}.$$

Applying (8.3) with  $X = [0, u]$ , and  $f(x) = 2v/u^2$ , we deduce that the above is

$$\leq \frac{1}{u^2} \int_0^u 2v \left\| \left( \frac{1}{v}(1 * \sigma)(v) \right) \frac{1 + \chi(u-v)}{2} \right\|_{\mathcal{R}} dv,$$

which by (8.4) is

$$\leq \frac{1}{u^2} \int_0^u 2v \left\| \frac{1}{v}(1 * \sigma)(v) \right\|_{\mathcal{R}} dv.$$

It follows that  $u\alpha(u) \geq (2 * \alpha)(u)$ , and so by Lemma 3.1,  $\alpha(u)$  is always non-negative, as desired. This completes the proof of Theorem 8.

### 8c. Proof of Corollary 4.

If  $S = [-1, 1]$  then  $\frac{1+s}{2} \in [0, 1]$  for all  $s \in S$ , and so it follows that  $\mathcal{R} = [0, 1]$  here. Hence  $\Gamma_0([-1, 1]) \subset [0, 1]$ . Since  $\Gamma_0([-1, 1]) \supset \mathcal{E}([-1, 1]) = [0, 1]$ , it follows that  $\Gamma_0([-1, 1]) = [0, 1]$ , proving part (i).

Part (ii) is proved in the same way as Corollary 3(ii): Take  $\chi(t) = 1$  for  $t \leq 1$  and  $\chi(t) = \alpha$  for  $t > 1$ . Then by Theorem 3.3,  $\sigma(t) = 1 - (1 - \alpha) \log t$  for  $1 \leq t \leq 2$ . Hence, for  $1 \leq u \leq 2$ ,

$$(8.5) \quad \frac{1}{u} \int_0^u \sigma(t)dt = 1 - (1 - \alpha) \frac{1}{u} \int_1^u \log t dt = 1 - (1 - \alpha) \left( \log u - 1 + \frac{1}{u} \right)$$

belongs to  $\Lambda_0(S)$ , as desired. If  $0 < \text{Ang}(S) < \pi/2$ , then take  $1 < u \leq 2$ , and  $\alpha \in S$  such that  $\text{Ang}(\alpha) = \text{Ang}(S)$ . The argument given in §6b (proof of Corollary 3(ii)) shows that such elements of  $\Lambda_0(S)$  are not in  $\Gamma_\Theta(S)$ .

Notice that taking  $\alpha \in S$  such that  $\text{Ang}(\alpha) = \text{Ang}(S)$  in the construction (8.5), we obtain that  $\text{Ang}(S) \leq \text{Ang}(\Gamma_0(S))$ .<sup>9</sup> We now show that  $\text{Ang}(\mathcal{R}) \leq \text{Ang}(S)$ , so that by Theorem 8, we have  $\text{Ang}(\Gamma_0(S)) = \text{Ang}(S)$ . Suppose  $\text{Ang}(S) = \frac{\pi}{2} - \delta$ , so that  $S$  is contained in the convex hull of  $\{1\} \cup \{e^{i\theta} : 2\delta \leq |\theta| \leq \pi\}$ . Each product  $\prod_{j=1}^m \frac{1+s_j}{2}$ , where  $s_j$  is in the convex hull of  $S$ , is easily expressed as a convex combination of elements of the form  $\prod_{j=1}^n \frac{1+e^{i\theta_j}}{2}$  where  $2\delta \leq |\theta_j| \leq \pi$ . Hence  $\mathcal{R}$  is contained in the convex hull of 1 and points of the form  $\prod_{j=1}^n \frac{1+e^{i\theta_j}}{2} = \prod_{j=1}^n \cos(\theta_j/2)e^{i\theta_j/2}$  where  $2\delta \leq |\theta_j| \leq \pi$ . Such a product has magnitude  $\leq (\cos \delta)^n \leq \cos \delta$  if  $n \geq 1$ . Thus  $\mathcal{R}$  is in the convex hull of  $\{1\} \cup \{|z| \leq \cos \delta\}$ . If  $|z| \leq \cos \delta$  then  $\text{Ang}(z) \leq \arcsin(|z|) \leq \frac{\pi}{2} - \delta$ , and so it follows that  $\text{Ang}(\mathcal{R}) \leq \frac{\pi}{2} - \delta$ . This proves (iii).

To prove (iv), we first observe that  $f(x) := (\cos x)^{\frac{1}{x}}$  is decreasing in  $(0, \frac{\pi}{2}]$ . Differentiating  $f$  logarithmically, we need to show that  $-(\log \cos x)/x^2 - \tan x/x \leq 0$ , or equivalently, that  $g(x) := x \tan x + \log \cos x \geq 0$ . Now  $g'(x) = x \sec^2 x$  is positive in  $(0, \frac{\pi}{2}]$ , and so  $g(x) \geq g(0) = 0$ , as desired. It follows that if  $\delta \leq \theta \leq \frac{\pi}{2}$  then  $\cos \theta \leq (\cos \delta)^{\frac{\theta}{\delta}}$ .

From the proof of (iii), we know that  $\mathcal{R}$  is contained in the convex hull of 1 and products of the form  $\prod_{j=1}^n \cos(\theta_j/2)e^{i\theta_j/2}$  where each  $\theta_j \in [2\delta, \pi]$ . If such a product has argument  $\nu$ , then we must have  $\sum_{j=1}^n \theta_j \geq 2\nu$ . By the previous paragraph, the magnitude of such a product is  $\leq \prod_{j=1}^n (\cos \delta)^{\frac{\theta_j}{2\delta}} \leq (\cos \delta)^{\frac{\nu}{\delta}}$ . Thus  $\mathcal{R}$  is contained in the set  $\{z : |z| \leq (\cos \delta)^{\frac{|\arg z|}{\delta}}\}$ , which proves (iv).

## 9. QUADRATIC RESIDUES AND NONRESIDUES REVISITED: PROOF OF THEOREM 9

Throughout this section  $D$  denotes a fundamental discriminant.

**Proposition 9.1.** *Let  $B$  be fixed, and  $X$  be large, and suppose  $1 \leq z \leq \frac{1}{4}(\log X)$ . Let  $f(n)$  be a completely multiplicative function satisfying  $f(p) = \pm 1$  for  $p \leq z$ , and  $f(p) = 0$  for  $p > z$ . Put  $P = 4 \prod_{p \leq z} p$  and let  $a \pmod{P}$  be an arithmetic progression (with  $a \equiv 1$ , or  $5 \pmod{8}$ ) such that  $\left(\frac{a}{p}\right) = f(p)$  for each  $p \leq z$ . With  $\mathcal{N}(X; a, P)$  denoting the number of fundamental discriminants  $0 < D \leq X$  with  $D \equiv a \pmod{P}$ , we have*

$$\frac{1}{\mathcal{N}(X; a, P)} \sum_{\substack{0 < D \leq X \\ D \equiv a \pmod{P}}} \sum_{n \leq (\log X)^B} \left(\frac{D}{n}\right) = \sum_{n \leq (\log X)^B} f(n) + O\left(\frac{(\log X)^B}{z}\right).$$

*Proof.* We write  $n = rs$  where each prime dividing  $r$  is  $\leq z$ , and each prime dividing  $s$  is

<sup>9</sup>Alternately, this follows because  $\text{Ang}(\Gamma_0(S)) \geq \text{Ang}(\mathcal{E}(S)) = \text{Ang}(S)$ .

$> z$ . Thus  $\left(\frac{D}{n}\right) = f(r)\left(\frac{D}{s}\right)$ , and so

$$(9.1) \quad \sum_{\substack{0 < D \leq X \\ D \equiv a \pmod{P}}} \sum_{n \leq (\log X)^B} \left(\frac{D}{n}\right) = \sum_{\substack{r \leq (\log X)^B \\ p|r \implies p \leq z}} f(r) \sum_{\substack{s \leq (\log X)^B / r \\ p|s \implies p > z}} \sum_{\substack{0 < D \leq X \\ D \equiv a \pmod{P}}} \left(\frac{D}{s}\right).$$

The  $s = 1$  term in (9.1) contributes

$$(9.2) \quad \sum_{\substack{r \leq (\log X)^B \\ p|r \implies p \leq z}} f(r) \mathcal{N}(X; a, P) = \mathcal{N}(X; a, P) \sum_{n \leq (\log X)^B} f(n).$$

The terms  $s > 1$  with  $s = \square$  ( $\square$  denotes the square of an integer) contribute

$$(9.3) \quad \begin{aligned} &\ll \mathcal{N}(X; a, P) \sum_{\substack{r \leq (\log X)^B \\ p|r \implies p \leq z}} \sum_{\substack{1 < s \leq (\log X)^B / r \\ s \in \mathbb{Z}^2 \\ p|s \implies p > z}} 1 \\ &\ll \mathcal{N}(X; a, P) \sum_{r \leq (\log X)^B / z^2} \sqrt{\frac{(\log X)^B}{r}} \ll \mathcal{N}(X; a, P) \frac{(\log X)^B}{z}. \end{aligned}$$

Finally we consider the contribution of the terms  $s \neq \square$  to (9.1). For such an  $s$ ,  $\left(\frac{\cdot}{s}\right)$  is a non-principal character of conductor  $\leq s$ , and so we may expect substantial cancellation in the sum over  $D$  in (9.1). Indeed, we have using  $\mu(m)^2 = \sum_{l^2|m} \mu(l)$

$$\sum_{\substack{0 < D \leq X \\ D \equiv a \pmod{P}}} \left(\frac{D}{s}\right) = \sum_{\substack{m \leq X \\ m \equiv a \pmod{P}}} \mu(m)^2 \left(\frac{m}{s}\right) = \sum_{l \leq \sqrt{X}} \mu(l) \sum_{\substack{m \leq X \\ m \equiv a \pmod{P} \\ l^2|m}} \left(\frac{m}{s}\right).$$

By the Pólya-Vinogradov inequality the inner sum over  $m$  above is  $\ll \sqrt{s} \log s$ . Hence the sum over  $D$  above is  $\ll \sqrt{Xs} \log s$ . This demonstrates that the  $s \neq \square$  terms in (9.1) contribute an amount

$$\ll \sum_{r \leq (\log X)^B} \sum_{s \leq (\log X)^B / r} \sqrt{Xs} \log s \ll X^{\frac{1}{2} + \epsilon}.$$

Combining this with the estimates (9.2), and (9.3), we see by (9.1) that

$$\frac{1}{\mathcal{N}(X; a, P)} \sum_{\substack{0 < D \leq X \\ D \equiv a \pmod{P}}} \sum_{n \leq (\log X)^B} \left(\frac{D}{n}\right) = \sum_{n \leq (\log X)^B} f(n) + O\left(\frac{(\log X)^B}{z} + \frac{X^{\frac{1}{2} + \epsilon}}{\mathcal{N}(X; a, P)}\right).$$



Since

$$\mathcal{N}(X; a, P) \sim \frac{X}{P} \frac{6}{\pi^2} \prod_{p|z} \left(1 - \frac{1}{p^2}\right)^{-1},$$

the second error term above is  $\ll P/X^{\frac{1}{2}-\epsilon}$ . Since  $z \leq \frac{1}{4} \log X$ ,  $P \ll X^{\frac{1}{4}+\epsilon}$  by the prime number theorem, and so the second error term above is  $\ll X^{-\frac{1}{4}+\epsilon}$ , which is subsumed by the error term of the Proposition.

Armed with Proposition 9.1, we now show that  $\beta(B) \leq \gamma(B)$  for all  $B$ . Let  $X$  be large, and choose  $z = \frac{1}{4} \log X$ . By Proposition 9.1, we know that there is a fundamental discriminant  $D$  with  $X/\log X \ll D \leq X$ , such that

$$(9.4) \quad \frac{1}{(\log D)^B} \sum_{n \leq (\log D)^B} \left(\frac{D}{n}\right) \leq \frac{1}{(\log D)^B} \sum_{n \leq (\log D)^B} f(n) + o(1),$$

where  $f$  is any completely multiplicative function as in Proposition 9.1. Suppose we are given  $\chi \in \mathcal{C}(u)$ . Put  $y = z^{\frac{1}{u}}$ , and choose  $f \in \mathcal{F}(\{0, \pm 1\})$  as in the converse of Proposition 1. Thus choose  $f$  so that  $f(p) = 1$  for  $p \leq y$ ,  $f(p) = 0$  for  $p > y^u = z$ , and such that for almost all  $0 \leq t \leq u$ ,

$$\left| \chi(t) - \frac{1}{\vartheta(y^t)} \sum_{p \leq y^t} f(p) \log p \right| \leq \epsilon.$$

From Proposition 1 (Converse) it follows that the right hand side of (9.4) is  $\sigma(Bu) + O(u^{B\epsilon} - 1) + O(u/\log y) + o(1)$ . Letting  $\epsilon \rightarrow 0$ , and  $X \rightarrow \infty$  (so that  $y \rightarrow \infty$ ), it follows that  $\beta(B) \leq \sigma(Bu)$ . Now varying  $u$ , and  $\chi \in \mathcal{C}(u)$ , we deduce that  $\beta(B) \leq \gamma(B)$ .

To complete the proof of Theorem 9, it remains now to show that  $-\rho(B) \leq \gamma(B) < 0$ . We first show that

$$(9.5) \quad |\sigma(Bu)| \leq \rho(B) \quad \text{for all } B \text{ and all } \chi \in \mathcal{C}(u).$$

To prove (9.5), suppose  $\chi \in \mathcal{C}(u)$  is given, and put  $a(B) = \rho(B) - |\sigma(Bu)|$ . Since  $\rho(B) = 1$  for  $B \leq 1$ , it follows that  $a(B) \geq 0$  for  $B \leq 1$ . Define  $b(t) = 1$  for  $t \leq 1$ , and  $b(t) = 0$  for  $t > 1$ . From the definition of the Dickman function  $B\rho(B) = (b * \rho)(B)$ , and so we have

$$\begin{aligned} B a(B) &= B\rho(B) - |B\sigma(Bu)| = (b * \rho)(B) - \frac{1}{u} \left| \int_{(B-1)u}^{Bu} \sigma(t) \chi(Bu - t) dt \right| \\ &\geq (b * \rho)(B) - \int_{B-1}^B |\sigma(ut)| dt = (b * a)(B). \end{aligned}$$

From Lemma 3.1 it follows that  $a(B) \geq 0$  always, which establishes (9.5).

From (9.5) we see that  $-\rho(B) \leq \gamma(B)$ , and it remains now to show that  $\gamma(B) < 0$ . We prove this by considering the following example: Put  $\chi_-(t) = 1$  for  $t \leq 1$ ,  $\chi_-(t) = -1$  for  $1 \leq t \leq 2$ , and  $\chi_-(t) = 0$  for  $t > 2$ , so that  $\chi_- \in \mathcal{C}(u)$  for all  $u \geq 2$ . Hence, if  $\sigma_-$  denotes the solution to  $w\sigma_-(w) = \sigma_- * \chi_-$  then  $\gamma(B) \leq \min_{u \geq 2} \sigma_-(Bu) = \min_{w \geq 2B} \sigma_-(w)$ . We

will now show that  $\sigma_-(w)$  changes sign infinitely often; hence there are arbitrarily large  $w$  with  $\sigma_-(w) < 0$  which shows that  $\gamma(B) < 0$  for all  $B$ .

Suppose  $\sigma_-(w)$  maintains sign from some point on: precisely, suppose  $|\sigma_-(w_0)| > 0$  and that  $\sigma_-(w)$  has the same sign as  $\sigma_-(w_0)$  for all  $w \geq w_0$ . Define  $F(w) = \int_{w-1}^w \sigma_-(t) dt$ . Note that

$$(9.6) \quad w\sigma_-(w) = (\sigma_- * \rho_-)(w) = \int_{w-1}^w \sigma_-(t) dt - \int_{w-2}^{w-1} \sigma_-(t) dt = F(w) - F(w-1).$$

Since  $F(w)$  has the same sign as  $\sigma_-(w_0)$  for all  $w \geq w_0 + 1$ , we deduce from (9.6) that  $|F(w+1)| = |F(w) + (w+1)\sigma_-(w+1)| = |F(w)| + |(w+1)\sigma_-(w+1)| \geq |F(w)|$  for all  $w \geq w_0 + 1$ . Hence

$$\liminf_{n \rightarrow \infty} |F(w_0 + n)| \geq |F(w_0 + 1)| = \int_{w_0}^{w_0+1} |\sigma_-(t)| dt > 0.$$

However, from (9.5) we see that

$$|F(w)| = \int_{w-1}^w |\sigma_-(t)| dt \leq \int_{w-1}^w \rho(t/2) dt \leq \rho((w-1)/2)$$

and so  $|F(w)| \rightarrow 0$  as  $w \rightarrow \infty$ . This contradiction proves that  $\sigma_-$  must change sign infinitely often, and completes our proof of Theorem 9.

#### ACKNOWLEDGEMENTS

We'd like to thank Richard Hall and Hugh Montgomery for exciting our interest in this problem, Mark Watkins for the question that inspired Theorem 9 and Malcolm Adams, Kevin Clancey, Steven Finch, Jim Haglund, Seva Lev and Carl Pomerance for several useful remarks. We are also grateful to the referee for some useful suggestions on style.

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