Multiplicative number theory

Andrew Granville
Université de Montréal

K. Soundararajan
Stanford University

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THE SMALL SIEVE

1.1 List of sieving results used

FLS

Lemma 1.1 ***(The Fundamental Lemma of Sieve Theory)*** *If $(am, q) = 1$ and all of the prime factors of m are $\leq z$ then*

$$\sum_{\substack{x<n\leq x+qy \\ (n,m)=1 \\ n\equiv a \pmod q}} 1 = \left\{1 + O(u^{-u-2})\right\} \frac{\phi(m)}{m} y + O(\sqrt{y}),$$

where $y = z^u$.

FLS1

Corollary 1.2 *If $(am, q) = 1$ and all of the prime factors of m are $\leq x^{1/u}$ then*

$$\sum_{\substack{n\leq x \\ (n,m)=1 \\ n\equiv a \pmod q}} \log n = \left\{1 + O(u^{-u-2})\right\} \frac{\phi(m)}{m} \frac{x}{q} (\log x - 1) + O(\sqrt{x} \log x).$$

The proof of this and the subsequent corollaries are left as exercises. One approach here is to begin by writing $\log n = \int_1^n \frac{dt}{t}$ and then swap the order of the summation and the integral.

FLS2

Corollary 1.3 *If χ is a character mod q and all of the prime factors of m are $\leq z = y^{1/u}$ and coprime with q, then*

$$\sum_{\substack{x<n\leq x+qy \\ (n,m)=1}} \chi(n) \ll \frac{1}{u^u} \frac{\phi(mq)}{mq} qy + q\sqrt{y}.$$

Let $p(n), P(n)$ be the smallest and largest prime factors of n, respectively.

FLS3

Corollary 1.4 *If $(a, q) = 1$ and z is chosen so that $q = z^{O(1)}$ and $z \leq y$ then*

$$\left| \sum_{\substack{x<n\leq x+qy \\ n\equiv a \pmod q \\ p(n)>z}} 1 \right| \ll \frac{q}{\phi(q)} \frac{y}{\log z}.$$

1.2 Shiu's Theorem

Suppose that $0 \le f(n) \le 1$. Corollary cor2.3 ?? states that the mean value of f up to x is $\ll \mathcal{P}(f;x)$. Shiu's Theorem states that an analogous result is true for the mean value of f in short intervals, in arithmetic progressions, and even in both:

Shiu **Theorem 1.5** *If* $(a,q)=1$ *then*

$$\left| \frac{1}{y} \sum_{\substack{x<n\le x+qy \\ n\equiv a \pmod q}} f(n) \right| \ll \prod_{\substack{p\le y \\ p\nmid q}} \left(1-\frac{1}{p}\right)\left(1+\frac{|f(p)|}{p}\right).$$

This is $\asymp \mathcal{P}(|f|\chi_0;y) \asymp \exp\left(-\sum_{p\le y,\ p\nmid q} \frac{1-f(p)}{p}\right)$.

Proof Let $g(p)=|f(p)|$ where $p\le y$, and $g(p^k)=1$ otherwise. Then $|\sum_n f(n)| \le \sum_n |f(n)| \le \sum_n g(n)$, and proving the result for g implies it for f.

Write $n = p_1^{k_1} p_2^{k_2}\ldots$ with $p_1 < p_2 < \ldots$, and let $d = p_1^{k_1}p_2^{k_2}\ldots p_r^{k_r}$ where $d \le y^{1/2} < dp_{r+1}^{k_{r+1}}$. Therefore $n = dm$ with $p(m) > z_d := \max\{P(d), y^{1/2}/d\}$, $(d,q)=1$ and $g(n)\le g(d)$. Now, if we fix d then m is in an interval $(x/d, x/d + qy/d]$ of an arithmetic progression $a/d \pmod q$ containing $y/d + O(1)$ integers. Note that $z_d \le \max\{d, y^{1/2}/d\} \le y^{1/2} \le y/d$, and so we may apply Corollary FLS3 1.4 to show that there are $\ll qy/d\phi(q)\log(P(d)+y^{1/2}/d)$ such m. This implies that

$$\sum_{\substack{x<n\le x+qy \\ n\equiv a \pmod q}} g(n) \le \frac{qy}{\phi(q)} \sum_{\substack{d\le y^{1/2} \\ (d,q)=1}} \frac{g(d)}{d\log(P(d)+y^{1/2}/d)}.$$

For those terms with $d \le y^{1/2-\epsilon}$ or $P(d) > y^{\epsilon}$, we have $\log(P(d)+y^{1/2}/d) \ge \epsilon \log y$, and so they contribute

$$\ll \frac{qy}{\phi(q)} \prod_{p\le y}\left(1-\frac{1}{p}\right) \sum_{\substack{d\le y^{1/2} \\ (d,q)=1}} \frac{g(d)}{d} \ll y \prod_{\substack{p\le y \\ p\nmid q}}\left(1-\frac{1}{p}\right)\left(1+\frac{g(p)}{p}\right),$$

the upper bound claimed above. We are left with the $d > y^{1/2-\epsilon}$ for which $P(d) \asymp 2^r$ for some $r,\ 1\le r\le k = [\epsilon \log y]$. Hence we obtain an upper bound:

$$\frac{qy}{\phi(q)} \sum_{r=1}^{k} \frac{1}{r} \sum_{\substack{d>y^{1/2-\epsilon} \\ (d,q)=1 \\ P(d)\asymp 2^r}} \frac{g(d)}{d} \ll \frac{qy}{\phi(q)} \left(\frac{1}{k} \sum_{\substack{d>y^{1/2-\epsilon} \\ (d,q)=1 \\ P(d)\le 2^k}} \frac{g(d)}{d} + \sum_{r=1}^{k} \frac{1}{r^2} \sum_{\substack{d>y^{1/2-\epsilon} \\ (d,q)=1 \\ P(d)\le 2^r}} \frac{g(d)}{d} \right).$$

For the first term we proceed as above. For the remaining terms we use Corollary 3.4.2, with $u_r := (1/2-\epsilon)\log y/(r\log 2)$, to obtain

$$\ll \frac{qy}{\phi(q)} \sum_{r=1}^{k} \frac{1}{r^2} \prod_{\substack{p \le 2^r \\ p \nmid q}} \left(1+\frac{g(p)}{p}\right) \frac{1}{u_r^{u_r+1}} \ll y \sum_{r=1}^{k} \frac{1}{r u_r^{u_r}} \prod_{\substack{p\le y \\ p\nmid q}} \left(1-\frac{1}{p}\right)\left(1+\frac{g(p)}{p}\right).$$

Finally note that u_r is decreasing, so that $\sum_{R/2<r\le R} 1/(r u_r^{u_r}) \ll 1/u_R^{u_R}$; moreover $u_{2R} = u_R/2$ and so $\sum_{1\le r\le k} 1/(r u_r^{u_r}) \ll 1/u_k^{u_k} \ll 1$, and the result follows. □

1.3 Consequences

Define

$$\rho_q(f) := \prod_{\substack{p\le q \\ p\nmid q}} \left(1-\frac{1}{p}\right)\left(1+\frac{|f(p)|}{p}\right) \text{ and } \rho_q'(f) = \frac{\phi(q)}{q}\rho_q(f).$$

(Note that $\rho_q(f)$ is an upper bound in Theorem Shiu 1.5 provided $y \ge q$.) We also define

$$\log_S(n) := \sum_{\substack{d\in S \\ d|n}} \Lambda(d),$$

where S might be an interval $[a, b]$, and we might write "$\le Q$" in place of "'$[2, Q]$", or "$\ge R$" in place of "'$[R, \infty)$". Note that $\log n = \log_{[2,n]} n$.

Small.1 **Lemma 1.6** *Suppose that $x \ge Q^{2+\epsilon}$ and $Q \ge q$. Then, for any character χ* (mod q),

$$\left| \sum_{\substack{n\in\mathcal{N} \\ n\equiv a \pmod q}} f(n)\overline{\chi}(n)\mathcal{L}(n) \right| \ll \rho_q(f)\frac{x}{q} = \rho_q'(f)\frac{x}{\phi(q)},$$

where $\mathcal{L}(n) = 1, \log(x/n)$, $\frac{\log_{\le Q} n}{\log Q}$ or $\frac{\log_{\ge x/Q} n}{\log Q}$, and $\mathcal{N} = \{n : Y < n \le Y + x\}$ for $Y = 0$ in the second and fourth cases, and for any Y in the other two cases.

Proof The first estimate follows from Shiu's Theorem for $x \ge q^{1+\epsilon}$. One can deduce the second since $\sum_{n\le x} a_n \log(x/n) = \int_{1\le T\le x} \frac{1}{T} \sum_{n\le T} a_n \, dT$ for any a_n.

If d is a power of the prime p then let $f_d(n)$ denote $f(n/p^a)$ where $p^a \| n$, so that if $n = dm$ then $|f(n)| \le |f_d(m)|$. Therefore if $x > Qq^{1+\epsilon}$ then, for the third estimate, times $\log Q$, we have, again using Shiu's Theorem,

$$\le \sum_{\substack{Y<md\le Y+x \\ md\equiv a \pmod q \\ d\le Q}} |f(md)|\Lambda(d) \le \sum_{\substack{d\le Q \\ (d,q)=1}} \Lambda(d) \sum_{\substack{Y/d<m\le (Y+x)/d \\ m\equiv a/d \pmod q}} |f_d(m)|$$

$$\ll \sum_{\substack{d\le Q \\ (d,q)=1}} \frac{\Lambda(d)}{d} \rho_q(f_d)\frac{x}{q} \ll \rho_q(f)\frac{x}{q}\log Q.$$

In the final case, writing $n = mp$ where p is a prime $> x/Q$ (and note that $p^2 \nmid n$ as $p > x/Q > \sqrt{x}$), we have

$$\leq \sum_{\substack{m \leq Q \\ (m,q)=1}} |f(m)| \sum_{\substack{x/Q < p \leq x/m \\ p \equiv a/m \pmod q}} \log p \ll \sum_{\substack{m \leq Q \\ (m,q)=1}} |f(m)| \frac{x/m}{\phi(q)} \ll \rho_q(f) \frac{x}{q} \log Q.$$

by the Brun-Titchmarsh theorem, and then applying partial summation to Shiu's Theorem. □

By (SumSqs ??) we immediately deduce

Small.2 **Corollary 1.7** *With the hypotheses of Lemma Small.1 1.6 we have*

$$\sum_{\chi \pmod q} \left| \sum_{n \leq x} f(n) \overline{\chi}(n) \mathcal{L}(n) \right|^2 \ll (\rho_q'(f)x)^2.$$

Small.3 **Lemma 1.8** *If $\Delta > q^{1+\epsilon}$ then for any $D \geq 0$ we have*

$$\sum_{\chi \pmod q} \left| \sum_{D \leq d \leq D+\Delta} f(d) \overline{\chi}(d) \Lambda(d) \right|^2 \ll \Delta^2.$$

Proof We expand the left side using (SumSqs ??) to obtain

$$\phi(q) \sum_{(b,q)=1} \left| \sum_{\substack{d \equiv b \pmod q \\ D \leq d \leq D+\Delta}} f(d) \Lambda(d) \right|^2 \leq \phi(q) \sum_{(b,q)=1} \left| \sum_{\substack{d \equiv b \pmod q \\ D \leq d \leq D+\Delta}} \Lambda(d) \right|^2 \ll \Delta^2,$$

by the Brun-Titchmarsh theorem. □

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THE PRETENTIOUS LARGE SIEVE

2.1 Mean values of multiplicative functions, on average

Define

$$S_\chi(x) := \sum_{n \le x} f(n)\overline{\chi}(n),$$

and order the characters $\chi_1, \chi_2, \ldots \pmod q$ so that the $|S_{\chi_j}(x)|$ are in descending order. Our main result is an averaged version of (HalExplic2 ??) for f twisted by all the characters $\chi \pmod q$, but with a better error term:

PLSk **Corollary 2.1** *Suppose that $x \ge Q^{2+\epsilon}$ and $Q \ge q^{2+\epsilon}\log x$. Then*

$$\sum_{\substack{\chi \pmod q \\ \chi \ne \chi_1, \chi_2, \ldots, \chi_{k-1}}} \left|\frac{1}{x} S_\chi(x)\right|^2 \ll \left(e^{O(\sqrt{k})} \rho_q'(f) \left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}} \log\left(\frac{\log x}{\log Q}\right)\right)^2,$$

where the implicit constants are independent of f. If $k = 1$, f is real and ψ_1 is not, then we can replace the exponent 0 with $1 - \frac{1}{\sqrt{2}}$.

Let $\mathcal{C}_q$ be any subset of the set of characters $\pmod q$, and define

$$L = L(\mathcal{C}_q) := \frac{1}{\log x} \max_{\chi \in \mathcal{C}_q} \max_{|t| \le \log^2 x} |F_\chi(1+it)|,$$

where

$$F_\chi(s) := \prod_{p \le x} \left(1 + \frac{f(p)\overline{\chi}(p)}{p^s} + \frac{f(p^2)\overline{\chi}(p^2)}{p^{2s}} + \ldots\right).$$

Our main result is the following:

PLSG **Theorem 2.2** *Suppose that $x \ge Q^{2+\epsilon}$ and $Q \ge q^{2+\epsilon}\log x$. Then*

$$\sum_{\chi \in \mathcal{C}_q} \left|\frac{1}{x} S_\chi(x)\right|^2 \ll \left(\left(L(\mathcal{C}_q) + \rho_q'(f)\frac{\log Q}{\log x}\right)\log\left(\frac{\log x}{\log Q}\right)\right)^2.$$

Corollary PLSk 2.1 follows immediately from Theorem PLSG 2.2 and Proposition kRepulsion ??.

To prove Theorem PLSG 2.2 we begin with an averaged version of (MeanAveraged ??), which was used in the proof of Halasz's Theorem. Notice that if we simply sum up the square of (MeanAveraged ??) for $S = S_\chi$, for each $\chi \pmod q$, then we would get the next lemma but with the much weaker error term $\phi(q)$.

AvLogWt **Lemma 2.3** *Suppose that* $x \ge Q^{2+\epsilon}$ *and* $Q \ge q$. *Then*

$$\log^2 x \sum_{\chi \in \mathcal{C}_q} \left| \frac{1}{x} S_\chi(x) \right|^2 \ll \sum_{\chi \in \mathcal{C}_q} \left(\int_Q^{x/Q} \left| \frac{1}{t} S_\chi(t) \right| \frac{dt}{t} \right)^2 + \left(\rho_q'(f) \log Q \right)^2 .$$

Proof Let $z = x/Q$. We follow the proof in section MeanF(n) ?? for the main terms, but deal with the error terms differently. By Corollary Small.2 1.7 we have

$$\sum_{\chi \pmod q} \left| \sum_{n \le x} f(n) \overline{\chi}(n) \log(x/n) \right|^2 \ll (\rho_q'(f) x)^2,$$

$$\text{and} \quad \sum_{\chi} \left| \sum_{n \le x} f(n) \overline{\chi}(n) (\log_{\le Q} n + \log_{> x/Q} n) \right|^2 \ll (\rho_q'(f) x \log Q)^2,$$

so that, using the identity $\log x = \log(x/n) + \log_{\le Q} n + \log_{> x/Q} n + \log_{(Q, x/Q)} n$,

$$\sum_{\chi \in \mathcal{C}_q} |S_\chi(x) \log x|^2 \ll \sum_{\chi \in \mathcal{C}_q} \left| \sum_{n \le x} f(n) \overline{\chi}(n) \log_{(Q, x/Q)} n \right|^2 + (\rho_q'(f) x \log Q)^2.$$

Now for $g = f\overline{\chi}$ we have

$$\begin{aligned} &\sum_{n \le x} g(n) \log_{(Q,x/Q)} n - \sum_{Q < p < x/Q} g(p) \log p \sum_{m \le x/p} g(m) \\ &= \sum_{\substack{Q < p^k < x/Q \\ k \ge 2}} \log p \sum_{m \le x/p^k} g(mp^k) + \sum_{Q < p < x/Q} \log p \sum_{m \le x/p} (g(mp) - g(p) g(m)). \end{aligned}$$

The last term is 0 unless $p^2 | m$, so this last bound is, in absolute value,

$$\le x \sum_{\substack{Q < p^k < x/Q \\ k \ge 2}} \frac{\log p}{p^k} + 2x \sum_{Q < p < x/Q} \frac{\log p}{p^2} \ll \frac{x}{Q^{1/2}}.$$

We now bound our main term as in section MeanF(n) ??; though now we let $z = y + \sqrt{y}$ so we obtain the error term $x/\sqrt{y}$ in the equation before (MeanAveraged ??). Summing over such dyadic intervals this yields

$$\left| \sum_{Q < p < x/Q} g(p) \log p \sum_{m \le x/p} g(m) \right| \ll \int_Q^{x/Q} |S_\chi(x/t)| dt + \frac{x}{Q^{1/2}}.$$

The result follows from the change of variable $t \to x/t$ since $Q \ge q$ and $\rho_q'(f) \log Q \gg 1$.

□

In the next Lemma we create a convolution to work with, as well as removing the small primes.

AvConvol **Lemma 2.4** *Suppose that $x \geq Q^{2+\epsilon}$ and $Q \geq q^{2+\epsilon} \log x$. Then*

$$\sum_{\chi \in \mathcal{C}_q} \left(\int_Q^x \left| \frac{1}{t} S_\chi(t) \right| \frac{dt}{t} \right)^2 \ll \sum_{\chi \in \mathcal{C}_q} \left(\int_Q^x \left| \sum_{n \leq t} f(n) \overline{\chi}(n) \log_{>Q} n \right| \frac{dt}{t^2 \log t} \right)^2$$

$$+ \left(\rho_q'(f) \log Q \cdot \log \left(\frac{\log x}{\log Q} \right) \right)^2 .$$

Proof We expand using the fact that $\log t = \log(t/n) + \log_{\leq Q} n + \log_{>Q} n$; and the Cauchy-Schwarz inequality so that, for any function $c_\chi(t)$,

$$\sum_\chi \left(\int_Q^x c_\chi(t) \frac{dt}{t^2 \log t} \right)^2 \leq \int_Q^x \frac{dt}{t \log t} \cdot \int_Q^x \sum_\chi c_\chi(t)^2 \frac{dt}{t^3 \log t}$$

By Corollary Small.2 1.7 we then have

$$\int_Q^x \sum_\chi \left| \sum_{m \leq t} f(m) \overline{\chi}(m) \log(t/m) \right|^2 \frac{dt}{t^3 \log t} \ll \rho_q'(f)^2 \int_Q^x \frac{dt}{t \log t} \ll \rho_q'(f)^2 \log \left(\frac{\log x}{\log Q} \right)$$

and

$$\int_Q^x \sum_\chi \left| \sum_{m \leq t} f(m) \overline{\chi}(m) \log_{\leq Q} m \right|^2 \frac{dt}{t^3 \log t} \ll \int_Q^x \left(\rho_q'(f) t \log Q \right)^2 \frac{dt}{t^3 \log t},$$

and the result follows. □

Now we prove the mean square version of Halasz's Theorem, which is at the heart of the pretentious large sieve.

AvParsev **Proposition 2.5** *If $x > Q^{1+\epsilon}$ and $Q \geq q^{1+\epsilon}$ then*

$$\sum_{\chi \in \mathcal{C}_q} \left(\int_Q^x \left| \sum_{Q \leq n \leq t} f(n) \chi(n) \log_{>Q} n \right| \frac{dt}{t^2 \log t} \right)^2$$

$$\ll \log \left(\frac{\log x}{\log Q} \right) \left(M^2 \log \left(\frac{\log x}{\log Q} \right) + \frac{\phi(q)}{T} \frac{\log Q}{Q} + \frac{\log^3 x}{T^2} \right)$$

where $M := \max_{\chi \in \mathcal{C}_q} \ \max_{|u| \leq 2T} |F_\chi(1 + iu)|$.

Proof (Revisiting the proof of Halasz's Theorem (particularly Proposition keyProp ??)). For a given $g = f\overline{\chi}$ and Q we define

$$h(n) = \sum_{\substack{md=n \\ d>Q}} g(m)g(d)\Lambda(d),$$

so that $G(s)(G'_{>Q}(s)/G_{>Q}(s)) = -\sum_{n\ge1} h(n)/n^s$ for $\mathrm{Re}(s) > 1$. Now

$$\left|\sum_{n\le t} g(n)\log_{>Q} n - \sum_{n\le t} h(n)\right| \le 2\sum_{p^b>Q}\log p \sum_{\substack{n\le t \\ p^{b+1}|n}} 1 \le 2t\sum_{b\ge1}\sum_{\substack{p^b>Q \\ p^{b+1}\le t}} \frac{\log p}{p^{b+1}} \ll \frac{t\log t}{Q},$$

by the prime number theorem. This substitution leads to a total error, in our estimate, of

$$\ll |\mathcal{C}_q| \left(\int_{Qq}^{x} \frac{t\log t}{Q}\ \frac{dt}{t^2\log t}\right)^2 \ll \frac{q}{Q^2}\log^2\left(\frac{\log x}{\log Q}\right) \ll \frac{1}{q}\log^2\left(\frac{\log x}{\log Q}\right),$$

which is smaller than the first term in the given upper bound, since $M \gg 1/\log q$.

Now we use the fact that

$$\frac{1}{\log t} \ll \int_{1/\log x}^{1/\log Q} \frac{d\alpha}{t^{2\alpha}}$$

whenever $x \ge t \ge Q$, as $x > Q^{1+\epsilon}$, so that

$$\int_2^x \left|\sum_{n\le t} h(n)\right| \frac{dt}{t^2\log t} \ll \int_{1/\log x}^{1/\log Q} \left(\int_2^x \Big|\sum_{n\le t} h(n)\Big| \frac{dt}{t^{2+2\alpha}}\right) d\alpha.$$

Now, Cauchying, but otherwise proceeding as in the proof of Proposition keyProp ?? (with $f(n)\log n$ there replaced by $h(n)$ here), the square of the left side is

$$\ll \int_{1/\log x}^{1/\log Q} \frac{d\alpha}{\alpha} \cdot \int_{1/\log x}^{1/\log Q} \alpha\cdot\frac{1}{2\pi\alpha}\int_{-\infty}^{\infty} \Big|\frac{G(G'_{>Q}/G_{>Q})(1+\alpha+it)}{1+\alpha+it}\Big|^2 dt d\alpha.$$

The integral in the region with $|t| \le T$ is now

$$\le \max_{|t|\le T} |G(1+\alpha+it)|^2 \int_1^\infty \Big|\sum_{Q<n\le t} g(n)\Lambda(n)\Big|^2 \frac{dt}{t^{3+2\alpha}}.$$

If we take $g = f\overline{\chi}$ and sum this over all characters $\chi \in \mathcal{C}_q$ then we obtain an error

$$\le \max_{\substack{|t|\le T \\ \chi\in\mathcal{C}_q}} |F_\chi(1+\alpha+it)|^2 \int_Q^\infty \sum_{\chi \pmod q} \Big|\sum_{Q<n\le t} f(n)\overline{\chi}(n)\Lambda(n)\Big|^2 \frac{dt}{t^{3+2\alpha}}$$

$$\ll \max_{\substack{|t|\le T \\ \chi\in\mathcal{C}_q}} |F_\chi(1+\alpha+it)|^2 \int_Q^\infty \frac{dt}{t^{1+2\alpha}} \ll \frac{1}{\alpha} \max_{\substack{|t|\le T \\ \chi\in\mathcal{C}_q}} |F_\chi(1+\alpha+it)|^2,$$

by Lemma Small.3 1.8 as $t \ge Q \ge q^{1+\epsilon}$.

For that part of the integral with $|t| > T$, summed over all twists of f by characters $\chi \pmod q$, we now proceed as in the proof of Proposition **??**. We obtain $\phi(q)$ times (**??**), with $f(\ell)\log\ell$ replaced by $h(\ell)$ for $\ell = m$ and n, but now with the sum over $m \equiv n \pmod q$ with $m, n \geq Q$. Observing that $|h(\ell)| \leq \log\ell$, we proceed analogously to obtain, in total

$$\ll \frac{\phi(q)}{T}\frac{(\log Q)^2}{Q} + \frac{\phi(q)}{q}\cdot\frac{1}{\alpha^4 T^2}.$$

The result follows by collecting the above. □

Proof of Theorem 2.2: The result follows by taking $T = \frac{1}{2}\log^2 x$ in Proposition 2.5, and then combining this with Lemmas 2.3 and 2.4, since $\rho_q'(f)\log q \gg 1$. □

PLSRange

Corollary 2.6 *Fix $\epsilon > 0$. There exists an integer $k \ll 1/\epsilon^2$ such that if $x \geq q^{4+5\epsilon}$ then*

$$\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_1,\chi_2,\dots,\chi_k}} \left|\frac{1}{y}S_{\chi_j}(y)\right|^2 \ll e^{O(1/\epsilon)}\left(\rho_q'(f)\left(\frac{\log Q}{\log y}\right)^{1-\epsilon}\right)^2,$$

where $Q = (q\log x)^2$, for any y in the range

$$\log x \geq \log y \geq \log x \Big/ 2\left(\frac{\log x}{\log Q}\right)^{\epsilon/2},$$

where the implicit constants are independent of f.

Proof Select k to be the smallest integer for which $1/\sqrt{k} < 3\epsilon$. Let $\mathcal{C}_q$ be the set of all characters mod q except $\chi_1, \chi_2, \dots, \chi_k$. Write $x = Q^B$, so that $y = Q^C$ where $B \geq C \geq \frac{1}{2}B^{1-\epsilon/2}$, and apply Theorem 2.2 with $x = y$. Then, by (**??**) and Proposition **??** we have

$$L_y \ll L_x\left(\frac{\log x}{\log y}\right)^2 \ll e^{O(1/\epsilon)}\rho_q'(f)\frac{1}{B^{1-3\epsilon}}B^{\epsilon} \ll e^{O(1/\epsilon)}\rho_q'(f)\frac{1}{C^{1-4\epsilon}},$$

and the result follows. Note that by bounding L_y in terms of L_x, we can have the same exceptional characters $\chi_1, \chi_2, \dots, \chi_k$ for each y in our range. □

3

MULTIPLICATIVE FUNCTIONS IN ARITHMETIC PROGRESSIONS

It is usual to estimate the mean value of a multiplicative function in an arithmetic progression in terms of the mean value of the multiplicative function on all the integers. This approximation is the summand corresponding to the principal character when we decompose our sum in terms of the Dirichlet characters mod q. In what follows we will instead compare our mean value with the summands for the k characters which best correlate with f. So define

$$E_f^{(k)}(x;q,a) := \sum_{\substack{n\le x\\ n\equiv a \pmod q}} f(n) - \frac{1}{\phi(q)}\sum_{j=1}^{k}\chi_j(a)\sum_{n\le x} f(n)\overline{\chi}_j(n).$$

The trivial upper bound $|E_f^{(k-1)}(x;q,a)| \ll k\rho_q'(f)x/\phi(q)$ can be obtained by bounding each sum in the definition using the small sieve. We now improve this:

FnsInAPs **Theorem 3.1** *For any given $k \ge 2$ and sufficiently large x, if $x \ge X \ge \max\{x^{1/2}, q^{6+7\epsilon}\}$ then*

$$|E_f^{(k-1)}(X;q,a)| \ll e^{C\sqrt{k}}\ \frac{\rho_q'(f)X}{\phi(q)}\ \left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}}\log\left(\frac{\log x}{\log Q}\right),$$

where $Q = (q\log x)^5$ and the implicit constants are independent of f and k. If f is real and χ_1 is not then we can extend this to $k = 1$ with exponent $1 - \frac{1}{\sqrt{2}}$.

To prove this we need the following technical tool, deduced from Corollary PLSRange 2.6.

LinearPLS **Proposition 3.2** *Fix $\varepsilon > 0$. For given $x = q^A$ there exists $K \ll \epsilon^{-3}\log\log A$ such that if $x \ge X \ge x^{1/2}$ and $Q = (q\log x)^5$ then*

$$\frac{1}{\log x}\sum_{\substack{\chi \pmod q\\ \chi\ne\chi_j,\ j=1,\dots,K}}\left|\frac{1}{X}\sum_{n\le X} f(n)\overline{\chi}(n)\log_{[Q,x/Q]} n\right| \ll e^{O(1/\epsilon)}\rho_q'(f)\left(\frac{\log Q}{\log x}\right)^{1-\epsilon}.$$

Proof Let $\log x_i = 2^{(1+\epsilon/3)^i+1}\log q$ for $0 \le i \le IA$, with I chosen to be the smallest integer for which $x_I > x/Q$, so that $I \ll (1/\epsilon)\log\log A$. In order to apply Corollary PLSRange 2.6 with $x = x_i$ we must exclude the characters $\chi_{j,i},\ 1 \le j \le k$, for $1 \le i \le I$. Let $\chi_1, \chi_2, \ldots, \chi_K$ be the union of these sets of characters, so that $K \le k(I+1) \ll \epsilon^{-3} \log\log A$. Therefore, for all $y \in [Q, x/Q]$, we have

$$\sum_{\substack{\chi \pmod q \\ \chi \ne \chi_1, \chi_2, \ldots, \chi_K}} \left| \frac{1}{y} S_{\chi_j}(y) \right|^2 \ll e^{O(1/\epsilon)} \left(\rho_q'(f) \left(\frac{\log Q}{\log y} \right)^{1-\epsilon} \right)^2 . \tag{3.1}$$ PLSuniform

We rewrite the sum in the Proposition as

$$\sum_{\substack{\chi \pmod q \\ \chi \ne \chi_j,\ j=1,\ldots,K}} \left| \sum_{\substack{dm \le X \\ Q \le d \le x/Q}} f(m)\overline{\chi}(m) f(d)\overline{\chi}(d)\Lambda(d) \right| ,$$

and split this into subsums, depending on the size of d. This is bounded by a sum of sums of the form

$$\sum_{\substack{\chi \pmod q \\ \chi \ne \chi_j,\ j=1,\ldots,K}} \left| \sum_{D \le d \le D+\Delta} f(d)\overline{\chi}(d)\Lambda(d) \sum_{m \le X/d} f(m)\overline{\chi}(m) \right| .$$

where $Q \le D \le x/Q$ with $\Delta \approx \frac{D \log(q \log(X/D))}{q \log(X/D)}$. If we approximate the last sum here with the range $m \le X/D$, then we can Cauchy to obtain

$$\left(\sum_{\substack{\chi \pmod q \\ \chi \ne \chi_j,\ j=1,\ldots,K}} \left| \sum_{D \le d \le D+\Delta} f(d)\overline{\chi}(d)\Lambda(d) \sum_{m \le X/D} f(m)\overline{\chi}(m) \right| \right)^2 \tag{3.2}$$

$$\le \sum_{\chi \pmod q} \left| \sum_{D \le d \le D+\Delta} f(d)\overline{\chi}(d)\Lambda(d) \right|^2 \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_j,\ j=1,\ldots,k-1}} \left| \sum_{m \le X/D} f(m)\overline{\chi}(m) \right|^2 \tag{3.3}$$

$$\ll e^{O(1/\epsilon)} \left(\Delta \cdot \rho_q'(f) \frac{X}{D} \left(\frac{\log Q}{\log(X/D)} \right)^{1-\epsilon} \right)^2 , \tag{3.4}$$ OneTermBound

by Lemma 1.8 [Small.3] and (3.1) [PLSuniform]. Summing the square root of this over the D/Δ such intervals for d in $[D, 2D)$ yields an upper bound

$$\ll e^{O(1/\epsilon)} \rho_q'(f) X \left(\frac{\log Q}{\log(X/D)} \right)^{1-\epsilon} ;$$

and then summing this over $D = X/Q2^j$ for $0 \le j \le J \asymp \log X$ we obtain the claimed upper bound.

Finally the error in replacing the range $m \le X/d$ by $m \le X/D$ is

$$\le \sum_{\substack{X/d<m\le x/D \\ (m,q)=1}} |f(m)|\chi_0(m) \le \sum_{\substack{X/(D+\Delta)<m\le x/D \\ (m,q)=1}} |f(m)|\chi_0(m) \ll \rho_q'(f)\frac{X\Delta}{D^2},$$

so an upper bound for the contribution in $[D, 2D)$ is

$$\ll \rho_q'(f)\frac{X\Delta\phi(q)}{D}\sum_{D\le d<2D}\frac{\Lambda(d)}{d} \ll \rho_q'(f)X\frac{\log Q}{\log(X/D)},$$

which is smaller than the other error term. □

Proof of Theorem FnsInAPs **3.1** : Fix $\epsilon > 0$ sufficiently small with $1/\sqrt{k} > \epsilon$. By applying Lemma Small.1 1.6 , with $\chi = \chi_0$ we have

$$\log x \sum_{\substack{n\le x \\ n\equiv a \pmod q}} f(n) = \sum_{\substack{n\le x \\ n\equiv a \pmod q}} f(n)\log_{[Q,x/Q]} n + O\left(\rho_q'(f)\frac{x}{\phi(q)}\log Q\right).$$

Multiplying this by $\overline{\chi}(a)$, and summing over a we obtain

$$\log x\sum_{n\le x} f(n)\overline{\chi}(n) = \sum_{n\le x} f(n)\overline{\chi}(n)\log_{[Q,x/Q]} n + O\left(\rho_q'(f)x\log Q\right);$$

so that

$$\begin{aligned} E_f^{(K)}(x;q,a) &= \frac{1}{\phi(q)}\sum_{j=K+1}^{\phi(q)}\chi_j(a)\sum_{n\le x} f(n)\overline{\chi}_j(n)\frac{\log_{[Q,x/Q]} n}{\log x} + O\left(K\rho_q'(f)\frac{x}{\phi(q)}\frac{\log Q}{\log x}\right) \\ &\ll e^{O(1/\epsilon)}\frac{\rho_q'(f)x}{\phi(q)}\left(\frac{\log Q}{\log x}\right)^{1-\epsilon}, \end{aligned}$$

by Proposition LinearPLS 3.2 , where $K \ll \epsilon^{-3}\log\log A$. By Cauchying and then Corollary PLSk 2.1, we obtain

$$\begin{aligned} |E_f^{(k)}(x;q,a) - E_f^{(K)}(x;q,a)| &\le \frac{1}{\phi(q)}\sum_{j=k+1}^{K}\left|S_{\chi_j}(x)\right| \\ &\le \frac{1}{\phi(q)}\left(K\sum_{j=k+1}^{K}\left|S_{\chi_j}(x)\right|^2\right)^{1/2} \ll e^{O(\sqrt{k})}\rho_q'(f)\frac{x}{\phi(q)}\left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}}, \end{aligned}$$

since $K \ll \log\log A$, and $1 - \frac{1}{\sqrt{k+1}} > 1 - \frac{1}{\sqrt{k}}$. Applying the same argument again, we also obtain

$$|E_f^{(k-1)}(x;q,a) - E_f^{(k)}(x;q,a)| \ll e^{C\sqrt{k}}\frac{\rho_q'(f)x}{\phi(q)}\left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}}\log\left(\frac{\log x}{\log Q}\right).$$

The result follows from using the triangle inequality and adding the last three inequalities. □

4

PRIMES IN ARITHMETIC PROGRESSION

PNTapsk **Theorem 4.1** *For any $k \geq 2$ and $x \geq q^2$ there exists an ordering $\chi_1, \dots$ of the non-principal characters $\chi \pmod q$ such that, for $Q = (q \log x)^2$,*

$$\sum_{\substack{n \leq y \\ n \equiv a \pmod q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \leq y} \Lambda(n) - \frac{1}{\phi(q)} \sum_{j=1}^{k-1} \chi_j(a) \sum_{n \leq y} \Lambda(n) \overline{\chi}_j(n)$$

$$\ll e^{C\sqrt{k}} \; \frac{x}{\phi(q)} \left(\frac{\log Q}{\log x} \right)^{1-\frac{1}{\sqrt{k}}} \log^3 \left(\frac{\log x}{\log Q} \right).$$

PNTaps1 **Corollary 4.2** *There exists a character $\chi \pmod q$ such that if $x \geq q^2$ then*

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \leq x} \Lambda(n) - \frac{\chi(a)}{\phi(q)} \sum_{n \leq x} \Lambda(n) \overline{\chi}(n) \ll \frac{x}{\phi(q)} \left(\frac{\log Q}{\log x} \right)^{1-\frac{1}{\sqrt{2}}-\varepsilon}.$$

where $Q = (q \log x)^2$. We may remove the χ term unless χ is a real-valued character.

Remark 4.3 *Can we obtain the error in terms of $1/|L(1+it, \chi)|/\log x$? And when χ is real, probably $t = 0$.*

Proof of Theorem PNTapsk **5.1** We may assume that $x \geq q^B$ for B sufficiently large, else the result follows from the Brun-Titchmarsh Theorem.

Let $g(.)$ be the totally multiplicative function for which $g(p) = 0$ for $p \leq Q$ and $g(p) = 1$ for $p > Q$, and then $f = \mu g$, so that we have the following variant of von Mangoldt's formula (Lammu **??**),

$$\Lambda_Q(n) := \sum_{dm=n} f(d) g(m) \log m = \begin{cases} \Lambda(n) & \text{if } p|n \implies p > Q, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\sum_{\substack{n \leq x \\ n \equiv b \pmod q}} (\Lambda(n) - \Lambda_Q(n)) \leq \sum_{\substack{n \leq x \\ p|n \implies p \leq Q}} \Lambda(d) \ll \sum_{p \leq Q} \log x \ll Q \frac{\log x}{\log Q}.$$

by the Brun-Titchmarsh theorem. Denote the left side of the equation in the Theorem as $E^{(k-1)}_{\Lambda,+}(x; q, a)$, and note that all of these sums can be expressed as mean-values of $\sum_{n \leq x, \; n \equiv b \pmod q} \Lambda(n)$, as b varies. Hence

$$E^{(k-1)}_{\Lambda,+}(x;q,a) - E^{(k-1)}_{\Lambda_Q,+}(x;q,a) \ll Q\frac{\log x}{\log Q}.$$

Now

$$\sum_{\substack{n\le x\\ n\equiv a \pmod q}} \Lambda_Q(n) = \sum_{\substack{d\le x\\ (d,q)=1}} f(d) \sum_{\substack{m\le x/d\\ m\equiv a/d \pmod q}} g(m)\log m. \qquad (4.1)$$

LQexpand

Similar decompositions for the $\sum_n \Lambda_Q(n)\overline{\chi}_j(n)$ imply that $E^{(k-1)}_{\Lambda_Q,+}(x;q,a)$ equals the sum of $f(d)$ over $d\le x$ with $(d,q)=1$, times

$$\sum_{\substack{m\le x/d\\ m\equiv a/d \pmod q}} g(m)\log m - \frac{1}{\phi(q)}\sum_{j=0}^{k-1}\chi_j(a/d)\sum_{(b,q)=1}\overline{\chi}_j(b)\sum_{\substack{m\le x/d\\ m\equiv b \pmod q}} g(m)\log m.$$

By Corollary FLS1 1.2 (with m the product of the primes $\le Q$ that do not divide q) this last quantity is

$$\ll \left(\frac{k}{u^{u+2}}\frac{x}{d\phi(q)\log Q} + k\sqrt{\frac{x}{d}}\right)\log x/d$$

where $x/d = Q^u$. Let R be the product of the primes $\le Q$. We deduce that the sum over d in a range $x/Q^{2u} < d \le x/Q^u$ with $f(d)\ne 0$, is

$$\ll k \sum_{\substack{x/Q^{2u}<d\le x/Q^u\\ (d,R)=1}} \left(\frac{1}{u^{u+2}}\frac{x}{d\phi(q)\log Q} + \sqrt{\frac{x}{d}}\right)\log x/d \ll \frac{k}{u^u}\frac{x}{\phi(q)} + \frac{kux}{Q^{u/2}}$$

by Corollary FLS3 1.4 (for the sum over d), provided $u\le \nu := \log\left(\frac{\log x}{\log Q}\right)$. Summing this up over $u=2,4,8,\dots,\nu$, the sum over d in the range $Q^2<d\le x/Q^{2\nu}$ is

$$\ll \frac{x}{\phi(q)}\left(\frac{\log Q}{\log x}\right)^2.$$

The same argument works to give a much better upper bound for the terms with $d\le Q^2$, though removing the condition $(d,R)=1$ in the sum above. Hence we are left to deal with those $d>x/Q^\nu$, which implies that $m\le x/d<Q^\nu$.

The remaining sum in (LQexpand 5.1) is

$$\sum_{\substack{m<Q^\nu\\ (m,q)=1}} g(m)\log m \sum_{\substack{x/Q^\nu<d\le x/m\\ d\equiv a/m \pmod q}} f(d).$$

There are analogous sums for the remaining terms in $E^{(k-1)}_{\Lambda_Q,+}(x;q,a)$ and so we need to bound

$$\sum_{\substack{m<Q^\nu\\(m,q)=1}} g(m)\log m\ (E_{f,+}^{(k-1)}(x/m;q,a/m)-E_{f,+}^{(k-1)}(x/Q^\nu;q,a/m)).$$

To do so we need to apply Theorem 2.2 with $\mathcal{C}_q$ to be the set of all characters mod q, less $\chi_0,\chi_1,\dots,\chi_{k-1}$. Then we can deduce Corollary 2.1 though now with $\chi\neq\chi_0,\dots,\chi_{k-1}$ as the condition on the sum (but otherwise the same). We can then similarly modify Corollary 2.6 and finally obtain Theorem 3.1 with $E_f^{(k-1)}$ replaced by $E_{f,+}^{(k-1)}$. Therefore we obtain the bound

$$\begin{aligned}\sum_{\substack{m<Q^\nu\\(m,q)=1}} g(m)\log m\ |E_{f,+}^{(k-1)}(x/m;q,a/m)-E_{f,+}^{(k-1)}(x/Q^\nu;q,a/m)|\\ \ll e^{C\sqrt{k}}\ \frac{\rho_q'(f)x}{\phi(q)}\ \left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}}\nu\sum_{\substack{m<Q^\nu\\(m,q)=1}} g(m)\frac{\log m}{m}\\ \ll e^{C\sqrt{k}}\ \frac{\rho_q'(f)x}{\phi(q)}\ \left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}}\nu\frac{(\nu\log Q)^2}{\log Q}.\end{aligned}$$

by Corollary 1.4, and the result follows since $\rho_q'(f)\ll 1/\log Q$. (*This means we need to change the sieving to go up to Q throughout rather than q.*) □

Proof of Corollary 5.2 We let $k=2$ in Theorem 11.1 to deduce the first part. If χ is not real valued, then we know that

$$\left|\sum_{n\le x}\Lambda(n)\overline{\chi}(n)\right|=\left|\sum_{n\le x}\Lambda(n)\chi(n)\right|\le |E_\Lambda^{(3)}(x;q,a)-E_\Lambda^{(2)}(x;q,a)|$$

and the result follows from Theorem 5.1. □

5

PRIMES IN ARITHMETIC PROGRESSION

PNTapsk **Theorem 5.1** *For any $k \geq 2$ and $x \geq q^2$ there exists an ordering $\chi_1, \dots$ of the non-principal characters $\chi \pmod q$ such that, for $Q = (q \log x)^2$,*

$$\sum_{\substack{n \leq y \\ n \equiv a \pmod q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \leq y} \Lambda(n) - \frac{1}{\phi(q)} \sum_{j=1}^{k-1} \chi_j(a) \sum_{n \leq y} \Lambda(n) \overline{\chi}_j(n)$$

$$\ll e^{C\sqrt{k}} \frac{x}{\phi(q)} \left(\frac{\log Q}{\log x} \right)^{1 - \frac{1}{\sqrt{k}}} \log^3 \left(\frac{\log x}{\log Q} \right).$$

PNTaps1 **Corollary 5.2** *There exists a character $\chi \pmod q$ such that if $x \geq q^2$ then*

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \leq x} \Lambda(n) - \frac{\chi(a)}{\phi(q)} \sum_{n \leq x} \Lambda(n) \overline{\chi}(n) \ll \frac{x}{\phi(q)} \left(\frac{\log Q}{\log x} \right)^{1 - \frac{1}{\sqrt{2}} - \varepsilon}.$$

where $Q = (q \log x)^2$. We may remove the χ term unless χ is a real-valued character.

Remark 5.3 *Can we obtain the error in terms of $1/|L(1+it,\chi)|/\log x$? And when χ is real, probably $t = 0$.*

Proof of Theorem 5.1 (PNTapsk) We may assume that $x \geq q^B$ for B sufficiently large, else the result follows from the Brun-Titchmarsh Theorem.

Let $g(.)$ be the totally multiplicative function for which $g(p) = 0$ for $p \leq Q$ and $g(p) = 1$ for $p > Q$, and then $f = \mu g$, so that we have the following variant of von Mangoldt's formula (??) (Lammu),

$$\Lambda_Q(n) := \sum_{dm = n} f(d) g(m) \log m = \begin{cases} \Lambda(n) & \text{if } p|n \implies p > Q, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\sum_{\substack{n \leq x \\ n \equiv b \pmod q}} (\Lambda(n) - \Lambda_Q(n)) \leq \sum_{\substack{n \leq x \\ p|n \implies p \leq Q}} \Lambda(d) \ll \sum_{p \leq Q} \log x \ll Q \frac{\log x}{\log Q}.$$

by the Brun-Titchmarsh theorem. Denote the left side of the equation in the Theorem as $E_{\Lambda,+}^{(k-1)}(x; q, a)$, and note that all of these sums can be expressed as mean-values of $\sum_{n \leq x,\ n \equiv b \pmod q} \Lambda(n)$, as b varies. Hence

$$E_{\Lambda,+}^{(k-1)}(x;q,a) - E_{\Lambda_Q,+}^{(k-1)}(x;q,a) \ll Q\frac{\log x}{\log Q}.$$

Now

$$\sum_{\substack{n\le x\\ n\equiv a \pmod q}} \Lambda_Q(n) = \sum_{\substack{d\le x\\ (d,q)=1}} f(d) \sum_{\substack{m\le x/d\\ m\equiv a/d \pmod q}} g(m)\log m. \qquad (5.1)$$ LQexpand

Similar decompositions for the $\sum_n \Lambda_Q(n)\overline{\chi}_j(n)$ imply that $E_{\Lambda_Q,+}^{(k-1)}(x;q,a)$ equals the sum of $f(d)$ over $d\le x$ with $(d,q)=1$, times

$$\sum_{\substack{m\le x/d\\ m\equiv a/d \pmod q}} g(m)\log m - \frac{1}{\phi(q)}\sum_{j=0}^{k-1}\chi_j(a/d)\sum_{(b,q)=1}\overline{\chi}_j(b)\sum_{\substack{m\le x/d\\ m\equiv b \pmod q}} g(m)\log m.$$

By Corollary FLS1 1.2 (with m the product of the primes $\le Q$ that do not divide q) this last quantity is

$$\ll \left(\frac{k}{u^{u+2}}\frac{x}{d\phi(q)\log Q} + k\sqrt{\frac{x}{d}}\right)\log x/d$$

where $x/d = Q^u$. Let R be the product of the primes $\le Q$. We deduce that the sum over d in a range $x/Q^{2u} < d \le x/Q^u$ with $f(d)\ne 0$, is

$$\ll k\sum_{\substack{x/Q^{2u}<d\le x/Q^u\\ (d,R)=1}}\left(\frac{1}{u^{u+2}}\frac{x}{d\phi(q)\log Q} + \sqrt{\frac{x}{d}}\right)\log x/d \ll \frac{k}{u^u}\frac{x}{\phi(q)} + \frac{kux}{Q^{u/2}}$$

by Corollary FLS3 1.4 (for the sum over d), provided $u\le \nu := \log\left(\frac{\log x}{\log Q}\right)$. Summing this up over $u = 2,4,8,\dots,\nu$, the sum over d in the range $Q^2 < d \le x/Q^{2\nu}$ is

$$\ll \frac{x}{\phi(q)}\left(\frac{\log Q}{\log x}\right)^2.$$

The same argument works to give a much better upper bound for the terms with $d\le Q^2$, though removing the condition $(d,R)=1$ in the sum above. Hence we are left to deal with those $d > x/Q^\nu$, which implies that $m \le x/d < Q^\nu$.

The remaining sum in LQexpand (5.1) is

$$\sum_{\substack{m<Q^\nu\\ (m,q)=1}} g(m)\log m \sum_{\substack{x/Q^\nu<d\le x/m\\ d\equiv a/m \pmod q}} f(d).$$

There are analogous sums for the remaining terms in $E_{\Lambda_Q,+}^{(k-1)}(x;q,a)$ and so we need to bound

$$\sum_{\substack{m<Q^\nu \\ (m,q)=1}} g(m)\log m \ (E_{f,+}^{(k-1)}(x/m;q,a/m) - E_{f,+}^{(k-1)}(x/Q^\nu;q,a/m)).$$

To do so we need to apply Theorem 2.2 with $\mathcal{C}_q$ to be the set of all characters mod q, less $\chi_0,\chi_1,\dots,\chi_{k-1}$. Then we can deduce Corollary 2.1 though now with $\chi\neq\chi_0,\dots,\chi_{k-1}$ as the condition on the sum (but otherwise the same). We can then similarly modify Corollary 2.6 and finally obtain Theorem 3.1 with $E_f^{(k-1)}$ replaced by $E_{f,+}^{(k-1)}$. Therefore we obtain the bound

$$\begin{aligned}\sum_{\substack{m<Q^\nu \\ (m,q)=1}} g(m)\log m \ |E_{f,+}^{(k-1)}(x/m;q,a/m) - E_{f,+}^{(k-1)}(x/Q^\nu;q,a/m)| \\ \ll e^{C\sqrt{k}}\ \frac{\rho_q'(f)x}{\phi(q)}\ \left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}}\ \nu \sum_{\substack{m<Q^\nu \\ (m,q)=1}} g(m)\frac{\log m}{m} \\ \ll e^{C\sqrt{k}}\ \frac{\rho_q'(f)x}{\phi(q)}\ \left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}}\ \nu\frac{(\nu\log Q)^2}{\log Q}.\end{aligned}$$

by Corollary 1.4, and the result follows since $\rho_q'(f)\ll 1/\log Q$. (*This means we need to change the sieving to go up to Q throughout rather than q.*) □

Proof of Corollary 5.2 We let $k=2$ in Theorem 11.1 to deduce the first part. If χ is not real valued, then we know that

$$\left|\sum_{n\le x}\Lambda(n)\overline{\chi}(n)\right| = \left|\sum_{n\le x}\Lambda(n)\chi(n)\right| \le |E_\Lambda^{(3)}(x;q,a) - E_\Lambda^{(2)}(x;q,a)|$$

and the result follows from Theorem 5.1. □

6

LINNIK'S THEOREM

In this section we complete the proof of Linnik's famous theorem:

Linnik **Theorem 6.1** *There exist constants $c, L > 0$ such that for any coprime integers a and q there is a prime $\equiv a \pmod q$ that is $< cq^L$.*

There are several proofs of this in the literature, none easy. Here we present a new proof as a consequence of the Pretentious Large Sieve, as developed in the previous few sections. Corollary PNTaps1 5.2 implies that if there are no primes $\equiv a \pmod q$ up to x, a large power of Q, then the vast majority of primes satisfy $\chi(p) = -\chi(a)$. The difficult part of our current proof is to now show that $\chi(a) = 1$ (*which surely should not be difficult!*):

LinkNoSieg **Proposition 6.2** *Suppose that $x \geq q^A$ where A is chosen sufficiently large. If*

$$\left| \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \leq x} \Lambda(n) \right| \gg \frac{x}{\phi(q)}$$

then there exists a real character $\chi \pmod q$ such that $\chi(a) = -1$, and

$$\sum_{\substack{Q < p \leq x \\ \chi(p) = 1}} \frac{1}{p} \ll \log \log \left(\frac{\log x}{\log Q} \right).$$

LinkSiegCond **Corollary 6.3** *If there are no primes $p \equiv a \pmod q$ with $Q < p \leq x$ then there exists a real character $\chi \pmod q$ such that $\chi(a) = -1$, and*

$$\sum_{\substack{Q < p \leq x \\ \chi(p) = 1}} \frac{1}{p} \ll 1.$$

HalRevisited **Lemma 6.4** *(Halasz's Theorem for sieved functions) Let f be a multiplicative function with the property that $f(p^k) = 0$ whenever $p \leq Q$. If $x \geq Q$ then*

$$\left| \frac{1}{x} \sum_{n \leq x} f(n) \right| \ll \frac{1}{\log Q} (1 + M) e^{-M} + \frac{1}{T} + \frac{1}{\log x} \left(1 + \frac{1}{\log Q} \log \left(\frac{\log x}{\log Q} \right) \right).$$

where $M := \min_{|t| \leq T} \sum_{Q < p \leq x} \frac{1 - Re(f(p)p^{-it})}{p}$.

Proof (sketch) We suitably modify the proof of Halasz's Theorem (HalExplic1 **??**). We begin by following the proof of Proposition keyProp **??**. First note that $S(N) = 1$ for all $N \leq Q$, so we can reduce the range in the integral for α, throughout the proof of Proposition keyProp **??**, to $\frac{1}{\log x} \leq \alpha \leq \frac{1}{\log Q}$. Moreover in the first displayed equation we can change the error term from $\ll \frac{N}{\log N}$ to $\ll \frac{1}{\log Q}\frac{N}{\log N}$ for $N \geq Q$ by sieving. This allows us to replace the error term in the second displayed equation from $\ll \log\log x$ to $\ll 1 + \frac{1}{\log Q}\log\left(\frac{\log x}{\log Q}\right)$. Hence we can restate Proposition keyProp **??** with the range for α, and the $\log\log x$ in the error term, changed in this way.

Now we use the bound $|F(1+\alpha+it)| \leq |F(1+iu)| + O\left(\frac{\alpha}{T}\frac{\log x}{\log Q}\right)$ throughout this range, as in Lemma OffLineOn **??**; and we also note that, in our range for α, $|F(1+\alpha+it)| \ll 1/(\alpha \log Q)$. We then proceed as in the proof of (HalExplic2 **??**), but now splitting the integral at $1/L\log Q \log x$ to obtain the result, since $L \log Q \asymp e^{-M}$. □

Proof of Proposition LinkNoSieg **6.2** Write $\nu := \log\left(\frac{\log x}{\log Q}\right)$. We return to the proof of Theorem PNTapsk 5.1, and show, under our hypothesis here, that there exists y in the range $x^{1/2} < y \leq x$ for which

$$\left|\sum_{n\leq y} f(n)\overline{\chi}(n)\right| \gg \frac{y}{\nu^2 \log Q}.$$

For, if not, the proof there implies that

$$\left|\sum_{n\leq x} \Lambda(n)\overline{\chi}(n)\right| = o\left(\frac{x}{\phi(q)}\right),$$

which, by Corollary PNTaps1 5.2, contradicts our hypothesis.

Taking $f = f\overline{\chi}$ in Lemma HalRevisited 6.4, and comparing our upper and lower bounds for $S_\chi(y)$ we deduce that

$$\sum_{Q<p\leq x} \frac{1 + \text{Re }(\chi(p)p^{it})}{p} \ll \log \nu.$$

Let $T := \{z : \ |z| = 1, \text{ and } \frac{\pi}{3} < \arg(z) < \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} < \arg(z) < \frac{5\pi}{3}\}$. We must have $|t| \ll \nu/\log x$ else $p^{it} \in T$ (and hence $\chi(p)p^{it} \in T$) for enough of the primes in $(x^{c/\nu}, x]$ that the previous estimate cannot hold. Therefore

$$\sum_{\substack{Q<p\leq x\\ \chi(p)=1}} \frac{1}{p} = \frac{1}{2}\sum_{Q<p\leq x} \frac{1+\text{Re}(\chi(p))}{p} \ll \sum_{Q<p\leq x} \frac{1+\text{Re}(\chi(p)p^{it}) + |p^{it}-1|}{p} \ll \log \nu.$$

□

Proof of Corollary 6.3 By Corollary 5.2 we know that for all y in the range $Q \le y \le x$ we have

$$\sum_{p \le y} \Lambda(n)(\chi(p) + \chi(a)) \ll y \left(\frac{\log Q}{\log y} \right)^{1/5}.$$

By partial summation, we deduce that

$$\sum_{Q < p \le x} \frac{\chi(a) + \chi(p)}{p} \ll 1.$$

Comparing this to the conclusion of Proposition 6.2, we deduce that $\chi(a) = -1$ and we obtain the result. □

Proposition 6.5 *If the hypotheses of Corollary 6.3 hold for $x = q^A$ where A is sufficiently large, and if $\chi(a) = 1$ then there are primes $\le x$ that are $\equiv a \pmod q$.*

6.1 Binary Quadratic Forms

Let us suppose that χ is induced from the quadratic character $(./D)$ so that D must be squarefree. We re-write this as $(d/.) = (./D)$ where $d = (-1)^{(D-1)/4}D$, so that $d \equiv 1 \pmod 4$. Suppose that a, b, c are integers for which $b^2 - 4ac = d$ and define the *binary quadratic form* $ax^2 + bxy + cy^2$, which has *discriminant* d. Now $(a, b, c)^2 | d$, which is squarefree, and so $(a, b, c) = 1$. We will study the values $am^2+bmn+cn^2$ when m and n are integers, and in particular the prime values. To begin with we look at divisibility. First note that $(m, n)^2$ divides $am^2+bmn+cn^2$, so we proceed by replacing m by $m/(m, n)$, and n by $n/(m, n)$, and hence we may assume that m and n are coprime.

We now show that if odd prime p divides $am^2 + bmn + cn^2$ then $(d/p) = 0$ or 1. If p divides n then $0 \equiv am^2 + bmn + cn^2 \equiv am^2 \pmod p$ and so p divides a as $(m, n) = 1$. Therefore $d = b^2 - 4ac \equiv b^2 \pmod p$ and hence $(d/p) = 0$ or 1. If $m \nmid n$ then $4ap$ divides $4a(am^2 + bmn + cn^2) = (2am + bn)^2 - dn^2$, and so

$$\left(\frac{2am + bn}{p} \right)^2 = \left(\frac{(2am + bn)^2}{p} \right) = \left(\frac{dn^2}{p} \right) = \left(\frac{d}{p} \right) \left(\frac{n}{p} \right)^2 = \left(\frac{d}{p} \right),$$

implying that $(d/p) = 0$ or 1.

Exercise 6.1 Show that if p is an odd prime then

$$1 - \frac{1}{p^2} \#\{m, n \pmod p) : am^2 + bmn + cn^2 \equiv 0 \pmod p\} = \left(1 - \frac{1}{p}\right) \left(1 - \frac{(d/p)}{p}\right).$$

We wish to show that $am^2 + bmn + cn^2$ takes on many prime values, that is not many composite values. If $am^2 + bmn + cn^2 \le x$ is composite then it certainly has a prime factor $\le \sqrt{x}$ so we will count the number of such values with no small prime factor. To explain our method in an intuitive fashion we will proceed assuming that $d < 0 < a$ (so that $am^2 + bmn + cn^2$ only takes non-negative values); when we give the actual proof we will use sieve weights that are easier to work with but more difficult to understand.

The small sieve shows us that if $x = y^u$ then for $M = \prod_{p \le y} p$

$$\#\{m, n \in \mathbb{Z} : \ N := am^2 + bmn + cn^2 \le x, \ (N, M) = 1\} =$$

$$= \{1 + O(u^{-u})\} \prod_{p \le y} \left(1 - \frac{1}{p}\right) \left(1 - \frac{(d/p)}{p}\right) X + O(\sqrt{X}),$$

where $X := \#\{m, n \in \mathbb{Z} : \ N := am^2 + bmn + cn^2 \le x\} = \pi x/\sqrt{d} + O(\sqrt{x})$.

We will use this estimate when y is a small power of x, and then obtain a lower bound by subtracting the number of such integers divisible by a prime in $(y, x^{1/2}]$.

The trick is that if prime ℓ is in this range with $(d/\ell) = 1$ then ℓ can be written as the value of a binary quadratic form of discriminant d in one of two (essentially different) ways, and then N/ℓ similarly. Hence to count the number of such N/ℓ we can use use the same estimate, though in this case we use the above simply as an upper bound, particularly as $N/\ell \ge \sqrt{x}$. Hence

$$\#\{m, n \in \mathbb{Z} : \ N := am^2 + bmn + cn^2 \le x, \ (N, M) = 1, \ell | N\}$$

$$\ll \prod_{p \le y} \left(1 - \frac{1}{p}\right) \left(1 - \frac{(d/p)}{p}\right) \frac{X}{\ell}.$$

Hence in total, we have

$$\#\{m, n \in \mathbb{Z} : \ N := am^2 + bmn + cn^2 \le x, \ N \text{is prime}\}$$

$$\gg \left\{ 1 - \sum_{\substack{y < \ell \le x^{1/2} \\ (d/\ell) = 1}} \frac{2}{\ell} - \epsilon \right\} \prod_{p \le y} \left(1 - \frac{1}{p}\right) \left(1 - \frac{(d/p)}{p}\right) X,$$

where say $u \gg 1/\epsilon$.

From the first equation in the proof of Corollary LinkSiegCond 6.3 we deduce that that if there are no primes $\equiv a \pmod q$ up to x then

$$\sum_{\substack{y < \ell \le x^{1/2} \\ (d/\ell) = 1}} \frac{1}{\ell} \ll \left(\frac{\log Q}{\log y} \right)^{1/5};$$

hence if $x = q^L$ where L/u is sufficiently large then $\sum_{y \le p \le x^{1/2}} (1 + (d/p))/p \le 1/2$; and so, from the above, we know that there are many prime values of our binary quadratic form.

6.2 Finishing the proof of Linnik's Theorem

To obtain a complete proof without proving all sorts of results about binary quadratic forms (and of positive and negative discriminant), we can proceed working (more-or-less) only with the character χ, though based on what we know about binary quadratic forms. The extra observation to add to the analysis of the previous section is that we should work with the values of all binary quadratic forms of discriminant d, simultaneously, since Gauss showed that the total number of "inequivalent" representations of n is then $\sum_{m|n}\chi(m)$. Hence let $w(n)=\sum_{m|n}\chi(m)$, so that $w(p)=1+\chi(p)$. We define

$$A(x;q,a)=\sum_{\substack{n\le x\\ n\equiv a \pmod q}} w(n).$$

Exercise 6.2 Show that if f is totally multiplicative and $g=1*f$ then

$$g(mn)=\sum_{d|(m,n)}\mu(d)f(d)g(m/d)g(n/d).$$

As usual $A_m(x;q,a):=\sum_n w(n)$ where the sum is over $n\le x$ with $M|n$ and $n\equiv a \pmod q$. Hence, using the exercise with $f=\chi$, if $(m,q)=1$ then

$$\begin{aligned}
A_m(x;q,a) &= \sum_{\substack{N\le x/m\\ N\equiv a/m \pmod q}} w(mN) = \sum_{\substack{N\le x/m\\ N\equiv a/m \pmod q}}\ \sum_{d|(m,N)}\mu(d)\chi(d)w(m/d)w(N/d)\\
&= \sum_{d|m}\mu(d)\chi(d)w(m/d)\sum_{\substack{N\le x/m\\ N\equiv a/m \pmod q\\ d|N}} w(N/d)\\
&= \sum_{d|m}\mu(d)\chi(d)w(m/d)A(x/md;q,a/md).
\end{aligned}$$

Now $w(n)=\sum_{m|n,\ m\le\sqrt{n}}\chi(m)+\sum_{m|n,\ m<\sqrt{n}}\chi(n/m)$. Therefore

$$\begin{aligned}
A(x;q,a) &= \sum_{\substack{n\le x\\ n\equiv a \pmod q}}\left(\sum_{m|n,\ m\le\sqrt{n}}\chi(m)+\sum_{m|n,\ m\le\sqrt{n}}\chi(n/m)\right)\\
&= \sum_{\substack{m\le x\\ (m,q)=1}}\chi(m)\sum_{\substack{m^2\le n\le x\\ n\equiv a \pmod q\\ m|n}}1+\chi(a)\sum_{\substack{m\le x\\ (m,q)=1}}\overline{\chi}(m)\sum_{\substack{m^2<n\le x\\ n\equiv a \pmod q\\ m|n}}1\\
&= \frac{1}{q}\sum_{\substack{m\le\sqrt{x}\\ (m,q)=1}}(\chi(m)+\chi(a)\overline{\chi}(m))\left(\frac{x}{m}-m+O(1)\right).
\end{aligned}$$

Now $\sum_{m \pmod q}(kq+m)\chi(m) = \sum_{m \pmod q} m\chi(m) \ll q^{3/2}$. Moreover $\sum_{m\le M}\chi(m)/m = L(1,\chi)+O(q/M)$, and so $A(x;q,a) = (1+\chi(a))L(1,\chi)x/q + O(q\sqrt{x})$ since χ is real. Hence, if m is squarefree and coprime to q, and $\chi(a)=1$ then

$$A_m(x;q,a) = L(1,\chi)\frac{x}{mq}\sum_{d|m}\frac{\mu(d)\chi(d)}{d}w(m/d)(1+\chi(a/md)) + O\left(\frac{q}{m}\sum_{d|m}w(m/d)\sqrt{mx/d}\right)$$
$$= 2L(1,\chi)\frac{x}{mq}\prod_{p|m}\left(1+\chi(p)\left(1-\frac{1}{p}\right)\right) + O\left(\frac{q}{m}\sqrt{x}\prod_{p|m}(1+(1+\chi(p))\sqrt{p})\right).$$

Hence if we write $A_m(x;q,a) = (g(m)/m)A(x;q,a) + r_m(x;q,a)$ then g is a multiplicative function with $g(p) = 1+\chi(p)\left(1-\frac{1}{p}\right)$ and

$$\sum_{m\le M}|r_m(x;q,a)| \ll q\sqrt{Mx}\sum_{m\le M}\frac{1}{m}\prod_{p|m}(1+\chi(p)+1/\sqrt{p}) \ll q\sqrt{Mx}\log^2 M.$$

Sieving Lemma

Lemma 6.6 *(Standard sieving lemma) Suppose that a_n are a set of real weights supported on a finite set of integers n. Let $A(x)=\sum_n a_n$ and suppose that there exists a non-negative multiplicative function $g(.)$ such that*

$$A_m(x) = \sum_{n:\ m|n} a_n = \frac{g(m)}{m}A(x) + r_m(x)$$

for all squarefree m, for which there exists $K,\kappa>0$ such that

$$\prod_{y<p\le z}\left(1-\frac{g(p)}{p}\right)^{-1} \le K\left(\frac{\log z}{\log y}\right)^{\kappa},$$

for all $2\le y<z\le x$. Let P be a given set of primes, and $P(z)$ be the product of the elements of P that are $\le z$. Then

$$\sum_{\substack{n\le x\\(n,P(z))=1}} a_n = \left\{1+O_{K,\kappa}(e^{-u})\right\}\prod_{\substack{p\in P\\p\le z}}\left(1-\frac{g(p)}{p}\right)\sum_{n\le x}a_n + O\left(\sum_{\substack{m|P(z)\\m\le z^u}}|r_m(x)|\right)$$

Above we let $x \gg q^5/L(1,\chi)^2$ and $z=x^{\epsilon}$, with u large and ϵu small, and then apply Lemma 6.6 with $\kappa=2$ to obtain

$$\sum_{\substack{n\le x\\n\equiv a \pmod q\\(n,P(z))=1}} w(n) = \left\{1+O(e^{-u})\right\}\prod_{p\le z}\left(1-\frac{1}{p}\right)\left(1-\frac{\chi(p)}{p}\right)A(x;q,a).$$

Now for each primes $p, z<p\le\sqrt{x}$ we must remove from the left side those n divisible by p. For each prime p write $n=Np$ and so we get an upper bound from

$w(p)$ times the sum of $w(N)$ over $N \leq x/p,\ N \equiv a/p \pmod q$ and $(N, P(z)) = 1$. Since $x/p \geq \sqrt{x}$, we can get an upper bound from the same estimate, of the right side with x/p in place of x; that is divided by p. Hence we deduce that

$$\sum_{\substack{p \leq x \\ p \equiv a \pmod q \\ p \text{ prime}}} w(p) = \left\{ 1 + O\left(e^{-u} + \sum_{z < p \leq \sqrt{x}} \frac{1 + \chi(p)}{p} \right) \right\} \prod_{p \leq z} \left(1 - \frac{1}{p}\right) \left(1 - \frac{\chi(p)}{p}\right) A(x; q, a).$$

In the last section we explained that $\sum_{z < p \leq \sqrt{x}} \frac{1+\chi(p)}{p} \ll \left(\frac{\log Q}{\log z}\right)^{1/5}$, and hence we have proved that

$$\pi(x; q, a) = \{1 + o_{L \to \infty}(1)\} \prod_{p \leq z} \left(1 - \frac{1}{p}\right) \left(1 - \frac{\chi(p)}{p}\right) L(1, \chi)\ \frac{x}{q},$$

where $x = q^L$ and $z = q^{\sqrt{L}}$.