

# THE DISTRIBUTION OF THE ZEROES OF RANDOM TRIGONOMETRIC POLYNOMIALS

ANDREW GRANVILLE AND IGOR WIGMAN

ABSTRACT. We study the asymptotic distribution of the number  $Z_N$  of zeros of random trigonometric polynomials of degree  $N$  as  $N \rightarrow \infty$ . It is known that as  $N$  grows to infinity, the expected number of the zeros is asymptotic to  $\frac{2}{\sqrt{3}} \cdot N$ . The asymptotic form of the variance was predicted by Bogomolny, Bohigas and Leboeuf to be  $cN$  for some  $c > 0$ . We prove that  $\frac{Z_N - \mathbb{E}Z_N}{\sqrt{cN}}$  converges to the standard Gaussian. In addition, we find that the analogous result is applicable for the number of zeros in short intervals.

## 1. INTRODUCTION

The distribution of zeros of random functions for various ensembles is one of the most studied problems. Of the most significant and important among those is the ensemble of random trigonometric polynomials, as the distribution of its zeros occurs in a wide range of problems in science and engineering, such as nuclear physics (in particular, random matrix theory), statistical mechanics, quantum mechanics, theory of noise etc.

**1.1. Background.** Understanding the distribution of zeros of random functions was first pursued by Littlewood and Offord [LO1], [LO2] and [LO3]. They considered, in particular, the distribution of the number of real roots of polynomials

$$(1) \quad P_N(x) = a_0 + a_1x + \dots + a_Nx^N,$$

of degree  $N$  with random coefficients  $a_n$ , as  $N \rightarrow \infty$ . For the coefficients  $a_n$  taking each of the values  $1, -1$  with equal probability  $1/2$ , they showed that the number  $Z_{P_N}$  of zeros of  $P_N(x)$  satisfies

$$(2) \quad Z_{P_N} \sim \frac{2}{\pi} \ln N$$

for  $(1 - o_{N \rightarrow \infty}(1))2^N$  of the vectors  $\vec{a} \in \{\pm 1\}^N$ . Later, Erdos and Offord [EO] refined their estimate.

Kac [K] proved that the expected number of zeros  $Z_{P_N}$  of the random polynomials (1) of degree  $N$ , this time  $a_n$  being Gaussian i.i.d. with mean 0 and variance 1, is asymptotic to the same expression (2). His result was generalized by Ibragimov and Maslova [IM1] and [IM2], who treated any distributions of the coefficients  $a_n$ , provided that they belong to the domain of attraction of the normal law: if each  $\mathbb{E}a_n = 0$  then the expectation is

---

I.W. is supported by a CRM ISM fellowship, Montréal and the Knut and Alice Wallenberg Foundation, grant KAW.2005.0098, Stockholm.

again asymptotic to (2), though, if  $\mathbb{E}a_n \neq 0$  one expects half as many zeros as in the previous case, that is

$$\mathbb{E}Z_{P_N} \sim \frac{1}{\pi} \ln N.$$

Maslova [M1] also established the only heretofore known asymptotics for the variance of the number of *real* zeros  $Z$ ,

$$\text{Var}Z_{P_N} \sim \frac{4}{\pi} \left(1 - \frac{2}{\pi}\right) \cdot \ln N$$

for the ensemble (1) of random functions. In her next paper [M2], she went further to establish an even more striking result, proving the normal limiting distribution for  $Z_{P_N}$ , as  $N \rightarrow \infty$ .

The case of random *trigonometric* polynomials was considered by Dunnage [DN]. Let  $T_N : [0, 2\pi] \rightarrow \mathbb{R}$  be defined by

$$(3) \quad T_N(t) = \sum_{n=1}^N a_n \cos nt,$$

where  $a_n$  are standard Gaussian i.i.d, and  $Z_{T_N}$  be the number of zeros of  $T_N$  on  $[0, 2\pi]$ . Dunnage proved that as  $N \rightarrow \infty$ ,  $\mathbb{E}Z_{T_N}$  is asymptotic to

$$\mathbb{E}Z_{T_N} \sim \frac{2}{\sqrt{3}}N,$$

and, moreover, that the distribution is concentrated around the expectation in the same sense as Littlewood and Offord mentioned earlier.

The variance of the zeros for (3) was shown by Farahmand to be

$$O(N^{24/13} \log^{16/13} N)$$

in [F1] and then  $O(N^{3/2})$  in [F2]. Either of those estimates imply that the distribution of  $Z_{T_N}$  concentrates around the mean.

Qualls [Q] considered a slightly different class of trigonometric polynomials,

$$X_N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^N \left( a_n \sin nt + b_n \cos nt \right).$$

Let  $Z_{X_N}$  be the number of the zeros of  $X_N$  on  $[0, 2\pi]$ . Applying the theory of stationary processes on  $X_N$ , one finds that

$$\mathbb{E}Z_{X_N} = 2\sqrt{\frac{(N+1)(2N+1)}{6}} \sim \frac{2}{\sqrt{3}}N,$$

similar to (3). Qualls proved that

$$\left| Z_{X_N} - \mathbb{E}Z_{X_N} \right| \leq C \cdot N^{3/4}$$

for some  $C > 0$  with probability  $1 - o_{N \rightarrow \infty}(1)$  (improving on earlier work of Farahmand).

Bogomolny, Bohigas and Leboeuf [BBL]<sup>1</sup> argued that the *variance* of  $Z_{X_N}$  satisfies

$$\text{Var}(Z_{X_N}) \sim cN,$$

<sup>1</sup>We wish to thank Jonathan Keating for pointing out this reference

as  $N \rightarrow \infty$ , where  $c$  is a positive constant approximated by

$$(4) \quad c \approx 0.55826$$

(this is equivalent to the formula (3.34) in [BBL] and the numeric value of  $\Delta \approx 0.44733$  immediately afterwards; one should bear in mind that they normalize the random variable to have unit expectancy).

In this paper we study the distribution of the random variable  $Z_{X_N}$  in more detail. We will find the asymptotics of the variance  $\text{Var}(Z_{X_N})$  as well as prove the central limit theorem for the distribution of  $Z_{X_N}$  (see section 1.2). We guess, but have not proved, that the same result may be true for Dunnage's ensemble (3).

The zeros of random *complex analytic* functions were examined in a series of papers by Sodin-Tsirelson (see e.g. [ST]), and Shiffman-Zelditch (see e.g. [SZ]).

**1.2. Statement of results.** Let  $X_N : [0, 2\pi] \rightarrow \mathbb{R}$  be Qualls' ensemble

$$(5) \quad X_N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^N a_n \sin nt + b_n \cos nt,$$

where  $a_n$  and  $b_n$  are standard Gaussian i.i.d.

As usual, given a random variable  $Y$ , we define  $\mathbb{E}(Y)$  to be the expectation of  $Y$ . For example, for any *fixed*  $t \in [0, 2\pi]$  and  $N$ , one has

$$\mathbb{E}(X_N(t)^2) = 1.$$

Note that  $X_N$  and its derivatives have at most  $2N$  zeros, fact that we will find useful later. Above we noted Qualls' result that

$$(6) \quad \mathbb{E}(Z_{X_N}) = 2\sqrt{\lambda_2},$$

where

$$\lambda_2 := \frac{1}{N} \sum_{n=1}^N n^2 = \frac{(2N+1)(N+1)}{6}.$$

We prove the central limit theorem for  $Z_{X_N}$ :

**Theorem 1.1.** *There exists a constant  $c > 0$  such that the distribution of*

$$\frac{Z_{X_N} - \mathbb{E}Z_{X_N}}{\sqrt{cN}}$$

*converges weakly to the standard Gaussian  $N(0, 1)$ . The variance is asymptotic to*

$$(7) \quad \text{Var}(Z_{X_N}) \sim cN,$$

*as  $N \rightarrow \infty$ , as predicted by Bogomolny-Bohigas-Leboeuf.*

We can compute the value of the constant  $c$  in Theorem 1.1 as

$$c = \frac{4}{3\pi}c_0 + \frac{2}{\sqrt{3}},$$

with

$$(8) \quad c_0 = \int_0^\infty \left[ \frac{(1-g(x)^2) - 3g'(x)^2}{(1-g(x)^2)^{3/2}} (\sqrt{1-R^{*2}} + R^* \arcsin R^*) - 1 \right],$$

where we denote

$$(9) \quad g(x) := \frac{\sin x}{x}$$

and

$$(10) \quad R^*(x) := \frac{g''(x)(1-g(x)^2) + g(x)g'(x)^2}{\frac{1}{3}(1-g(x)^2) - g'(x)^2}.$$

More generally, for  $0 \leq a < b \leq 2\pi$  one defines  $Z_{X_N}(a, b)$  to be the number of zeros of  $X_N$  on the subinterval  $[a, b] \subseteq [0, 2\pi]$ . It is easy to generalize the computation of the expectation (6) for this case as

$$\mathbb{E}Z_{X_N}(a, b) = \frac{\sqrt{\lambda_2}}{\pi} \cdot (b - a)$$

A priori, it seems that the behaviour of the number of zeros of  $X_N$  in short intervals  $[a_N, b_N]$ , shrinking as  $N \rightarrow \infty$ , should be more erratic than on the full interval. Surprisingly, just as in the previous case, we are able to find a precise asymptotics for the variance  $\text{Var}Z_N(a_N, b_N)$ , and prove a central limit theorem, provided that  $[a_N, b_N]$  does not shrink too rapidly. We have the following Theorem:

**Theorem 1.2.** *Let  $0 \leq a_N < b_N \leq 2\pi$  be any sequences of numbers with  $N \cdot (b_N - a_N) \rightarrow \infty$ . Then as  $N \rightarrow \infty$ ,*

$$\text{Var}(Z_{X_N}(a_N, b_N)) \sim c \cdot \frac{(b_N - a_N)}{2\pi} N,$$

where  $c$  is the same constant as in Theorem 1.1. Moreover,

$$\frac{Z_{X_N}(a_N, b_N) - \mathbb{E}Z_{X_N}(a_N, b_N)}{\sqrt{c \frac{(b_N - a_N)}{2\pi} N}}$$

converges weakly to the standard Gaussian  $N(0, 1)$ .

The proof of Theorem 1.2 is identical to the proof of Theorem 1.1, and in this paper we will give only the proof of Theorem 1.1.

The multi-dimensional analogue of (5) (for dimension  $d \geq 2$ ), was considered by Rudnick and Wigman [RW]. For example, for  $d = 2$ , they study the zero set of

$$(11) \quad X_n(\vec{x}) = \sum_{\|\vec{\lambda}\|^2=n} a_{\vec{\lambda}} \cos(2\pi\langle \vec{x}, \vec{\lambda} \rangle) + b_{\vec{\lambda}} \sin(2\pi\langle \vec{x}, \vec{\lambda} \rangle),$$

for  $n \in \mathbb{Z}$ ,  $\vec{x} = (x_1, x_2) \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ , and  $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ . This is the ensemble of *random eigenfunctions* of the Laplacian on the torus, with eigenvalue

$$E := -4\pi^2 n,$$

and we denote its multiplicity by  $N$ . The zero set of (11) is a random curve  $C_X$  on the torus, called the *nodal line*, and one is interested in the distribution of its length  $L_{C_X}$ . The authors compute the expectation  $\mathbb{E}L_{C_X}$  of the length of  $C_X$  to be

$$\mathbb{E}L = a \cdot \sqrt{E} \text{ and } \text{Var}L = O_{N \rightarrow \infty} \left( \frac{E}{\sqrt{N}} \right)$$

for some constant  $a > 0$ .

**1.3. Plan of the paper.** To prove the central limit theorem we will first need to establish the asymptotic form (7) for the variance. This is done throughout sections 2 and 3: in section 2 we develop an integral formula for the second moment of the number of zeros, and in section 3 we exploit it to study the asymptotics of the variance.

Sections 4 and 5 are dedicated to the proof of the main statement of Theorem 1.1, that is the central limit theorem. While the proof is contained in section 4, a certain result, required by the proof, is proven throughout section 5.

**1.4. On the proof of Theorem 1.1.** As an initial step for the central limit theorem, we will have to find the asymptotics (7) for the variance. This is done throughout sections 2 and 3.

While computing the asymptotic of the variance of  $Z_{X_N}$ , we determined that the *covariance function*  $r_N$  of  $X_N$  has a scaling limit  $r_\infty(x) = \frac{\sin x}{x}$ , which proved useful for the purpose of computing the asymptotics. Rather than scaling  $r_N$ , one might consider scaling  $X_N$ .

We realize, that the above *should* mean, that the distribution of  $Z_{X_N}$  is intimately related to the distribution of the number  $\tilde{Z}_N$  of the zeros on (roughly)  $[0, N]$  of a certain Gaussian stationary process  $Y(x)$ , defined on the real line  $\mathbb{R}$ , with covariance function  $r = r_\infty$  (see section 4.1). Intuitively, this should follow, for example, from the approach of [GS], see e.g. Theorem 9.2.2, page 450. Unfortunately, this approach seems to be difficult to make rigorous, due to the different scales of the processes involved.

The latter problem of the distribution of the number of the zeros (and various other functionals) on growing intervals is a classical problem in the theory of stochastic processes. Malevich [ML] and subsequently Cuzick [CZ] prove the central limit theorem for  $\tilde{Z}_N$ , provided that  $r$  lies in some (rather wide) class of functions, which include  $r_\infty$ . Their result was generalized in a series of papers by Slud (see e.g. [SL]), and the two-dimensional case was treated by Kratz and Leon [KL].

We modify the proof of Malevich-Cuzick to suit our case. There are several marked differences between our case and theirs. In their work, one has to deal with growing sums of identically distributed (but by no means independent) random variables (which will be referred to as a *linear system*); to prove the central limit theorem one applies a result due to Diananda [DN]. In our case, we deal with *triangular systems* (to be defined), applying a theorem of Berk [BR]. For more details about the proof, see section 4.3.

**1.5. Acknowledgements.** The second author wishes to thank Zeev Rudnick for suggesting the problem as well as his help and support while conducting the research. In addition, he wishes to thank Mikhail Sodin, Ildar A. Ibragimov, Pär Kurlberg and Iosif Polterovich for many fruitful discussions. We are grateful to Phil Sosoe for conducting some empirical experiments available in the longer version of this paper. The authors wish to thank the anonymous referees for many useful comments and suggestions how to improve the paper.

## 2. A FORMULA FOR THE SECOND MOMENT

**Proposition 2.1.** *We have*

$$(12) \quad \mathbb{E}(Z_{X_N}^2) - \mathbb{E}(Z_{X_N}) = \frac{2}{\pi} \int_0^{2\pi} \frac{\lambda_2(1-r^2) - (r')^2}{(1-r^2)^{3/2}} \left( \sqrt{1-\rho^2} + \rho \arcsin \rho \right) dt,$$

where

$$(13) \quad \lambda_2 = \lambda_{2,N} = \frac{1}{N} \sum_{n=1}^N n^2 = \frac{(N+1)(2N+1)}{6},$$

$$(14) \quad r(t) = r_{X_N}(t) = \frac{1}{N} \sum_{n=1}^N \cos nt = \frac{1}{2N} \left[ \frac{\sin(N+1/2)t - \sin t/2}{\sin(t/2)} \right]$$

and

$$(15) \quad \rho = \rho_N(t) = \frac{r''(1-r^2) + (r')^2 r}{\lambda_2(1-r^2) - (r')^2}.$$

A similar but less explicit formula was obtained by Steinberg et al. [SSWZ]. The rest of this section is dedicated to the proof of this result.

The ensemble  $X_N$  is a centered stationary Gaussian process (meaning that the finite dimensional distributions are Gaussian with zero mean). An explicit computation with the double angle formula shows that its covariance function is

$$r_{X_N}(t_1, t_2) = r_{X_N}(t_2 - t_1),$$

with the function on the right side as defined in (14).

Let  $I$  be an interval and  $X : I \rightarrow \mathbb{R}$  be a mean zero stationary process with covariance function  $r$ . We assume that  $r(0) = 1$  (i.e.  $X$  has unit variance) and furthermore that the sample functions of  $X$  are a.s. sufficiently smooth (e.g. twice differentiable) so that its sets of zeros is discrete. We have

$$(16) \quad |r(t)| \leq 1$$

for every  $t \in I$  by the Cauchy-Schwartz inequality. We denote  $Z$  to be the number of zeros of  $X$  on  $I$ ;  $Z$  is a random variable.

In general, we have the following celebrated Kac-Rice formula (see e.g. [CL]) for the expectation of  $Z$ :

$$\mathbb{E}Z = \frac{|I|}{\pi} \sqrt{\lambda_2},$$

where  $|I|$  is the length of  $I$  (finite or infinite), and  $\lambda_2 = -r''(0)$ . As mentioned earlier, it was exploited by Qualls to compute the expected number (6) of zeros of trigonometric polynomials.

In this section we find a formula for the second moment  $\mathbb{E}Z_X^2$  of the number of zeros of any Gaussian stationary process  $X$  on  $I$ , assuming that its covariance function  $r$  is smooth. To determine  $\mathbb{E}(Z_X^2)$ , we naturally encounter the distribution of the random vector

$$(17) \quad V = V_{t_1, t_2} := (X(t_1), X(t_2), X'(t_1), X'(t_2)).$$

for some fixed  $t_1, t_2 \in I$ . As an artifact of the stationarity of  $X$ , the distribution of  $V$  depends only on  $t := t_2 - t_1$ . The covariance matrix of  $V$  is

$$(18) \quad \Sigma = \Sigma(t) := \begin{pmatrix} 1 & r(t) & 0 & r'(t) \\ r(t) & 1 & -r'(t) & 0 \\ 0 & -r'(t) & \lambda_2 & -r''(t) \\ r'(t) & 0 & -r''(t) & \lambda_2 \end{pmatrix} =: \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}.$$

The random vector  $V$  has a multivariate normal distribution with mean zero and covariance matrix  $\Sigma$ . For  $t \in (0, 2\pi)$ ,  $\Sigma(t)$  is nonsingular (for  $X_N$  see [Q] and Remark 2.3).

**Lemma 2.2.** *Let  $X$  be a Gaussian stationary process, which almost surely has a continuous sample derivative such that the distribution of  $V$  is non-degenerate for  $t_1 \neq t_2$ . Then*

$$(19) \quad \begin{aligned} & \mathbb{E}(Z_X^2) - \mathbb{E}(Z_X) \\ &= \iint_{[0, 2\pi] \times [0, 2\pi]} dt_1 dt_2 \iint_{\mathbb{R}^2} |y_1| |y_2| \phi_{t_1, t_2}(0, 0, y_1, y_2) dt_1 dt_2, \end{aligned}$$

where  $\phi_{t_1, t_2}(u_1, u_2, v_1, v_2)$  is the probability density function of  $V_{t_1, t_2}$ .

**Remark 2.3.** The original formulation of Lemma 2.2 from [CL] assumes that the Fourier transform of  $r$  has a continuous component, a stronger condition than stated. However, their proof works just as well in the less restrictive case as formulated above. Qualls proved that the trigonometric polynomials (5) satisfy this assumption and thus we may apply Lemma 2.2 to (5). Qualls' argument can be generalized to higher moments: we use it to bound the third moment in Proposition A.1.

**Remark 2.4.** Let  $\psi_{t_1, t_2}$  be the probability density function of the random vector  $(X'(t_1), X'(t_2))$  conditional on  $X(t_1) = X(t_2) = 0$ . Then we have

$$(20) \quad \phi_{t_1, t_2}(0, 0, y_1, y_2) = \frac{\psi_{t_1, t_2}(y_1, y_2)}{2\pi\sqrt{1 - r(t_2 - t_1)^2}}$$

(see also (25)). Therefore we may rewrite (19) as

$$\mathbb{E}(Z_X^2) - \mathbb{E}(Z_X) = \iint_{I \times I} \frac{\mathbb{E}[|X'(t_1)X'(t_2)| | X(t_1) = X(t_2) = 0]}{\sqrt{1 - r(t_2 - t_1)^2}} \frac{dt_1 dt_2}{2\pi}.$$

We use this representation in the proof of Proposition 4.3, as well as its analogue for the third moment in the proof of Proposition A.1.

We use Lemma 2.2 to derive the following.

**Corollary 2.5.** *Under the assumptions of Lemma 2.2, one has*

$$(21) \quad \mathbb{E}(Z_{X_N}^2) - \mathbb{E}(Z_{X_N}) = \iint_{I \times I} \frac{\lambda_2(1 - r^2) - (r')^2}{(1 - r^2)^{3/2}} (\sqrt{1 - \rho^2} + \rho \arcsin \rho) \frac{dt_1 dt_2}{\pi^2},$$

where  $r = r_X(t_2 - t_1)$ , and  $\rho = \rho_X(t_2 - t_1)$  with

$$\rho_X(t) = \frac{r''(t)(1 - r(t)^2) + r'(t)^2 r(t)}{\lambda_2(1 - r(t)^2) - r'(t)^2}.$$

*Proof.* Direct matrix multiplication confirms that

$$\Sigma^{-1} = \begin{pmatrix} (A - BC^{-1}B^t)^{-1} & -A^{-1}B(C - B^tA^{-1}B)^{-1} \\ -C^{-1}B^t(A - BC^{-1}B^t)^{-1} & (C - B^tA^{-1}B)^{-1} \end{pmatrix},$$

so if  $\Omega$  is the  $2 \times 2$  “reduced covariance matrix”, that is  $\Omega^{-1}$  is the bottom right corner of  $\Sigma^{-1}$ , then

$$(22) \quad \Omega = C - B^tA^{-1}B.$$

The matrix  $\Omega$  is the covariance matrix of the random vector  $(X'(t_1), X'(t_2))$  conditioned upon  $X(t_1) = X(t_2) = 0$ .

Computing (22) explicitly, we have

$$\Omega = \mu\Omega_1,$$

where

$$\Omega_1 = \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$$

with  $\rho$  given by (15) and

$$(23) \quad \mu := \frac{\lambda_2(1 - r^2) - (r')^2}{1 - r^2} > 0.$$

Since  $\Omega$  is a covariance matrix, we have

$$(24) \quad |\rho| \leq 1$$

by the Cauchy-Schwartz inequality.

The easy to check identity

$$\Sigma = \begin{pmatrix} A & 0 \\ B^t & I \end{pmatrix} \cdot \begin{pmatrix} I & A^{-1}B \\ 0 & \Omega \end{pmatrix},$$

yields

$$(25) \quad \det \Sigma = \det A \det \Omega = (1 - r^2)\mu^2(1 - \rho^2).$$

Using (19) and the explicit form of the Gaussian density  $\phi_{t_1, t_2}$ , we obtain

$$(26) \quad \begin{aligned} \mathbb{E}(Z_X^2) - \mathbb{E}(Z_X) &= \iint_{I^2} dt_1 dt_2 \iint_{\mathbb{R}^2} |y_1||y_2| \frac{\exp(-\frac{1}{2}y\Omega^{-1}y^t)}{\sqrt{\det \Sigma}} \frac{dy_1 dy_2}{(2\pi)^2}, \\ &= \iint_{I^2} \frac{dt_1 dt_2}{\mu \sqrt{(1 - r^2)(1 - \rho^2)}} \iint_{\mathbb{R}^2} |y_1||y_2| \exp\left(-\frac{1}{2}\mu^{-1}y\Omega_1^{-1}y^t\right) \frac{dy_1 dy_2}{(2\pi)^2} \\ &= \iint_{I^2} \frac{\mu}{\sqrt{(1 - r^2)(1 - \rho^2)}} dt_1 dt_2 \iint_{\mathbb{R}^2} |z_1||z_2| \exp\left(-\frac{1}{2}z\Omega_1^{-1}z^t\right) \frac{dz_1 dz_2}{(2\pi)^2}, \end{aligned}$$

where  $y = (y_1, y_2)$ , making the change of coordinates  $z = \frac{y}{\sqrt{\mu}}$ . The inner integral is

$$(27) \quad \begin{aligned} &\iint_{\mathbb{R}^2} |z_1||z_2| \exp\left(-\frac{1}{2(1 - \rho^2)}(z_1^2 + 2\rho z_1 z_2 + z_2^2)\right) dz \\ &= 4(1 - \rho^2) \left(1 + \frac{\rho}{\sqrt{1 - \rho^2}} \arcsin \rho\right), \end{aligned}$$



computed by Bleher and Di [BD], appendix A. Substituting this in (26) gives our result.  $\square$

We are finally in the position to prove Proposition 2.1.

*Concluding the proof of Proposition 2.1.* We use Corollary 2.5 on the trigonometric polynomials  $X_N$ . The integrand in (21) depends only on  $t := t_2 - t_1$  (because of the stationarity of  $X_N$ ), which allows us to convert the double integral into a simple one. Proposition 2.1 then follows from the periodicity of the integrand.  $\square$

### 3. ASYMPTOTICS FOR THE VARIANCE

Formulas (12) and (6) imply the following formula for the variance

$$(28) \quad \text{Var}(Z_X) = J + \mathbb{E}(Z_X),$$

where

$$(29) \quad J := \frac{2}{\pi} \int_0^{2\pi} \left[ \frac{\lambda_2(1-r^2) - (r')^2}{(1-r^2)^{3/2}} \left( \sqrt{1-\rho^2} + \rho \arcsin \rho \right) - \lambda_2 \right] dt$$

#### 3.1. Evaluating the integral $J$ in (29).

**Proposition 3.1.**

$$(30) \quad J = \frac{4c_0}{3\pi} N \left( 1 + O\left(\frac{\log N}{N}\right)^{1/13} \right),$$

where  $c_0$  is defined by (8).

Our key observation is that  $r_{X_N}$  has a scaling limit, or, more precisely, we have

$$f_N(x) := r_{X_N}\left(\frac{x}{m}\right) = \frac{\sin x}{x} + \text{small error}$$

with  $m = N + \frac{1}{2}$  and  $x \in [0, m\pi]$  (treating  $[m\pi, 2m\pi]$  by symmetry), at least outside a small interval around the origin. It is therefore natural to change the integration variable in (29) from  $mt$  to  $x$ , which will recover the asymptotics for  $J$  being linear with  $m$ , and thus also with  $N$  (see (48))<sup>2</sup>.

We will argue that it is possible, up to an admissible error, to replace the  $f = f_N$  in the resulting integrand by  $g(x) := \frac{\sin x}{x}$ , the latter being  $N$ -independent. The decay at infinity of the new integrand (i.e. with  $f$  replaced by  $g$ ) imply that the integral will converge to a constant intimately related to (8).

We divide the new domain of integration  $[0, \pi m]$  (which is an artifact of changing the variable of integration  $x = mt$ ; we also use the symmetry around  $t = \pi$ ) into two ranges. Lemma 3.3 will bound the contribution of a small neighbourhood  $[0, \delta]$  of the origin. The main contribution to the integral (48) results from  $[\delta, \pi m]$ , where Lemma 3.4 will allow us to replace  $f_N$  in the integrand with  $N$ -independent  $g(x) = \frac{\sin x}{x}$ .

<sup>2</sup>In fact, rather than introducing a new parameter  $m$ , it is also possible to use  $x = Nt$ ; it results in a nastier computation.

**Notation 3.2.** In this manuscript we will use the notations  $A \ll B$  and  $A = O(B)$  interchangeably.

**Lemma 3.3.** *Let*

$$(31) \quad M(x) := \frac{\lambda'_2(1 - f(x)^2) - f'(x)^2}{(1 - f(x)^2)^{3/2}} (\sqrt{1 - R(x)^2} + R(x) \arcsin R(x)),$$

where

$$(32) \quad f(x) = f_N(x) = r_{X_N} \left( \frac{x}{m} \right) = \left( \frac{\sin x}{\sin \frac{x}{2m}} - 1 \right) \frac{1}{2m - 1},$$

$$(33) \quad R(x) = \rho_N \left( \frac{x}{m} \right) = \frac{f''(x)(1 - f(x)^2) + f(x)f'(x)^2}{\lambda'_2(1 - f(x)^2) - f'(x)^2}$$

and

$$(34) \quad \lambda'_2 = \frac{1 + \frac{1}{2m}}{3}.$$

There exists a universal  $\delta_0 > 0$ , such that for any

$$(35) \quad 0 < \delta < \delta_0,$$

we have the following estimate

$$\int_0^\delta M(x) dx = O(\delta^2),$$

where the constant involved in the “O”-notation is universal.

*Proof.* First we note that

$$(36) \quad |R(x)| \leq 1$$

by the definition (33) of  $R(x)$  and (24), so that the definition of  $M(x)$  makes sense.

We have to estimate  $f(x)$  and its derivative around the origin. Expanding  $f$  and  $f'$  into Taylor polynomial around  $x = 0$ , we have

$$f(x) = 1 + a_m x^2 + b_m x^4 + O(x^6),$$

and

$$f'(x) = 2a_m x + 4b_m x^3 + O(x^5).$$

with

$$a_m := -\frac{2m + 1}{12m} = O(1),$$

$$b_m := \frac{(2m + 1)(12m^2 - 7)}{2880m^3} = O(1),$$

and the constants in the ‘O’-notation being universal.

Thus,

$$1 - f(x)^2 = -2a_m^2 x^2 - (2b_m + a_m^2) x^4 + O(x^6) \gg x^2,$$

and

$$\begin{aligned} \lambda'_2(1 - f(x)^2) - f'(x)^2 &= \frac{1 + \frac{1}{2m}}{3} (-2a_mx^2 - (2b_m + a_m^2)x^4 + O(x^6)) \\ &\quad - 4x^2(a_m^2 + 4a_mb_mx^2 + O(x^4)) \\ &= \frac{64m^4 + 24m^3 - 108m^2 - 94m - 21}{8640m^4} x^4 + O(x^6) \ll x^4. \end{aligned}$$

Now

$$1 \leq \sqrt{1 - y^2} + y \arcsin y \leq \pi/2$$

for every  $y \in [-1, 1]$ , so combining the last three displayed equations, we obtain

$$M(x) \ll \frac{x^4}{x^3} = x,$$

and the Lemma follows.  $\square$

**Lemma 3.4.** *Let*

$$(37) \quad \delta > (m/2)^{-1/9}$$

and

$$\delta < x \leq \pi m.$$

Denote

$$M_1(x) := M(x) - \lambda'_2,$$

where  $M(x)$  is given by (31). Then, for  $m$  sufficiently large,

$$\begin{aligned} M_1(x) &= \frac{1}{3} \cdot \left[ \frac{(1 - g(x)^2) - 3g'(x)^2}{(1 - g(x)^2)^{3/2}} (\sqrt{1 - R^*(x)^2} + R^*(x) \arcsin R^*(x)) - 1 \right] \\ &\quad + O\left(\frac{1}{\delta^{12}mx} + \frac{1}{\delta^8 m^2}\right), \end{aligned}$$

where, as usual,  $g(x)$  and  $R^*$  are given by (9) and (10) respectively.

*Proof.* We will approximate  $f$  and its first couple of derivatives by  $g$  and its first couple of derivatives.

For  $\delta < x < \pi m$ , we have

$$(38) \quad \begin{aligned} f(x) &= \left( \frac{\sin x}{\frac{x}{2m}(1 + O(\frac{x^2}{m^2}))} - 1 \right) \frac{1}{2m - 1} \\ &= \left[ \frac{2m \cdot \sin(x)}{x} \left( 1 + O\left(\frac{x^2}{m^2}\right) \right) - 1 \right] \frac{1}{2m - 1} = g(x) + O\left(\frac{1}{m}\right), \end{aligned}$$

where to justify the second equality, we use the explicit coefficient of the second summand in the Taylor formula for the sine. For the derivatives we have, after performing a similar (but longer) computation shown in the longer version of this paper

$$(39) \quad f'(x) := g'(x) + O\left(\frac{x}{m^2}\right) + O\left(\frac{1}{mx}\right).$$

and

$$(40) \quad f''(x) = g''(x) + O\left(\frac{1}{\delta^2 mx} + \frac{x}{m^2}\right).$$

Next, we apply (38) to obtain

$$1 - f(x)^2 = (1 - g(x)^2) \cdot \left(1 + O\left(\frac{1}{\delta^2 mx}\right)\right),$$

where we used for  $x > \delta$ ,

$$1 - g(x)^2 \gg 1 - g(\delta)^2 = \frac{\delta^2 - \sin \delta^2}{\delta^2} \gg \delta^2.$$

Thus

$$(41) \quad (1 - f(x)^2)^{-3/2} = (1 - g(x)^2)^{-3/2} \cdot \left(1 + O\left(\frac{1}{\delta^2 mx}\right)\right),$$

where we used the assumption (37) to bound  $\frac{1}{\delta^2 mx}$  away from 1.

By the above, we have

$$(42) \quad \begin{aligned} \lambda'_2(1 - f(x)^2) - f'(x)^2 &= \lambda'_2(1 - g(x)^2) - g'(x)^2 + O\left(\frac{1}{\delta^2 mx}\right) \\ &= (\lambda'_2(1 - g(x)^2) - g'(x)^2) \left(1 + O\left(\frac{1}{\delta^6 mx}\right)\right), \end{aligned}$$

since for  $x > \delta$ ,

$$(43) \quad \lambda'_2(1 - g(x)^2) - g'(x)^2 \geq \frac{1}{3}(1 - g(x)^2) - g'(x)^2 \geq \frac{1}{3}(1 - g(\delta)^2) - g'(\delta)^2 \gg \delta^4.$$

Next, we have

$$f''(x)(1 - f(x)^2) + f(x)f'(x)^2 = g''(x)(1 - g(x)^2) + g(x)g'(x)^2 + O\left(\frac{1}{\delta^2 mx} + \frac{x}{m^2}\right),$$

using (37) again. Therefore,

$$R(x) = \frac{g''(x)(1 - g(x)^2) + g(x)g'(x)^2}{\lambda'_2(1 - g(x)^2) - g'(x)^2} + O\left(\frac{1}{\delta^6 mx} + \frac{x}{\delta^4 m^2}\right),$$

exploiting (37) once more as well as (43).

Note that

$$\begin{aligned} \frac{g''(x)(1 - g(x)^2) + g(x)g'(x)^2}{\lambda'_2(1 - g(x)^2) - g'(x)^2} &= \frac{g''(x)(1 - g(x)^2) + g(x)g'(x)^2}{\frac{1}{3}(1 - g(x)^2) - g'(x)^2 + O\left(\frac{1}{m}\right)} \\ &= \frac{g''(x)(1 - g(x)^2) + g(x)g'(x)^2}{\frac{1}{3}(1 - g(x)^2) - g'(x)^2} + O\left(\frac{1}{\delta^8 mx}\right), \end{aligned}$$

where we use twice (43) as well as (37) again.

All in all we obtain

$$(44) \quad R(x) = R^*(x) + O\left(\frac{1}{\delta^8 mx} + \frac{x}{\delta^4 m^2}\right),$$

where  $R^*$  is defined by (10). It is important to notice that (36) implies

$$|R^*(x)| \leq 1,$$

since for any *fixed*  $x$ ,

$$R(x) \rightarrow R^*(x),$$

as  $m \rightarrow \infty$  by (44) and (37).

Notice that

$$(45) \quad R^*(x) = O\left(\frac{1}{\delta^4 x}\right),$$

again by (43). Therefore for  $x \gg 1/\delta^4$ ,  $R^*(x)$  is bounded away from 1 so that for a small  $\epsilon > 0$  one has

$$\begin{aligned} \sqrt{1 - (R^* + \epsilon)^2} &= \sqrt{1 - R^*(x)} \cdot \sqrt{1 - \frac{2\epsilon R^*(x) + \epsilon^2}{1 - R^*(x)^2}} \\ &= \sqrt{1 - R^*(x)} \cdot \sqrt{1 - O(\epsilon R^*(x) + \epsilon^2)} = \sqrt{1 - R^*(x)} + O(\epsilon R^*(x) + \epsilon^2). \end{aligned}$$

It yields

$$\sqrt{1 - R(x)^2} = \sqrt{1 - R^*(x)^2} + O\left(\frac{1}{\delta^{12} m x^2} + \frac{1}{\delta^8 m^2}\right),$$

and similarly

$$R(x) \arcsin R(x) = R^*(x) \arcsin R^*(x) + O\left(\frac{1}{\delta^{12} m x^2} + \frac{1}{\delta^8 m^2}\right).$$

Therefore for  $x \gg 1/\delta^4$ , we have

$$(46) \quad \begin{aligned} &\sqrt{1 - R(x)^2} + R(x) \arcsin R(x) \\ &= \sqrt{1 - R^*(x)^2} + R^*(x) \arcsin R^*(x) + O\left(\frac{1}{\delta^{12} m x^2} + \frac{1}{\delta^8 m^2}\right). \end{aligned}$$

On the other hand, for  $\delta < x \ll \frac{1}{\delta^4}$ ,

$$\begin{aligned} &\sqrt{1 - R(x)^2} + R(x) \arcsin R(x) \\ &= \sqrt{1 - R^*(x)^2} + R^*(x) \arcsin R^*(x) + O\left(\frac{1}{\delta^8 m x} + \frac{x}{\delta^4 m^2}\right) \\ &= \sqrt{1 - R^*(x)^2} + R^*(x) \arcsin R^*(x) + O\left(\frac{1}{\delta^{12} m x} + \frac{1}{\delta^8 m^2}\right), \end{aligned}$$

since the derivative of the function

$$x \mapsto \sqrt{1 - x^2} + x \arcsin x,$$

namely  $\arcsin x$ , is bounded everywhere in  $[-1, 1]$ . Therefore (46) is valid for  $x > \delta$ .

Also we have

$$(47) \quad \sqrt{1 - R^*(x)^2} + R^*(x) \arcsin R^*(x) = 1 + O(R^*(x)^2) = 1 + O\left(\frac{1}{\delta^8 x^2}\right).$$

by (45).

Collecting (42) and (41), we have

$$\begin{aligned} &\frac{\lambda'_2(1 - f(x)^2) - f'(x)^2}{(1 - f(x)^2)^{3/2}} = \frac{\lambda'_2(1 - g(x)^2) - g'(x)^2}{(1 - g(x)^2)^{3/2}} \left(1 + O\left(\frac{1}{\delta^6 m x}\right)\right) \\ &= \frac{1/3(1 - g(x)^2) - g'(x)^2}{(1 - g(x)^2)^{3/2}} + \frac{1}{6m} + O\left(\frac{1}{\delta^6 m x}\right). \end{aligned}$$

Together with (46) it gives

$$\begin{aligned}
& \frac{\lambda'_2(1-f(x)^2) - f'(x)^2}{(1-f(x)^2)^{3/2}} \left( \sqrt{1-R(x)^2} + R(x) \arcsin R(x) \right) - \lambda'_2 \\
&= \frac{1/3(1-g(x)^2) - g'(x)^2}{(1-g(x)^2)^{3/2}} \left( \sqrt{1-R^*(x)^2} + R^*(x) \arcsin R^*(x) \right) \\
&+ \frac{1}{6m} \left( 1 + O\left(\frac{1}{\delta^8 x^2}\right) \right) - \frac{1}{3} - \frac{1}{6m} + O\left(\frac{1}{\delta^{12} m x} + \frac{1}{\delta^8 m^2}\right) \\
&= \frac{1}{3} \cdot \left[ \frac{(1-g(x)^2) - 3g'(x)^2}{(1-g(x)^2)^{3/2}} \left( \sqrt{1-R^*(x)^2} + R^*(x) \arcsin R^*(x) \right) - 1 \right] \\
&+ O\left(\frac{1}{\delta^{12} m x} + \frac{1}{\delta^8 m^2}\right),
\end{aligned}$$

by (47). □

*Proof of Proposition 3.1.* Noting that the integrand in (29) is symmetric around  $t = \pi$ , denoting  $m := N + \frac{1}{2}$  and changing the variable of integration  $mt$  to  $x$  in (29), we find that  $J$  is

$$(48) \quad J = \frac{4m}{\pi} \int_0^{\pi m} \left[ \frac{\lambda'_2(1-f(x)^2) - f'(x)^2}{(1-f(x)^2)^{3/2}} \left( \sqrt{1-R(x)^2} + R(x) \arcsin R(x) \right) - \lambda'_2 \right] dx,$$

where  $f$ ,  $R$  and  $\lambda'_2$  are defined in (32), (33) and (34).

We divide the interval into two ranges:  $I_1 := [0, \delta]$  and  $I_2 = [\delta, \pi m]$ , for some parameter  $\delta = \delta(m) > 0$ . On  $I_1$  we employ Lemma 3.3 to bound (from above) the total contribution of the integrand, whereas we invoke Lemma 3.4 to asymptotically estimate the integral on  $I_2$ . The constant  $\delta$  has to satisfy the constraint of Lemma 3.4, namely (37). The constraint of Lemma 3.3, (35), is satisfied for  $m$  sufficiently large, provided that  $\delta$  vanishes with  $m$ . To bound the contribution of  $\lambda'_2$  to the integral on  $I_1$ , we use the trivial estimate  $\lambda'_2 = O(1)$ .

Hence we obtain

$$\begin{aligned}
(49) \quad J &= \frac{4m}{3\pi} \int_0^{\pi m} \left[ \frac{(1-g(x)^2) - 3g'(x)^2}{(1-g(x)^2)^{3/2}} \left( \sqrt{1-R^*(x)^2} + R^*(x) \arcsin R^*(x) \right) - 1 \right] \\
&+ O\left(\frac{\log m}{\delta^{12}} + \frac{1}{\delta^8}\right) + O(\delta m).
\end{aligned}$$

Note that for a large  $x$  we have

$$(1-g(x)^2) - 3g'(x)^2 = 1 + O\left(\frac{1}{x^2}\right),$$

$$\frac{1}{(1-g(x)^2)^{3/2}} = 1 + O\left(\frac{1}{x^2}\right),$$

and the definition (10) implies

$$R^*(x) = O\left(\frac{1}{x^2}\right)$$

so that

$$\sqrt{1 - R^*(x)^2} + R^*(x) \arcsin R^*(x) = 1 + O(R^*(x)) = 1 + O\left(\frac{1}{x^2}\right).$$

Plugging the estimates above into the integrand of (49) shows that the integrand is  $O(\frac{1}{x^2})$ . Hence

$$\begin{aligned} J &= \frac{4N}{3\pi} \int_0^\infty \left[ \frac{(1 - g(x)^2) - 3g'(x)^2}{(1 - g(x)^2)^{3/2}} \left( \sqrt{1 - R^*(x)^2} + R^*(x) \arcsin R^*(x) \right) - 1 \right] \\ &\quad + O\left(\frac{\log N}{\delta^{12}} + \frac{1}{\delta^8}\right) + O(\delta N). \end{aligned}$$

We finally obtain the statement (30) of the present proposition upon choosing

$$\delta := \left(\frac{\log N}{N}\right)^{1/13}.$$

□

### 3.2. Concluding the proof of the variance part (7) of Theorem 1.1.

*Proof.* Proposition 2.1 together with Proposition 3.1 and (28) imply

$$\text{Var}(Z_X) = J + \mathbb{E}(Z_X) \sim \frac{4c_0}{3\pi}N + \frac{2}{\sqrt{3}}N = cN.$$

It then remains to show that  $c > 0$ . As mentioned earlier, Bogomolny-Bohigas-Lebeouf [BBL] estimated  $c \approx 0.55826$  and it is possible to use numerical methods to rigorously obtain  $c > 0$  (see the longer version of the present paper).

There exists a more systematic approach though. One can construct a Gaussian process  $Y_\infty(x)$  on  $\mathbb{R}$  with the covariance function  $r_\infty(x) = \frac{\sin x}{x}$  (see section 4.1). In this case, we denote again

$$\lambda_{2,\infty} = -r_\infty''(0) = \frac{1}{3}.$$

For  $T > 0$ , let  $Z_\infty(T)$  be the number of the zeros of  $Y_\infty$  on  $[0, T]$ .

By the general theory of stochastic processes developed in [CL], one has

$$\mathbb{E}Z_\infty(T) = \frac{T}{\pi}\lambda_{2,\infty}.$$

Using the same method we used to compute the variance of  $X_N$ , it is not difficult to see that

$$\text{Var}Z_\infty(T) \sim cT,$$

where  $c$  is the same constant as in Theorem 1.1, provided that  $c > 0$ . It was proved by Slud [SL], that it is indeed the case for a wide class of covariance

functions  $r(x)$  which contains our case  $r = r_\infty$ . Moreover, Slud (following Malevich [ML] and Cuzick [CZ]), established the central limit theorem for

$$\frac{Z_\infty(T) - \mathbb{E}Z_\infty(T)}{\sqrt{cT}}.$$

□

#### 4. PROOF OF THEOREM 1.1

In this section we pursue the proof of the central limit theorem. The main probabilistic tool we use is a result of Berk [BR], which establishes a central limit theorem for a triangular system of random variables defined below. We start however with a general remark about the zeros.

**4.1. A general remark about the distribution of the zeros.** Let  $Y_\infty : \mathbb{R} \rightarrow \mathbb{R}$  be a Gaussian stationary process with the covariance function  $r_\infty(x) = \frac{\sin x}{x}$ . Such a process exists<sup>3</sup>, since  $r_\infty$  has a nonnegative Fourier transform on  $\mathbb{R}$  (this is related to Bochner's theorem, see e.g. [CL], page 126). Moreover, we may assume with no loss of generality that  $Y_\infty$  is almost surely everywhere differentiable.

To define all the processes on the same space, we will assume that  $Y_N$  are defined on  $\mathbb{R}$  by periodicity. We have the convergence  $r_{Y_N}(x) \rightarrow r_{Y_\infty}(x)$ , as  $N \rightarrow \infty$ . This implies the convergence of all the finite-dimensional distributions of  $Y_N$  to the finite-dimensional distributions of  $Y_\infty$ . By theorem 9.2.2 [GS]<sup>4</sup>, for any continuous functional  $\phi : C([a, b]) \rightarrow \mathbb{R}$ , the distribution of  $\phi(Y_N)$  converges to the distribution of  $\phi(Y_\infty)$ . Thus one could model a “generic” statistic of  $X_N$  by the corresponding statistic of  $Y_\infty$  on intervals, growing linearly with  $N$ .

The convergence  $r_N \rightarrow r_\infty$  suggests that the distribution of the number of zeros of  $X_N$  on the fixed interval  $[0, 2\pi]$  is intimately related to the distribution of the number of zeros of the fixed process  $Y_\infty$  on growing intervals. The particular case of the latter problem when the process is Gaussian stationary (which is the case in this paper), has been studied extensively over the past decades.

**4.2. Triangular systems of random variables.** Let  $\tilde{\mathcal{L}} = \{l_k\}$  be a (finite or infinite) sequence of random variables (which we will refer as a *linear system*),  $K$  its length ( $K = \infty$  if  $\tilde{\mathcal{L}}$  is infinite), and  $\tilde{M} \geq 1$  an integer. We say that  $\tilde{\mathcal{L}}$  is  $\tilde{M}$ -dependent, if for every  $i, j \geq 1$  with  $i - j \geq \tilde{M}$ ,  $\{l_k : k \leq i\}$  and  $\{l_k : k > j\}$  are independent. For example, a 0-dependent linear system  $\tilde{\mathcal{L}}$  is independent.

One is usually interested in the distribution of the sums

$$S_N = \sum_{k=1}^N l_k$$

<sup>3</sup>One may construct  $Y_\infty$  using its spectral representation. Alternatively, one may use the Paley-Wiener construction  $Y(x) = \sum_{n \in \mathbb{Z}} a_n \frac{\sin(n-x)}{n-x}$ .

<sup>4</sup>One may easily check the additional Lipschitz-like condition required by that theorem



as  $N \rightarrow \infty$ , that is if  $K = \infty$ . In this situation one employs a special version of the CLT due to Diananda [DN]. For a process  $X(t)$  on  $\mathbb{R}$  we denote  $Z_X(T)$  to be the number of zeros of  $X$  on  $[0, T]$ . Cuzick [CZ] employed Diananda's result to prove the central limit theorem for  $Z_X(T)$  as  $T \rightarrow \infty$  for a wide class of stationary processes  $X$ . A more basic version of this theorem was applied earlier by Malevich [ML] to obtain a similar, but more restrictive result.

Diananda's result applies to finite sums of random variables of linear systems. The situation in our hands is somewhat different, namely, of a so-called *triangular* system (or array) of random variables. A triangular system of random variables is the correspondence  $K(N) : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ , together with a linear system

$$\tilde{\mathcal{L}}_N = \{z_{N,k} : 1 \leq k \leq K(N)\}$$

for each  $N \in \mathbb{N}$ . We will use the notation  $\tilde{\mathcal{Z}} = \{z_{N,k}\}$  to denote a triangular system.

Let  $\tilde{M} = \tilde{M}(N)$  be sequence of integers. We say that  $\tilde{\mathcal{Z}}$  is  $\tilde{M}$ -dependent, if  $\tilde{\mathcal{L}}_N$  is  $\tilde{M}(N)$ -dependent for every  $N$ .

Given a triangular system  $\tilde{\mathcal{Z}}$ , we are usually interested in the asymptotic distribution of the sums

$$S_N = \sum_{k=1}^{K(N)} z_{N,k},$$

as  $N \rightarrow \infty$ . Note that here, unlike the linear systems, both the number of the summands of  $S_N$  and the summands themselves depend on  $N$ . We have the following theorem due to Berk [BR], which establishes the asymptotic normality of  $S_N$  for  $\tilde{M}$ -dependent triangular systems:

**Theorem 4.1** (Berk [BR]). *Let  $\mathcal{Z} = \{z_{N,k} : 1 \leq k \leq K(N)\}$  be a  $\tilde{M}$ -dependent triangular system of mean zero random variables. Assume furthermore, that*

- (1) *For some  $\delta > 0$ ,  $\mathbb{E}|z_{N,k}|^{2+\delta} \leq A_1$ , where  $A_1 > 0$  is a universal constant.*
- (2) *For every  $N$  and  $1 \leq i < j \leq K(N)$ , one has*

$$\text{Var}(z_{N,i+1} + \dots + z_{N,j}) \leq (j - i)A_2$$

*for some universal constant  $A_2 > 0$ .*

- (3) *The limit*

$$\lim_{N \rightarrow \infty} \frac{\text{Var}(z_{N,1} + \dots + z_{N,K})}{K}$$

*exists and is nonzero. Denote the limit  $v > 0$ .*

- (4) *We have*

$$\tilde{M} = o\left(K^{\frac{\delta}{2\delta+2}}\right).$$

*Then*

$$\frac{z_{N,1} + \dots + z_{N,K}}{\sqrt{vK}}$$

*converges weakly to  $N(0, 1)$ .*

Note that condition 4 requires, in particular, that as  $N \rightarrow \infty$ , one has  $K \rightarrow \infty$ . Berk's result was recently generalized by Romano and Wolf [RWL].

**4.3. Plan of the proof of Theorem 1.1.** We define the scaled processes

$$Y_N(x) := X_N\left(\frac{x}{m}\right),$$

on  $[0, 2\pi m]$ , where we reuse the notation  $m = N + 1/2$  from the proof of Proposition 3.1. Let us denote their covariance function

$$r_N(x) = r_{Y_N}(x) = r_{X_N}\left(\frac{x}{m}\right) = f_N(x),$$

with  $f_N$  defined by (32). It is obvious that

$$Z_{X_N} = Z_{Y_N},$$

the number of zeros of  $Y_N(x)$  on

$$I'_N := [0, 2\pi m].$$

It will be sometimes more convenient for us to work with  $Y_N(x)$  and  $r_N(x)$  being defined (by periodicity) on

$$I_N := [-\pi m, \pi m]$$

rather than on  $I'_N$ .

We are interested in the number  $Z_{Y_N}$  of zeros of  $Y_N$  on intervals  $I_N$ , whose length grows linearly with  $N$ . Divide  $I_N$  into (roughly)  $N$  subintervals  $I_{N,k}$  of equal length with disjoint interiors (so that the probability of having a zero on an overlap is 0), and represent  $Z_{Y_N}$ , almost surely, as a sum of random variables  $Z_{N,k}$ , the number of zeros of  $Y_N$  on  $I_{N,k}$ . The stationarity of  $Y_N$  implies that for a fixed  $N$ ,  $Z_{N,k}$  are identically distributed (but by no means *independent*).

We, therefore, obtain a triangular system

$$\tilde{\mathcal{Z}} = \{\tilde{Z}_{N,k} = Z_{N,k} - \mathbb{E}Z_{N,k}\}$$

of mean zero random variables with growing rows. Just as  $Z_{N,k}$ , the random variables  $\tilde{Z}_{N,k}$  are identically distributed. We will easily show that  $\tilde{\mathcal{Z}}$  satisfies the conditions 2-3 of Theorem 4.1. Moreover, we will see later that condition 1 holds with  $\delta = 1$  (see Proposition A.1; here we deal with a more complicated case of mollified random variables defined below; an easier version of the same argument applies in case of  $\tilde{\mathcal{Z}}$ , however it does not give the CLT due to the lack of independence).

The main obstacle to this approach is that the random variables  $Z_{N,k}$  are *not independent* (and thus neither are  $\tilde{Z}_{N,k}$ ). In fact, we may give an explicit expression for

$$\text{Cov}(Z_{N,k_1}, Z_{N,k_2})$$

in terms of an integral, which involves  $r_N$  and its derivatives. The stationarity of  $Y_N$  implies that  $\text{Cov}(Z_{N,k_1}, Z_{N,k_2})$  depends on the difference  $k_2 - k_1$  only (that is, the discrete process  $Z_{N,k}$  with  $N$  fixed is stationary, provided that the continuous process  $Y_N$  is).

To overcome this obstacle, we notice that  $r_N(x)$  and its couple of derivatives are *small* outside a short interval around the origin, and, moreover,

their  $L^2$  mass is concentrated around the origin. This means that the dependencies between the values and the derivatives of  $Y_N(x)$  on  $I_{N,k_1}$  and those on  $I_{N,k_2}$  are “small” for  $|k_1 - k_2|$  sufficiently large. Thus the system  $\tilde{Z}$  is “almost  $M$ -independent”, provided that  $M = M(N)$  is large enough (it is sufficient to take any sequence  $M(N)$  growing to infinity; see Proposition 4.3).

One may then hope to exchange the process  $Y_N$  (and thus the system  $\tilde{Z}$ ) with a process  $Y_N^M$  (resp.  $\tilde{Z}^M$ ), where  $M = M(N)$  is a growing parameter, so that the distributions of the number of zeros  $Z_N = Z_{Y_N}$  and  $Z_N^M = Z_{Y_N^M}$  of  $Y_N$  and  $Y_N^M$  respectively, are asymptotically equivalent, and the above properties of the original system stay unimpaired. In addition, we require  $\tilde{Z}^M$  to be  $M$ -dependent (or, rather,  $\text{const} \cdot M$ -dependent). To prove the asymptotic equivalence of  $Z_N$  and  $Z_N^M$  we will evaluate the variance of the difference  $\text{Var}(Z_N - Z_N^M)$  (see Proposition 4.3).

To define  $Y_N^M$ , we introduce a function  $r_N^M = r_N \cdot S_M$ , where  $|S_M| \leq 1$  is a sufficiently smooth function supported on  $[-\text{const} \cdot M, \text{const} \cdot M]$  approximating the unity near the origin with a positive Fourier transform on the circle  $I_N$ . We require that  $r_N^M$  (and a couple of its derivatives) preserve 100% of the  $L^2$  mass of  $r_N$  (resp. a couple of its derivatives) (see Lemma 5.1). We then construct  $Y_N^M$  with covariance function  $r_N^M$  using some Fourier analysis on  $I_N$ . It is important to observe that the covariance function being supported essentially at  $[-M, M]$  means that  $\tilde{Z}^M$  is (roughly)  $M$ -independent, in the periodic sense.

To get rid of the long-range dependencies, which occur as a result of the periodicity of  $Y_N$ , we prove the central limit theorem for positive and negative zeros separately (see the proof of Proposition 4.4). Namely we define  $Z^{M,+}$  (resp.  $Z^{M,-}$ ) to be the number of zeros  $z$  of  $Y_N^M$  with  $z > 0$  (resp.  $z < 0$ ), and

$$Z^M = Z^{M,+} + Z^{M,-}$$

almost surely. We are going to prove the asymptotic normality of the distribution of  $Z^{M,+}$  and similarly, of  $Z^{M,-}$ . We will prove that this will imply the asymptotic normality of the sum  $Z^M$ .

Concerning the choice of  $M = M(N)$ , on one hand, to well approximate  $Y_N$ ,  $M$  has to grow to infinity with  $N$ . On the other hand, condition 4 of theorem 4.1 constrains the growth rate of  $M$  from above. The above considerations leave us a handy margin for  $M$ .

**4.4. Some conventions.** In this section we will use some Fourier analysis with functions defined on the circle  $I_N := [-\pi m, \pi m]$  (or equivalently,  $I'_N := [0, 2\pi m]$ ). We will adapt the following conventions. Let  $f : I_N \rightarrow \mathbb{R}$  be a real-valued function. For  $n \in \mathbb{Z}$ , we define

$$\hat{f}(n) = \int_{I_N} f(x) e^{-\frac{in}{m}x} \frac{dx}{\sqrt{2\pi m}}.$$

If  $f$  is a real valued *even* nice function, then

$$r(x) = \hat{r}(0) \cdot \frac{1}{\sqrt{2\pi m}} + \sum_{n=1}^{\infty} \sqrt{2} \hat{r}(n) \cdot \frac{\cos(\frac{nx}{m})}{\sqrt{\pi m}}.$$

With the above conventions, if  $f, g : I_N \rightarrow \mathbb{R}$  are two functions, then

$$(f \hat{\cdot} g)(n) = \frac{1}{\sqrt{2\pi m}} (\hat{f} * \hat{g})(n),$$

and

$$(f \hat{*} g)(n) = \sqrt{2\pi m} \hat{f}(n) \cdot \hat{g}(n).$$

For the real valued even functions, the Parseval identity is

$$\|f\|_{L^2(I_N)}^2 = \|\hat{f}\|_{l^2(\mathbb{Z})}^2 = \hat{f}(0)^2 + \sum_{n=1}^{\infty} 2\hat{f}(n)^2.$$

#### 4.5. Proof of Theorem 1.1.

*Proof of Theorem 1.1.* Let  $Y_N(x) = X_N(\frac{x}{m})$ , and for notational convenience, we assume by periodicity, that  $Y_N$  and its covariance function  $r_N$  are defined on  $I_N := [-\pi m, \pi m]$ . One may rewrite the definition of  $Y_N$  using

$$(50) \quad r_N(x) = \hat{r}_N(0) \cdot \frac{1}{\sqrt{2\pi m}} + \sum_{n=1}^{\infty} \sqrt{2} \hat{r}_N(n) \cdot \frac{1}{\sqrt{\pi m}} \cos\left(\frac{n}{m}x\right),$$

as

$$(51) \quad Y_N(x) = \sqrt{\hat{r}_N(0)} \frac{1}{(2\pi m)^{1/4}} a_0 + \sum_{n=1}^{\infty} 2^{1/4} \sqrt{\hat{r}_N(n)} \cdot \frac{1}{(\pi m)^{1/4}} \left( a_n \cos\left(\frac{n}{m}x\right) + b_n \sin\left(\frac{n}{m}x\right) \right),$$

where  $a_n$  and  $b_n$  are  $(0, 1)$  Gaussian i.i.d. One may compute  $\hat{r}_N$  to be

$$\hat{r}_N(n) = \begin{cases} \frac{\sqrt{\pi m}}{\sqrt{2N}}, & 1 \leq |n| \leq N \\ 0, & \text{otherwise} \end{cases} = \frac{\sqrt{\pi m}}{\sqrt{2N}} \chi_{1 \leq |n| \leq N}(n).$$

It is easy to identify (51) as the spectral form of  $Y_N$  on the circle, analogous to the well-known spectral theory on the real line (see e.g. [CL], section 7.5). The spectral representation proved itself as extremely useful and powerful while studying various properties of stationary processes.

Let  $0 < M < \pi m$  be a large parameter and  $\chi_{[-M, M]}$  be the characteristic function of  $[-M, M] \subseteq I_N$ . Define

$$S_M(x) = \frac{(\chi_{[-M, M]})^{*8}(x)}{CM^7},$$

where  $(\cdot)^{*l}$  stands for convolving a function  $l$  times to itself, and the universal constant  $C > 0$  is chosen so that  $S_M(0) = 1$ . The function  $S_M : I_N \rightarrow \mathbb{R}$  is a piecewise polynomial of degree 7 in  $\frac{|x|}{M}$ , independent of  $M$ . It is a 6-times continuously differentiable function supported at  $[-8M, 8M]$ . For  $|x| < 2M$ , for example,

$$(52) \quad S_M(x) = 1 + b_1 \left(\frac{x}{M}\right)^2 + b_2 \left(\frac{x}{M}\right)^4 + b_3 \left(\frac{x}{M}\right)^6 + b_4 \left(\frac{|x|}{M}\right)^7$$

for some constants  $b_1, \dots, b_4 \in \mathbb{R}$ , which may be easily computed.

We define the mollified covariance function  $r^M = r_N^M : I_N \rightarrow \mathbb{R}$  by

$$(53) \quad r_N^M(x) := r_N(x) \cdot S_M(x),$$

with the Fourier series given by

$$(54) \quad \hat{r}_N^M(n) = \frac{1}{\sqrt{2\pi m}} \cdot (\hat{r}_N * \hat{S}_M)(n) = \frac{1}{2N} (\chi_{1 \leq |n| \leq N} * \hat{S}_M)(n) \geq 0,$$

since

$$(55) \quad \hat{S}_M(n) = \frac{(2\pi m)^{7/2}}{CM^7} \cdot (\hat{\chi}_{[-M, M]}(n))^8 \geq 0.$$

One may compute explicitly the Fourier transform of  $\chi_{[-M, M]}$  to be

$$(56) \quad \hat{\chi}_{[-M, M]}(n) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{M}{\sqrt{m}}, & n = 0 \\ \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{m}}{n} \sin\left(\frac{nM}{m}\right), & n \neq 0 \end{cases}.$$

The nonnegativity of  $\hat{S}_M$  allows us to construct a process  $Y_N^M(x)$  on  $I_N$  with covariance function  $r_{Y_N^M} = r_N^M$  as

$$(57) \quad Y_N^M(x) = \sqrt{\hat{r}_N^M(0)} \frac{1}{(2\pi m)^{1/4}} a_0 + \sum_{n=1}^{\infty} \sqrt{\hat{r}_N^M(n)} \cdot \frac{2^{1/4}}{(\pi m)^{1/4}} \left( a_n \cos\left(\frac{n}{m}x\right) + b_n \sin\left(\frac{n}{m}x\right) \right),$$

the RHS being almost surely an absolutely convergent series, uniformly w.r.t.  $x$ .

**Remark 4.2.** We assume that the  $a_n$  and  $b_n$  for  $n \leq N$  in (57) are the same as in (51), so that  $Y_N^M(x)$  converges in  $L^2$  to  $Y_N(x)$  (see Lemma 5.5).

Let  $M = M(N)$  be any sequence of numbers growing to infinity, satisfying  $M = o(N^{1/4})$ . Proposition 4.4 then implies that as  $N \rightarrow \infty$ ,  $\frac{Z_N^M - \mathbb{E}Z_N^M}{\sqrt{cN}}$  is asymptotically normal. Proposition 4.3 states that

$$\text{Var}(Z_N^M - Z_N) = o(\text{Var}(Z_N)),$$

so that the distribution of  $\frac{Z_N - \mathbb{E}Z_N}{\sqrt{cN}}$  is asymptotically equivalent to that of  $\frac{Z_N^M - \mathbb{E}Z_N^M}{\sqrt{cN}}$ , which implies the statement of Theorem 1.1.  $\square$

**Proposition 4.3.** *Suppose that as  $N \rightarrow \infty$ , we have  $M \rightarrow \infty$ . Then one has*

$$\text{Var}(Z_N - Z_N^M) = o(N).$$

*The proof of Proposition 4.3 is given in section 5.*

**4.6. Proof of CLT for  $Z_N^M$ .** The main result of the present section is Proposition 4.4, which establishes the central limit theorem for the mollified random variable  $Z^M$ .

**Proposition 4.4.** *Suppose that as  $N \rightarrow \infty$ ,  $M = o(N^{1/4})$ . Then for  $N \rightarrow \infty$ , the random variables  $\frac{Z_N^M - \mathbb{E}Z_N^M}{\sqrt{cN}}$ , weakly converge to the standard Gaussian  $N(0, 1)$ .*

*Proof of Proposition 4.4.* Recall that  $Z_N^M$  is the number of zeros of  $Y_N^M$  on  $I_N = [-\pi m, \pi m]$ . We are going to prove a central limit theorem for the number of zeros on  $I_N^+ := [0, \pi m]$  (denote similarly  $I_N^- := [-\pi m, 0]$ ) only. We thereby denote  $Z_N^{M,+}$  (resp.  $Z_N^{M,-}$ ) the number of zeros of  $Y_N^M$  on  $I_N^+$  (resp.  $I_N^-$ ), and analogously,  $Z_N^+$  and  $Z_N^-$  will denote the number of zeros of  $Y_N$  on  $I_N^+$  and  $I_N^-$  respectively. This also implies a central limit theorem for  $Z_N^{M,-}$  by the stationarity. The problem is that  $Z_N^{M,+}$  and  $Z_N^{M,-}$  are not independent, so that writing  $Z_N^M = Z_N^{M,+} + Z_N^{M,-}$  a.a. does not imply the asymptotic normality for the sum. Therefore we will have to come up with a more gentle argument in the end of this proof.

Let  $L > 0$  be any integer, which we will keep fixed throughout the proof. We divide  $I_N^+$  into subintervals

$$I_{N,k} = \left[ (k-1) \cdot \frac{\pi m}{LN}, k \cdot \frac{\pi m}{LN} \right]$$

for  $1 \leq k \leq LN$ , and denote  $Z_{N,k}^M$  the number of zeros of  $Y_N^M(x)$  on  $I_{N,k}$ .

Recall that, as a function on  $[-\pi m, \pi m]$ ,  $r_N^M$  is supported on

$$[-8 \cdot M, 8 \cdot M].$$

Therefore, if  $i - j \geq 8LM$ , the random variables  $\{Z_{N,k}^M : k \leq i\}$  are independent of  $\{Z_{N,k}^M : k > j\}$ .

We apply Theorem 4.1 on the  $\tilde{M} = \text{const} \cdot M$ -dependent triangular system

$$\tilde{Z}^M = \{\tilde{Z}_{N,k}^M := Z_{N,k}^M - \mathbb{E}Z_{N,k}^M : 1 \leq k \leq K(N)\}$$

with  $K(N) = NL$ . Since with probability 1 neither of  $Y_N^M$  have zeros on the overlaps of  $I_{N,k}$ , we have

$$Z_N^{M,+} - \mathbb{E}Z_N^{M,+} = \sum_{k=1}^{NL} \tilde{Z}_{N,k}^M$$

almost surely, so that to finish the proof of this Proposition, it remains to check that  $\tilde{Z}^M$  satisfies conditions 1-4 of Theorem 4.1.

Proposition A.1 implies that condition 1 is satisfied with  $\delta = 1$ , provided that we choose  $L$  large enough. Since  $\tilde{M} \sim \text{const} \cdot M$  and  $K(N) \sim \text{const} \cdot N$ , the assumption  $M = o(N^{1/4})$  of the present Proposition is equivalent to condition 4.

Condition 3 of Theorem 4.1 is equivalent to  $\text{Var}(Z_N^{M,+}) \sim c_1 N$  for some  $c_1 > 0$ . An application of (7) together with Proposition 4.3 and the triangle inequality, imply that

$$\text{Var}(Z_N^M) \sim cN.$$

One may also derive the corresponding result for  $Z_N^{M,+}$ , starting from  $\text{Var}(Z_N^+) \sim \frac{\varepsilon}{2}N$  (the proof follows along the same lines as the proof of (7)) and using (62) with the triangle inequality.

It then remains to check that  $\tilde{Z}^M$  satisfies condition 2 of Theorem 4.1. Using the same approach we used in the course of the proof of (7), one may find out that

$$(58) \quad \begin{aligned} & \text{Var}(\tilde{Z}_{N,i+1}^M + \dots + \tilde{Z}_{N,j}^M) \\ &= \frac{2}{\pi^2} \int_0^{(j-i)\frac{\pi m}{LN}} \left[ \left( (j-i)\frac{\pi m}{LN} - x \right) \cdot \left( \frac{\lambda_{2,N}^{M'}(1-r(x)^2) - r'(x)^2}{(1-r(x)^2)^{3/2}} (\sqrt{1-\rho(x)^2} \right. \right. \\ & \quad \left. \left. + \rho(x) \arcsin \rho(x) \right) - \lambda_{2,N}^{M'} \right] dx + (j-i) \frac{m}{LN} \sqrt{\lambda_{2,N}^{M'}}, \end{aligned}$$

where we use the shortcuts  $r = r_N^M$ ,  $\lambda_{2,N}^{M'} = -r_N^{M''}(0)$ , and

$$\rho(x) = \rho_N^M(x) = \frac{r''(x)(1-r(x)^2) + r'(x)^2 r(x)}{\lambda_2^M(1-r(x)^2) - r'(x)^2}.$$

We have

$$(j-i) \frac{m}{LN} \sqrt{\lambda_{2,N}^{M'}} \ll (j-i),$$

since  $\frac{m}{N} \leq 2$  and  $\lambda_{2,N}^{M'} = O(1)$ . It remains therefore to bound the integral in (58), which we denote  $J$ . We write, denoting  $\tau := (j-i)\frac{\pi m}{LN}$ :

$$(59) \quad J \ll (j-i) \int_0^\tau \left[ \frac{\lambda_2^{M'}(1-r^2) - r'^2}{(1-r^2)^{3/2}} (\sqrt{1-\rho^2} + \rho \arcsin \rho) - \lambda_2^{M'} \right] dx.$$

It will suffice then to prove that the latter integral is uniformly bounded. Let  $K_N^M(x)$  be the integrand. Expanding  $K_N^M(x)$  into Taylor polynomial around the origin, as we did in the course of the proof of (7) (see the proof of Lemma 3.3), we find that  $K_N^M(x)$  is uniformly bounded on some fixed neighbourhood of the origin (say, on  $[0, c]$ ). We claim, that outside  $[0, c]$ , the integrand is rapidly decaying, uniformly with  $N$ .

It is easy to see that  $r(x)$  being supported at  $[0, \text{const} \cdot M]$  implies that  $K(x)$  is supported in  $[0, \text{const} \cdot M]$  as well (note that we exploit here the fact that by counting only the positive zeros we disregard the dependencies between zeros near  $-\pi m$  and  $\pi m$ ). Moreover, on  $[c, \text{const} \cdot M]$ ,  $|K_N^M(x)| \ll \frac{1}{x^2}$ , where the constant involved in the “ $\ll$ ”-notation is universal. Therefore the integral on the RHS of (59) is uniformly bounded, so that  $J \ll (j-i)$ , which verifies condition 2 of Berk’s theorem.

This concludes the proof of the asymptotic normality for  $Z_N^{M,+}$  (and also  $Z_N^{M,-}$ ). Having that result in our hands, we define the random variables  $\hat{Z}_N^{M,+}$  and  $\hat{Z}_N^{M,-}$  to be the number of zeros of  $Y_N^M$  on  $[8M, \pi m - 8M]$  and  $[-\pi m + 8M, -8M]$  respectively. The random variables  $\hat{Z}_N^{M,\pm}$  are *independent*, since  $r_N^M$  is supported on  $[-8M, 8M]$ .

In addition, let  $Z_{N,S}^{M,+}$ ,  $Z_{N,L}^{M,+}$ ,  $Z_{N,S}^{M,-}$  and  $Z_{N,L}^{M,-}$  be the number of zeros of  $Y_N^M$  on  $[0, 8M]$ ,  $[\pi m - 8M, \pi m]$ ,  $[-8M, 0]$  and  $[-\pi m, -\pi m + 8M]$  respectively.

We have

$$(60) \quad \begin{aligned} & \text{Var}Z_{N,S}^{M,+}, \text{Var}Z_{N,L}^{M,+}, \text{Var}Z_{N,S}^{M,-}, \text{Var}Z_{N,L}^{M,-} \ll M \\ & = o(\text{Var}Z_N^{M,+}), o(\text{Var}Z_N^{M,-}), o(\text{Var}Z_N^M) \end{aligned}$$

by condition (2) of Theorem 4.1 (which we validated). Therefore

$$\hat{Z}_N^{M,+} = Z_N^{M,+} - Z_{N,S}^{M,+} - Z_{N,L}^{M,+}$$

and

$$\hat{Z}_N^{M,-} = Z_N^{M,-} - Z_{N,S}^{M,-} - Z_{N,L}^{M,-}$$

converge to the Gaussian distribution.

The independence of  $\hat{Z}_N^{M,\pm}$  then implies the asymptotic normality of  $\hat{Z}_N^{M,+} + \hat{Z}_N^{M,-}$ , and finally we obtain the asymptotic normality of

$$Z_N^M = (\hat{Z}_N^{M,+} + \hat{Z}_N^{M,-}) + Z_{N,S}^{M,+} + Z_{N,L}^{M,+} + Z_{N,S}^{M,-} + Z_{N,L}^{M,-},$$

again by (60). □

## 5. PROOF OF PROPOSITION 4.3

**5.1. Introduction and the basic setting.** Recall that we have the processes  $Y_N(x)$  and  $Y_N^M(x)$ , defined on  $I_N = [-\pi m, \pi m]$  and are interested in the distribution of  $Z_N$  and  $Z_N^M$ , the number of zeros of  $Y_N$  and  $Y_N^M$  on  $I_N$  respectively. The goal of the present section is to prove the bound

$$(61) \quad \text{Var}(Z_N - Z_N^M) = o(N)$$

on the variance of the difference. For notational convenience, we will consider only the positive zeros, that is, let  $Z_N^+$  (resp.  $Z_N^{M,+}$ ) be the number of zeros of  $Y_N$  (resp.  $Y_N^M$ ) on  $I_N^+ = [0, \pi m]$ . We will prove that

$$(62) \quad \text{Var}(Z_N^+ - Z_N^{M,+}) = o(N),$$

and by the stationarity, it will also imply

$$(63) \quad \text{Var}(Z_N^- - Z_N^{M,-}) = o(N),$$

where we denoted the number of negative zeros in an analogous manner. Finally, (62) together with (63), will imply (61), by the triangle inequality.

Let  $S > 0$  and  $R > 0$  be a couple of large integral parameters. We divide  $I_N^+$  into  $K = 2Sm$  equal subintervals, so that

$$I_{N,k} = \left[ (k-1) \frac{2\pi m}{K}, k \frac{2\pi m}{K} \right]$$

for  $1 \leq k \leq K$ .

We then write the LHS of (62) as

$$(64) \quad \text{Var}(Z_N^+ - Z_N^{M,+}) = \sum_{k_1, k_2=1}^K \text{Cov}(Z_{N,k_1} - Z_{N,k_1}^M, Z_{N,k_2} - Z_{N,k_2}^M).$$

We divide the last summation into 3 different ranges. That is, we define

$$(65) \quad E_1 = \sum_{|k_1 - k_2| \leq 1},$$



$$(66) \quad E_2 = \sum_{2 \leq |k_1 - k_2| \leq SR},$$

and

$$(67) \quad E_3 = \sum_{|k_1 - k_2| \geq SR}^K,$$

and prove that for  $1 \leq i \leq 3$

$$\lim_{N \rightarrow \infty} \frac{E_i}{N} = 0.$$

**5.2. Preliminaries.** In addition to the covariance functions  $r = r_N$  and  $r^M = r_N^M$  of  $Y_N$  and  $Y_N^M$  respectively, defined on  $I_N$ , we introduce the joint covariance function

$$(68) \quad r^{M,0}(x) = r_N^{M,0}(x) = \mathbb{E}[Y_N(y)Y_N^M(y+x)],$$

which is a function of  $x$  indeed, by stationarity. Similarly to (16), one has  $|r^{M,0}| \leq 1$  by the Cauchy-Schwartz inequality.

Using the spectral form (51) (resp. (57)) of  $Y_N$  (resp.  $Y_N^M$ ), one may compute the Fourier expansion of  $r_N^{M,0}$  to be

$$(69) \quad \begin{aligned} r_N^{M,0}(x) &= \frac{\sqrt{\hat{r}(0) \cdot \hat{r}^M(0)}}{\sqrt{2\pi m}} + \sum_{n=1}^{\infty} \sqrt{\hat{r}(n)\hat{r}^M(n)} \cdot \frac{\sqrt{2}}{\sqrt{\pi m}} \cos\left(\frac{n}{m}x\right) \\ &= \sum_{n=1}^{\infty} \sqrt{\hat{r}(n)\hat{r}^M(n)} \cdot \frac{\sqrt{2}}{\sqrt{\pi m}} \cos\left(\frac{n}{m}x\right) \\ &= \frac{1}{N} \sum_{n=1}^N \sqrt{(\chi_{1 \leq |n| \leq N} * \hat{S}_M)(n)} \cdot \frac{1}{(2\pi m)^{1/4}} \cos\left(\frac{n}{m}x\right). \end{aligned}$$

In particular,  $r_N^{M,0}$  is *even*, and

$$(70) \quad \hat{r}_N^{M,0}(n) = \sqrt{\hat{r}_N(n)\hat{r}_N^M(n)} = \left(\frac{\pi m}{8}\right)^{1/4} \frac{1}{N} \cdot \chi_{1 \leq |n| \leq N}(n) \cdot \sqrt{(\chi_{1 \leq |n| \leq N} * \hat{S}_M)(n)}.$$

Recall that to determine the second moment  $\mathbb{E}Z_X^2$  of a process  $X$ , we naturally encountered the distribution of the random vector (17). Similarly, to evaluate the covariances in (64), one naturally encounters the distribution of vectors

$$W_1 = (Y_N^M(x_1), Y_N^M(x_2), Y_N^{M'}(x_1), Y_N^{M'}(x_2))$$

with probability density  $\phi_{N,M}^{x_1,x_2}(u_1, u_2, v_1, v_2)$  and

$$W_2 = (Y_N(x_1), Y_N^M(x_2), Y_N'(x_1), Y_N^{M'}(x_2))$$

with probability density  $\phi_{N,M,0}^{x_1,x_2}(u_1, u_2, v_1, v_2)$  for some  $x_1, x_2 \in I_N$ . As before, the distributions  $\phi_{N,M}^{x_1,x_2}$  and  $\phi_{N,M,0}^{x_1,x_2}$  depend only on  $x = x_2 - x_1$  by stationarity, and we will denote  $\phi_{N,M}^x = \phi_{N,M}^{x_1,x_2}$  and  $\phi_{N,M,0}^x = \phi_{N,M,0}^{x_1,x_2}$  accordingly.

Similarly to the mean zero Gaussian distribution with covariance matrix (18) of the random vector (17), both  $W_1$  and  $W_2$  are mean zero Gaussian with covariance matrices

$$(71) \quad \Sigma_{N,M}(x) = \begin{pmatrix} 1 & r_N^M(x) & 0 & r_N^{M'}(x) \\ r_N^M(x) & 1 & -r_N^{M'}(x) & 0 \\ 0 & -r_N^{M'}(x) & \lambda_{2,N}^{M'} & -r_N^{M''}(x) \\ r_N^{M'}(x) & 0 & -r_N^{M''}(x) & \lambda_{2,N}^{M'} \end{pmatrix}$$

and

$$(72) \quad \Sigma_{N,M,0}(x) = \begin{pmatrix} 1 & r_N^{M,0}(x) & 0 & r_N^{M,0'}(x) \\ r_N^{M,0}(x) & 1 & -r_N^{M,0'}(x) & 0 \\ 0 & -r_N^{M,0'}(x) & \lambda'_{2,N} & -r_N^{M,0''}(x) \\ r_N^{M,0'}(x) & 0 & -r_N^{M,0''}(x) & \lambda_{2,N}^{M'} \end{pmatrix},$$

where, as usual, we denote

$$\lambda'_{2,N} := -r_N^{M,0''}(0); \quad \lambda_{2,N}^{M'} := -r_N^{M''}(0).$$

Similarly to  $\Sigma(t)$  in (18), both  $\Sigma_{N,M}(x)$  and  $\Sigma_{N,M,0}(x)$  are nonsingular for  $x \neq 0$ , and so

$$(73) \quad \phi_{N,M}^x(w) = \frac{1}{(2\pi)^2 \sqrt{\det \Sigma_{N,M}(x)}} e^{-\frac{1}{2} w \Sigma_{N,M}(x)^{-1} w^t}$$

and

$$(74) \quad \phi_{N,M,0}^x(w) = \frac{1}{(2\pi)^2 \sqrt{\det \Sigma_{N,M,0}(x)}} e^{-\frac{1}{2} w \Sigma_{N,M,0}(x)^{-1} w^t}.$$

We denote

$$(75) \quad \tilde{\phi}_{N,M}^x(v_1, v_2) := \phi_{N,M}^x(0, 0, v_1, v_2); \quad \tilde{\phi}_{N,M,0}^x(v_1, v_2) := \phi_{N,M,0}^x(0, 0, v_1, v_2)$$

and define the random vector

$$(V_1 = V_1(x), V_2 = V_2(x)) = (Y_N'(0), Y_N^{M'}(x))$$

conditioned upon  $Y_N(0) = Y_N^M(x) = 0$  with probability density function  $\psi_{N,M,0}^x(v_1, v_2)$ . The random vector  $(V_1, V_2)$  has a mean zero Gaussian distribution with covariance matrix

$$(76) \quad \Omega_{N,M,0}^x = \begin{pmatrix} \lambda'_{2,N} - \frac{r_N^{M,0'}(x)^2}{1-r_N^{M,0}(x)^2} & -r_N^{M,0''}(x) - \frac{r_N^{M,0}(x) \cdot r_N^{M,0'}(x)^2}{1-r_N^{M,0}(x)^2} \\ -r_N^{M,0''}(x) - \frac{r_N^{M,0}(x) \cdot r_N^{M,0'}(x)^2}{1-r_N^{M,0}(x)^2} & \lambda_{2,N}^{M'} - \frac{r_N^{M,0'}(x)^2}{1-r_N^{M,0}(x)^2} \end{pmatrix},$$

The matrix  $\Omega_{N,M,0}^x$  is regular for  $x \neq 0$ . We have, analogously to (20)

$$\tilde{\phi}_{N,M,0}^x(v_1, v_2) = \frac{\psi_{N,M,0}^x(v_1, v_2)}{2\pi \sqrt{1 - r_N^{M,0}(x)^2}}.$$

Similarly, let  $\psi_{N,M}^x(v_1, v_2)$  be the probability density function of  $(Y_N^{M'}(0), Y_N^{M'}(x))$  conditioned upon  $Y_N^M(0) = Y_N^M(x) = 0$ . One then has

$$(77) \quad \tilde{\phi}_{N,M}^x(v_1, v_2) = \frac{\psi_{N,M}^x(v_1, v_2)}{2\pi\sqrt{1 - r_N^M(x)^2}}.$$

### 5.3. Auxiliary Lemmas.

**Lemma 5.1.** *One has the following estimates*

(1)

$$\|r_N^M - r_N\|_{L^2(I_N)} = O\left(\frac{1}{\sqrt{M}}\right),$$

(2)

$$\|r_N^{M,0} - r_N\|_{L^2(I_N)} = O\left(\frac{1}{M^{1/4}}\right),$$

(3)

$$\|r_N^{M''} - r_N''\|_{L^2(I_N)} = O\left(\frac{1}{\sqrt{M}}\right),$$

(4)

$$\|r_N^{M,0''} - r_N''\|_{L^2(I_N)} = O\left(\frac{1}{M^{1/4}}\right).$$

(5)

$$\|r_N^{M'} - r_N'\|_{L^2(I_N)} = O\left(\frac{1}{\sqrt{M}}\right),$$

(6)

$$\|r_N^{M,0'} - r_N'\|_{L^2(I_N)} = O\left(\frac{1}{M^{1/4}}\right).$$

*Proof.* First, we notice that (5) (resp. (6)) follows from (1) with (3) (resp. (2) with (4)) by integration by parts and the Cauchy-Schwartz inequality.

By the symmetry of all the functions involved, it is sufficient to bound  $\|\cdot\|_{L^2(I_N^+)}$ . To establish (1), we note that for  $|x| \leq M$ , one has

$$(78) \quad S_M(x) = 1 + O\left(\frac{x}{M}\right)^2,$$

and both  $r_N$  and  $r_N^M$  are rapidly decaying for large  $x$ , since

$$|r_N(x)| \ll \frac{1}{x},$$

and  $S_M$  is bounded, with constants independent of  $N$  or  $M$ . Thus, we have

$$\begin{aligned} \|r_N^M - r_N\|_{L^2(I_N)}^2 &= \int_0^{\pi m} (r_N^M(x) - r_N(x))^2 dx \\ &= \int_0^{\pi m} (r_N(x)(1 - S_M(x)))^2 dx \ll \frac{1}{M^4} \int_0^M r_N(x)^2 x^4 dx + \int_M^{\pi m} r_N(x)^2 dx \\ &\ll \frac{1}{M^4} \int_0^M (r_N(x)^2 x^2) \cdot x^2 dx + \int_M^{\pi m} \frac{dx}{x^2} \ll \frac{1}{M^4} \int_0^M x^2 dx + \frac{1}{M} \ll \frac{1}{M}. \end{aligned}$$

It is easy to establish (3) using a similar approach.

To prove (2), we will use the Fourier series representation (50) of  $r_N$ , and its analogue (69) for  $r_N^{M,0}$  with Parseval's identity. We then have by (70)

$$\begin{aligned} \|r_N^{M,0} - r_N\|_{L^2(I_N)}^2 &= \|\hat{r}_N^{M,0} - \hat{r}_N\|_{l^2(\mathbb{Z})}^2 \\ &= 2 \sum_{n=1}^N \hat{r}_N(n) \cdot (\sqrt{\hat{r}_N(n)} - \sqrt{\hat{r}_N^M(n)})^2 \leq 2 \sum_{n=1}^N \hat{r}_N(n) \cdot |\hat{r}_N(n) - \hat{r}_N^M(n)|, \end{aligned}$$

since for  $a, b \geq 0$ ,

$$(79) \quad (a - b)^2 \leq |a^2 - b^2|.$$

Continuing, we use the Cauchy-Schwartz inequality to obtain

$$\begin{aligned} \|r_N^{M,0} - r_N\|_{L^2(I_N)}^2 &\ll \|\hat{r}_N\|_{l^2(\mathbb{Z})} \cdot \|\hat{r}_N - \hat{r}_N^M\|_{l^2(\mathbb{Z})} \\ &= \|\hat{r}_N\|_{l^2(\mathbb{Z})} \cdot \|r_N - r_N^M\|_{l^2(\mathbb{Z})} \ll \frac{1}{\sqrt{M}}, \end{aligned}$$

by (1) of the present Lemma, and the obvious estimate  $\|\hat{r}_N\| \ll 1$ . This proves part (2) of this Lemma.

It is now easy to establish part (4) of the present Lemma, using

$$\hat{f}''(n) = -\frac{n^2}{m^2} \hat{f}(n).$$

□

**Lemma 5.2.** *The functions  $r_N(x)$ ,  $r_N^M(x)$ ,  $r_N^{M,0}(x)$  and their first couple of derivatives are Lipschitz, uniformly with  $N$ , i.e. satisfy*

$$(80) \quad |h(x) - h(y)| \leq A|x - y|$$

for some universal constant  $A > 0$ .

*Proof.* The statement is clear for  $r_N(x) = \frac{1}{N} \sum_{n=1}^N \cos(\frac{n}{m}x)$ , as well as  $r_N^M(x) = r_N(x)S_M(x)$  (due to the fact that  $S_M$  and  $S_M'$  are bounded).

It then remains to prove the result for  $r_N^{M,0}$ . From the representation (69) it is clear that it would be sufficient to prove that the coefficients

$$\frac{1}{(2\pi m)^{1/4}} \cdot \sqrt{(\chi_{1 \leq |n| \leq N} * \hat{S}_M)(n)}$$

are uniformly bounded. We will bound the square

$$(81) \quad \frac{1}{(2\pi m)^{1/2}} \cdot (\chi_{1 \leq |n| \leq N} * \hat{S}_M)(n).$$

Using (55) with (56), we bound  $\hat{S}_M(n)$  by

$$(82) \quad \hat{S}_M(n) \ll \begin{cases} \frac{m^{15/2}}{M^7} \left( \frac{\sin\left(\frac{nM}{m}\right)}{n} \right)^8, & n \neq 0 \\ \frac{M}{\sqrt{m}}, & n = 0 \end{cases}$$

so that the coefficients (81) are bounded by

$$\begin{aligned} &\ll \frac{1}{\sqrt{N}} \cdot \frac{m^{15/2}}{M^7} \sum_{\substack{k=n-N \\ k \neq 0}}^{n+N} \left( \frac{\sin\left(\frac{kM}{m}\right)}{k} \right)^8 + \frac{1}{\sqrt{m}} \cdot \frac{M}{\sqrt{m}} \\ &\ll \frac{M}{N} \sum_{1 \leq |k| \leq \frac{N}{M}} \left( \frac{\sin\left(\frac{kM}{m}\right)}{\frac{kM}{m}} \right)^8 + \frac{N^7}{M^7} \sum_{|k| > \frac{N}{M}} \frac{1}{k^8} + \frac{M}{N} \\ &\ll \frac{M}{N} \cdot \frac{N}{M} + \frac{N^7}{M^7} \cdot \frac{1}{(N/M)^7} + 1 \ll 1. \end{aligned}$$

This proves that the squared coefficients (81) are uniformly bounded, and thus, that  $r_N^{M,0}$  satisfy the Lipschitz condition (80) with some universal constant  $A > 0$ . □

**Lemma 5.3.** *Let  $I = [a, b]$  be any interval and  $h : I \rightarrow \mathbb{R}$  a Lipschitz function satisfying (80). Then for every  $x \in I$ ,*

$$|h(x)| \leq 2A^{1/3} \|h\|_{L^2(I)}^{2/3},$$

provided that

$$(83) \quad b - a > \frac{\max_{x \in I} |h(x)|}{2A}.$$

*Proof.* Let  $x_0 \in I$  and

$$J := I \cap \left[ x_0 - \frac{|h(x_0)|}{2A}, x_0 + \frac{|h(x_0)|}{2A} \right].$$

Our assumption (83) implies that interval  $J$  has length  $|J| \geq \frac{|h(x_0)|}{2A}$ , and moreover, on  $J$  we have

$$|h(x)| \geq \frac{|h(x_0)|}{2}$$

by (80). Thus we have

$$\|h\|_{L^2(I)}^2 \geq \int_J h(x)^2 dx \geq |J| \cdot \frac{h(x_0)^2}{4} \geq \frac{|h(x_0)|}{2A} \cdot \frac{h(x_0)^2}{4} = \frac{1}{8A} |h(x_0)|^3,$$

which is equivalent to the statement of this Lemma. □

Lemmas 5.1, 5.2 and 5.3 together imply

**Corollary 5.4.** *For every  $x \in I_N$ , one has*

$$\begin{aligned} |r_N^M(x) - r_N(x)| &= O\left(\frac{1}{M^{1/3}}\right), \\ |r_N^{M,0}(x) - r_N(x)| &= O\left(\frac{1}{M^{1/6}}\right), \\ |r_N^{M'}(x) - r_N'(x)| &= O\left(\frac{1}{M^{1/3}}\right), \\ |r_N^{M,0'}(x) - r_N'(x)| &= O\left(\frac{1}{M^{1/6}}\right), \\ |r_N^{M''}(x) - r_N''(x)| &= O\left(\frac{1}{M^{1/3}}\right), \\ |r_N^{M,0''}(x) - r_N''(x)| &= O\left(\frac{1}{M^{1/6}}\right), \end{aligned}$$

uniformly w.r.t.  $x$  and  $N$ .

**Lemma 5.5.** *For every  $x \in I$ , we have*

$$\mathbb{E}(Y_N^M(x) - Y_N(x))^2 = O\left(\frac{1}{\sqrt{M}}\right)$$

with the constant involved in the  $O$ -notation universal.

*Proof.* By the stationarity we may assume that  $x = 0$ . We have by (51) and (57)

$$\mathbb{E}(Y_N^M(0) - Y_N(0))^2 = \frac{1}{\sqrt{2\pi m}} \hat{r}_N^M(0) + \frac{\sqrt{2}}{\sqrt{\pi m}} \sum_{n=1}^{\infty} \left( \sqrt{\hat{r}_N(n)} - \sqrt{\hat{r}_N^M(n)} \right)^2.$$

Since  $\hat{r}_N$  is supported in  $n \leq N$ , we have

$$(84) \quad \mathbb{E}(Y_N^M(x) - Y_N(x))^2 \leq \frac{\sqrt{2}}{\sqrt{\pi m}} \sum_{n=0}^{2N} \left( \sqrt{\hat{r}_N(n)} - \sqrt{\hat{r}_N^M(n)} \right)^2 + \frac{\sqrt{2}}{\sqrt{\pi m}} \sum_{n>2N} \hat{r}_N^M(n).$$

We use (79) again to bound the first summation of (84), getting

$$\begin{aligned} \frac{\sqrt{2}}{\sqrt{\pi m}} \sum_{n=0}^{2N} &\leq \frac{\sqrt{2}}{\sqrt{\pi m}} \sum_{n=1}^{2N} |\hat{r}_N(n) - \hat{r}_N^M(n)| \\ &\ll \frac{1}{\sqrt{N}} \cdot \sqrt{N} \left( \sum_{n=1}^{2N} (\hat{r}_N(n) - \hat{r}_N^M(n))^2 \right)^{1/2} \\ &\leq \|\hat{r}_N - \hat{r}_N^M\|_{l^2(\mathbb{Z})} \ll \frac{1}{\sqrt{M}}, \end{aligned}$$

by the Cauchy-Schwartz inequality, Parseval's identity and Lemma 5.1, part (1).

To bound the second summation in (84), we reuse the estimate (82) to obtain

$$\hat{S}_M(n) \ll \frac{N^{15/2}}{M^7} \cdot \frac{1}{n^8}, \quad n \neq 0$$

so that (54) implies

$$\begin{aligned} \hat{r}_N^M(n) &\ll \frac{1}{N} \sum_{k=-N}^N \hat{S}_M(n+k) \ll \frac{1}{N} \sum_{k=-N}^N \frac{N^{15/2}}{M^7} \cdot \frac{1}{(n+k)^8} \\ &\ll \frac{N^{13/2}}{M^7} \cdot N \frac{1}{(n/2)^8} \ll \frac{N^{15/2}}{M^7} \cdot \frac{1}{n^8}, \end{aligned}$$

and thus the second summation in (84) is bounded by

$$\frac{1}{\sqrt{N}} \sum_{n>2N} \frac{N^{15/2}}{M^7} \cdot \frac{1}{n^8} \ll \frac{1}{M^7}.$$

This concludes the proof of this lemma.  $\square$

**Lemma 5.6** (Cuzick [CZ], lemma 4). *Let  $V_1$  and  $V_2$  be a mean zero Gaussian pair of random variables and let*

$$\rho = \text{Cor}(V_1, V_2) := \frac{\mathbb{E}V_1V_2 - \mathbb{E}V_1\mathbb{E}V_2}{\sqrt{\text{Var}V_1 \cdot \text{Var}V_2}}.$$

Then

$$0 \leq \text{Cor}(|V_1|, |V_2|) \leq \rho^2.$$

**Remark 5.7.** Lemma 5.6 may be also obtained computing explicitly both sides of the inequality using the integral (27) due to Bleher and Di [BD].

#### 5.4. Proof of Proposition 4.3.

*Proof of Proposition 4.3.* Recall that  $Z_N$  (resp.  $Z_N^M$ ) is the number of the zeros of  $Y_N$  (resp.  $Y_N^M$ ) on  $I_N = [-\pi m, \pi m]$ . The process  $Y_N^M$  is given in its spectral form (57) with the RHS absolutely convergent, uniformly w.r.t.  $x \in I_N$ . The rapid decay of  $\hat{r}_N^M$  implies that  $Y_N^M$  is almost surely continuously differentiable, and we may differentiate (57) term by term.

As stated before, for notational convenience, rather than showing the original statement of the Proposition, we are going to prove (62). We want to bound  $E_i$  defined by (65)-(67) given the large integral parameters  $S$  and  $R$ .

5.4.1. *Bounding  $E_1$ .* For every  $x \in I_N$ , let  $\chi_N^x$  (resp.  $\chi_{N,M}^x$ ) be the indicator of the event  $\{Y_N(0)Y_N(x) < 0\}$  (resp.  $\{Y_N^M(0)Y_N^M(x) < 0\}$ ). Intuitively, for  $S$  large (i.e.  $I_{N,1}$  is short) one expects at most one zero of either  $Y_N$  or  $Y_N^M$  on  $I_{N,1}$ . Thus the number of zeros of  $Y_N$  (resp.  $Y_N^M$ ) on  $I_{N,1} = [0, \tau]$  with  $\tau := \frac{\pi}{2S}$  should, with high probability, equal  $\chi = \chi_N^\tau$  (resp.  $\chi^M = \chi_{N,M}^\tau$ ).

Using the Cauchy-Schwartz inequality, we have for every  $k_1, k_2$

$$\begin{aligned} &\text{Cov}(Z_{N,k_1} - Z_{N,k_1}^M, Z_{N,k_2} - Z_{N,k_2}^M) \\ &\leq \sqrt{\text{Var}(Z_{N,k_1} - Z_{N,k_1}^M)} \cdot \sqrt{\text{Var}(Z_{N,k_2} - Z_{N,k_2}^M)} = \text{Var}(Z_{N,1} - Z_{N,1}^M) \end{aligned}$$

by the stationarity. Therefore,

$$\begin{aligned} (85) \quad E_1 &\ll SN \text{Var}(Z_{N,1} - Z_{N,1}^M) \leq SN \mathbb{E}(Z_{N,1} - Z_{N,1}^M)^2 \\ &\ll SN \left( \mathbb{E}(Z_{N,1} - \chi)^2 + \mathbb{E}(\chi - \chi^M)^2 + \mathbb{E}(\chi^M - Z_{N,1}^M)^2 \right). \end{aligned}$$

We recognize the second summand of (85) as the probability  $Pr(\chi \neq \chi^M)$ . We may bound it as

$$(86) \quad \mathbb{E}(\chi - \chi^M)^2 = Pr(\chi \neq \chi^M) \leq Pr(\text{sgn}(Y_N(0)) \neq \text{sgn}(Y_N^M(0))) \\ + Pr(\text{sgn}(Y_N(\tau)) \neq \text{sgn}(Y_N^M(\tau))).$$

We bound the first summand of the RHS of (86), and similarly the second one. For every  $\epsilon > 0$ , we have

$$(87) \quad Pr(\text{sgn}(Y_N(0)) \neq \text{sgn}(Y_N^M(0))) \leq Pr(|Y_N(0)| < \epsilon) + Pr(|Y_N(0) - Y_N^M(0)| > \epsilon).$$

The first summand of (87) is bounded by

$$Pr(|Y_N(0)| < \epsilon) = O(\epsilon),$$

since  $Y_N(0)$  is  $(0, 1)$ -Gaussian, and the second one is

$$Pr(|Y_N(0) - Y_N^M(0)| > \epsilon) \ll \frac{1}{\sqrt{M}\epsilon^2},$$

by Lemma 5.5 and Chebyshev's inequality.

Hence, we obtain the bound

$$Pr(\text{sgn}(Y_N(0)) \neq \text{sgn}(Y_N^M(0))) = O\left(\epsilon + \frac{1}{\sqrt{M}\epsilon^2}\right),$$

and, similarly,

$$Pr(\text{sgn}(Y_N(\tau)) \neq \text{sgn}(Y_N^M(\tau))) = O\left(\epsilon + \frac{1}{\sqrt{M}\epsilon^2}\right).$$

Plugging the last couple of estimates into (86) yields that for every  $\epsilon > 0$

$$(88) \quad \mathbb{E}(\chi - \chi^M)^2 = O\left(\epsilon + \frac{1}{\sqrt{M}\epsilon^2}\right).$$

The RHS of (88) can be made arbitrarily small.

Now we treat the third summand of (85), and similarly, but easier, the first one. We have

$$(89) \quad \mathbb{E}(Z_{N,1}^M - \chi^M)^2 = \sum_{k=1}^{\infty} k^2 Pr(Z_{N,1}^M - \chi^M = k) \\ \leq \sum_{k=2}^{\infty} 2(k^2 - k) Pr(Z_{N,1}^M = k + \chi^M) \leq 2\mathbb{E}[(Z_{N,1}^M)^2 - Z_{N,1}^M],$$

and using the same approach as in (58) in addition to some easy manipulations yields

$$\mathbb{E}[(Z_{N,1}^M)^2 - Z_{N,1}^M] \ll \int_0^{\tau} (\tau - x) \cdot \tilde{K}(x) dx,$$

recalling the notation  $\tau := \frac{\pi}{2s}$ , where

$$\tilde{K}(x) = \frac{\lambda_{2,N}^M (1 - r^2) - r'^2}{(1 - r^2)^{3/2}} (\sqrt{1 - \rho^2} + \rho \arcsin \rho)$$



with notations as in (59). We saw already that  $\tilde{K}(x)$  is bounded, uniformly w.r.t.  $N$ , so that

$$\mathbb{E}[(Z_{N,1}^M)^2 - Z_{N,1}^M] = O(\tau^2) = O\left(\frac{1}{S^2}\right).$$

Plugging the last estimate into (89), we obtain the bound

$$(90) \quad \mathbb{E}(Z_{N,1}^M - \chi^M)^2 = O\left(\frac{1}{S^2}\right)$$

and similarly,

$$(91) \quad \mathbb{E}(Z_{N,1} - \chi)^2 = O\left(\frac{1}{S^2}\right)$$

as well.

Collecting the bounds for various summands of (85) we encountered i.e. (88), (90) and (91), we obtain the bound

$$|E_1| \ll NS \left( \epsilon + \frac{1}{\sqrt{M}\epsilon^2} + \frac{1}{S^2} \right),$$

or, equivalently,

$$(92) \quad \frac{|E_1|}{N} \ll \epsilon S + \frac{S}{\sqrt{M}\epsilon^2} + \frac{1}{S},$$

which could be made arbitrarily small.

5.4.2. *Bounding  $E_2$ .* We write  $E_2$  as

$$(93) \quad \begin{aligned} E_2 &= \sum \text{Cov}(Z_{N,k_1} - Z_{N,k_1}^M, Z_{N,k_2} - Z_{N,k_2}^M) = \sum \mathbb{E}Z_{N,k_1} \cdot (Z_{N,k_2} - Z_{N,k_2}^M) \\ &\quad - \sum \mathbb{E}Z_{N,k_1}^M \cdot (Z_{N,k_2} - Z_{N,k_2}^M) - \sum \mathbb{E}[Z_{N,k_1} - Z_{N,k_1}^M] \mathbb{E}[Z_{N,k_2} - Z_{N,k_2}^M] \\ &=: E_{2,1} - E_{2,2} - E_{2,t}, \end{aligned}$$

and bound each of the summands of (93) separately. In fact, we will only bound the contribution of the summands  $E_{2,t}$  and of the (slightly more difficult of the remaining two)  $E_{2,2}$ , bounding  $E_{2,1}$  in a similar manner. Note that the number of summands in each of the summations in (93), which equals the number of pairs  $(k_1, k_2)$  with  $2 \leq |k_1 - k_2| \leq RS$ , is of order  $K \cdot SR = NS \cdot SR = NS^2R$ .

We reuse the notation  $\tau := \frac{\pi}{2S}$ . Note that differentiating (53) yields

$$r_N^{M''}(0) = r_N''(0) + O\left(\frac{1}{M^2}\right).$$

using (78) near the origin. One then has

$$\begin{aligned} \mathbb{E}[Z_{N,k_1} - Z_{N,k_1}^M] \mathbb{E}[Z_{N,k_2} - Z_{N,k_2}^M] &= (\mathbb{E}[Z_{N,1} - Z_{N,1}^M])^2 \\ &\ll (\tau(r_N''(0) - r_N^{M''}(0)))^2 \ll \frac{1}{S^2M^4}, \end{aligned}$$

by the stationarity, formula (6) and its analogue for  $Y_N^M$ . Therefore

$$(94) \quad E_{2,t} \ll N \frac{R}{M^4}.$$

Note that for  $|k_1 - k_2| \geq 2$ , the intervals  $I_{N,k_1}$  and  $I_{N,k_2}$  are disjoint. Using a similar approach to [CL], we find that

$$\begin{aligned} \mathbb{E}Z_{N,k_1}^M \cdot (Z_{N,k_2} - Z_{N,k_2}^M) &= \int_{(k_2-k_1-1)\tau}^{(k_2-k_1+1)\tau} (\tau - |x - (k_2 - k_1)\tau|) \times \\ &\times \left[ \iint_{\mathbb{R}^2} |v_1| \cdot |v_2| (\tilde{\phi}_{N,M,0}^x(v_1, v_2) - \tilde{\phi}_{N,M}^x(v_1, v_2)) dv_1 dv_2 \right] dx, \end{aligned}$$

using the notations (75). We then bound  $E_{2,2}$  as

$$\begin{aligned} (95) \quad E_{2,2} &\leq NS^2R \cdot \max_{|k_1-k_2| \geq 2} \{ \mathbb{E}Z_{N,k_1}^M \cdot (Z_{N,k_2} - Z_{N,k_2}^M) \} \\ &\ll NR \max_{\tau \leq x \leq \pi m} \left\{ \iint_{\mathbb{R}^2} |v_1||v_2| \cdot |\tilde{\phi}_{N,M,0}^x(v_1, v_2) - \tilde{\phi}_{N,M}^x(v_1, v_2)| dv_1 dv_2 \right\}, \end{aligned}$$

where we used the obvious inequality

$$\max_{2 \leq |k_1-k_2| \leq RS} \leq \max_{|k_1-k_2| \geq 2}.$$

To bound the last integral, we exploit the fact that on any compact subset of  $\mathbb{R}^2$  we have

$$|\tilde{\phi}_{N,M,0}^x(v_1, v_2) - \tilde{\phi}_{N,M}^x(v_1, v_2)| \rightarrow 0$$

as  $N \rightarrow \infty$ , uniformly w.r.t.  $x > \tau$ , whereas outside both  $\tilde{\phi}_{N,M,0}^x$  and  $\tilde{\phi}_{N,M}^x$  are rapidly decaying. More precisely, let  $T > 0$  be a large parameter. We write

$$(96) \quad \iint_{\mathbb{R}^2} = \iint_{[-T,T]^2} + \iint_{\max\{|v_i|\} \geq T} =: J_1 + J_2.$$

While bounding  $J_2$ , we may assume with no loss of generality, that

$$|v_1| \geq T$$

on the domain of the integration. Let

$$J_{2,1} := \iint_{|v_1| \geq T} |v_1||v_2| \cdot |\tilde{\phi}_{N,M,0}^x(v_1, v_2)| dv_1 dv_2$$

and

$$J_{2,2} := \iint_{|v_1| \geq T} |v_1||v_2| \cdot |\tilde{\phi}_{N,M}^x(v_1, v_2)| dv_1 dv_2,$$

so that

$$(97) \quad J_2 \leq J_{2,1} + J_{2,2}.$$

Upon using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
(98) \quad J_{2,2} &\leq \int_{-\infty}^{\infty} dv_2 \int_{|v_1| \geq T} |v_1| |v_2| \cdot \tilde{\phi}_{N,M}^x(v_1, v_2) dv_1 \\
&\ll \left( \int_{-\infty}^{\infty} v_2^2 dv_2 \int_T^{\infty} \tilde{\phi}_{N,M}^x(v_1, v_2) dv_1 \right)^{1/2} \cdot \left( \int_{-\infty}^{\infty} dv_2 \int_T^{\infty} v_1^2 \tilde{\phi}_{N,M}^x(v_1, v_2) dv_1 \right)^{1/2} \\
&\leq \left( \frac{\mathbb{E}[Y_N^{M'}(x)^2 | Y_N^M(0) = Y_N^M(x) = 0]}{2\pi\sqrt{1-r_N^M(x)^2}} \right)^{1/2} \cdot \left( \int_{-\infty}^{\infty} dv_2 \int_T^{\infty} v_1^2 \tilde{\phi}_{N,M}^x(v_1, v_2) dv_1 \right)^{1/2},
\end{aligned}$$

by (77).

Computing explicitly, we have

$$(99) \quad \mathbb{E}[Y_N^{M'}(x)^2 | Y_N^M(0) = Y_N^M(x) = 0] = \lambda_{2,N}^{M'} - \frac{r_N^{M'}(x)^2}{1-r_N^M(x)^2} = O(1),$$

where  $\lambda_{2,N}^{M'} := -r_N^{M''}(0)$ , and, changing the order of integration,

$$\int_{-\infty}^{\infty} dv_2 \int_T^{\infty} v_1^2 \tilde{\phi}_{N,M}^x(v_1, v_2) dv_1 = \int_T^{\infty} v^2 \exp\left(-\frac{1}{2}\frac{v^2}{\sigma^2}\right) \frac{dv}{2\pi\sigma\sqrt{1-r_N^M(x)^2}},$$

where

$$\sigma^2 := \mathbb{E}[Y_N^{M'}(0)^2 | Y_N^M(0) = Y_N^M(x) = 0] = \lambda_{2,N}^{M'} - \frac{r_N^{M'}(x)^2}{1-r_N^M(x)^2}$$

as well. Continuing, we bound the integral by

$$\begin{aligned}
(100) \quad \int_{-\infty}^{\infty} dv_2 \int_T^{\infty} &\ll \frac{\sigma^2}{\sqrt{1-r_N^M(x)^2}} \int_{\frac{T}{\sigma}}^{\infty} v'^2 \exp\left(-\frac{1}{2}v'^2\right) dv' \\
&\ll \frac{\sigma^2}{T^2} \cdot \frac{\sigma^2}{\sqrt{1-r_N^M(x)^2}} \ll \frac{\sigma^4}{T^2\sqrt{1-r_N^M(x)^2}}
\end{aligned}$$

(say), by the rapid decay of the exponential.

Plugging (99) and (100) into (98), and using the crude estimate

$$1 - r_N^M(x) \gg \tau^2$$

for  $\tau \leq x \leq \pi m$ , we obtain the estimate

$$(101) \quad J_{2,2} \ll \frac{\left(\lambda_{2,N}^{M'} - \frac{r_N^{M'}(x)^2}{1-r_N^M(x)^2}\right)^{3/2}}{\sqrt{1-r_N^M(x)^2}} \cdot \frac{1}{T} \ll \frac{S}{T}.$$

Repeating all of the above for  $J_{2,1}$ , we obtain

$$(102) \quad J_{2,1} \ll \frac{\left(\lambda_{2,N'} - \frac{r_N^{M'}(x)^2}{1-r_N^{M,0}(x)^2}\right) \cdot \left(\lambda_{2,N}^{M'} - \frac{r_N^{M,0'}(x)^2}{1-r_N^{M,0}(x)^2}\right)^{1/2}}{\sqrt{1-r_N^{M,0}(x)^2}} \cdot \frac{1}{T} \ll \frac{S}{T},$$

using the same estimate

$$1 - r_N^{M,0}(x) \gg \tau^2,$$

which is easy to obtain using (69) and Lemmas 5.1 and 5.2.

Plugging the inequality (102) together with (101) into (97), we obtain

$$(103) \quad J_2 = O\left(\frac{S}{T}\right).$$

Now we are going to bound  $J_1$ . Recall the definition (75) of  $\tilde{\phi}_{N,M}^x$  and  $\tilde{\phi}_{N,M,0}^x$  with (73) and (74), and the covariance matrices (71), (72).

Corollary 5.4 implies

$$|\Sigma_N^M - \Sigma_N^{M,0}| = O\left(\frac{1}{M^{1/6}}\right)$$

(here and anywhere else the inequality  $M \leq y$  where  $M$  is a matrix and  $y$  is a number means that all the entries of  $M$  are  $\leq$  than  $y$ ). Expanding the determinants  $\det \Sigma_N^M$  and  $\det \Sigma_N^{M,0}$  into Taylor polynomial around the origin shows that they are bounded away from zero in the sense that for  $\tau < x < \pi m$ ,

$$\det \Sigma_N^M, \det \Sigma_N^{M,0} \gg \tau^A \gg \frac{1}{S^A}$$

for some constant  $A > 0^5$ .

Thus also

$$|(\Sigma_N^M)^{-1} - (\Sigma_N^{M,0})^{-1}| = O\left(\frac{S^A}{M^{1/6}}\right)$$

and

$$\left| \frac{1}{\sqrt{\det \Sigma_N^M}} - \frac{1}{\sqrt{\det \Sigma_N^{M,0}}} \right| \ll O\left(\frac{S^{2A}}{M^{1/6}}\right)$$

Substituting the estimates above into (73) and (74), and using (75), we obtain

$$\left| \tilde{\phi}_{N,M,0}^x(v_1, v_2) - \phi_{N,M}^x(v_1, v_2) \right| \ll \frac{S^{2A}}{M^{1/6}} + S^{A/2} \cdot \frac{T^2 S^A}{M^{1/6}} \ll \frac{S^{2A} T^2}{M^{1/6}},$$

uniformly for  $\tau \leq x \leq \pi m$  and  $|v_i| \leq T$ , where we used the trivial estimate  $|e^x - e^y| \leq |x - y|$  for  $x, y < 0$ .

Integrating the last estimate for  $|v_i| \leq T$  and substituting into the definition of  $J_1$ , we finally obtain

$$(104) \quad J_1 = O\left(\frac{S^{2A} T^6}{M^{1/6}}\right).$$

<sup>5</sup>An explicit computation shows that  $\det \Sigma_N^M(x) \gg x^8$  and also  $\det \Sigma_N^{M,0}(x) \gg x^8$ , with universal constants

Upon combining (104) and (103), and recalling (95) with (96), we finally obtain a bound for  $E_{2,2}$

$$(105) \quad E_{2,2} = NR \cdot \left( O\left(\frac{S^{2A}T^6}{M^{1/6}}\right) + O\left(\frac{S}{T}\right) \right),$$

and repeating the same computation for  $E_{2,1}$ , we may find that the same bound is applicable for  $E_{2,1}$

$$(106) \quad E_{2,1} = NR \cdot \left( O\left(\frac{S^{2A}T^6}{M^{1/6}}\right) + O\left(\frac{S}{T}\right) \right).$$

Using (106) together with (105) and (94), and noting (93), we finally obtain a bound for  $E_2$

$$E_2 = NR \cdot \left( O\left(\frac{S^{2A}T^6}{M^{1/6}}\right) + O\left(\frac{S}{T}\right) \right),$$

so that

$$(107) \quad \frac{E_2}{N} = O\left(\frac{RS^{2A}T^6}{M^{1/6}}\right) + O\left(\frac{RS}{T}\right),$$

which could be made arbitrarily small.

5.4.3. *Bounding  $E_3$ .* By the symmetry

$$\text{Cov}(Z_{N,k_1}, Z_{N,k_2}^M) = \text{Cov}(Z_{N,k_2}, Z_{N,k_1}^M)$$

we may rewrite  $E_3$  as

$$(108) \quad \begin{aligned} E_3 &= \sum \text{Cov}(Z_{N,k_1}, Z_{N,k_2}) - 2\text{Cov}(Z_{N,k_1}, Z_{N,k_2}^M) + \text{Cov}(Z_{N,k_1}^M, Z_{N,k_2}^M) \\ &=: E_{3,1} - 2E_{3,2} + E_{3,3}. \end{aligned}$$

First we treat the ‘‘mixed’’ term  $E_{3,2}$ , providing a similar treatment for the other terms. Assume with no loss of generality, that  $k_2 > k_1$ . Here we employ the random vector  $(V_1, V_2)$  defined in section 5.2. Using the theory developed in [CL], modified to treat the covariance (see also remark 2.4), we may write

$$\begin{aligned} \text{Cov}(Z_{N,k_1}, Z_{N,k_2}^M) &= \frac{1}{2\pi} \int_{(k_2-k_1-1)\tau}^{(k_2-k_1+1)\tau} \left[ (\tau - |x - (k_2 - k_1)\tau|) \times \right. \\ &\quad \left. \left( \frac{\mathbb{E}[|V_1(x)V_2(x)|]}{\sqrt{1 - r_N^{M,0}(x)^2}} - \mathbb{E}|Y_N'(0)|\mathbb{E}|Y_N^{M'}(x)| \right) \right] dx. \end{aligned}$$

where, as usual, we denote  $\tau := \frac{\pi}{2S}$ . Summing that up for  $|k_2 - k_1| \geq SR$ , and using the stationarity, we obtain the bound

$$E_{3,2} \ll N \int_{\frac{\pi R}{2}}^{\pi m} \left[ \frac{\mathbb{E}[|V_1(x)V_2(x)|]}{\sqrt{1 - r_N^{M,0}(x)^2}} - \mathbb{E}|Y_N'(0)|\mathbb{E}|Y_N^{M'}(x)| \right] dx,$$

so that

$$(109) \quad \frac{E_{3,2}}{N} \ll \int_{\frac{\pi R}{2}}^{\pi m} \left[ \frac{\mathbb{E}[|V_1(x)V_2(x)|]}{\sqrt{1-r_N^{M,0}(x)^2}} - \mathbb{E}|Y'_N(0)|\mathbb{E}|Y_N^{M'}(x)| \right] dx.$$

To bound the integral on the RHS of (109), we use the triangle inequality to write

$$(110) \quad \begin{aligned} \frac{E_{3,2}}{N} &\ll \int_{\frac{\pi R}{2}}^{\pi m} \frac{|Cov(|V_1|, |V_2|)|}{\sqrt{1-r_N^{M,0}(x)^2}} dx \\ &+ \int_{\frac{\pi R}{2}}^{\pi m} \left| \frac{\mathbb{E}|V_1| \cdot \mathbb{E}|V_2|}{\sqrt{1-r_N^{M,0}(x)^2}} - \mathbb{E}|Y'_N(0)|\mathbb{E}|Y_N^{M'}(x)| \right| dx =: J_{3,1} + J_{3,2}. \end{aligned}$$

For  $\frac{\pi R}{2} < x < \pi m$ ,  $R$  sufficiently large,  $r_N^{M,0}$  is bounded away from 1 (see Corollary 5.4), and therefore, while bounding  $J_{3,1}$ , we may disregard the denominator of the first integrand in (110). Note that if  $V$  is a mean zero Gaussian random variable, then

$$(111) \quad \mathbb{E}(|V|) = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\text{Var}(V)}.$$

and

$$(112) \quad \text{Var}(|V|) = \left(1 - \frac{2}{\pi}\right) \text{Var}(V).$$

Note also that for  $\frac{\pi R}{2} < x < \pi m$  the variances  $\text{Var}(V_1(x))$  and  $\text{Var}(V_2(x))$ , given by the diagonal entries of (76), are bounded away from 0. This follows from the decay of  $r_N^{M,0}(x)$  and  $r_N^{M,0'}(x)$  for large values of  $x$ , due to Corollary 5.4. Thus, an application of Lemma 5.6 yields

$$0 \leq Cov(|V_1|, |V_2|) \leq \frac{(1 - \frac{2}{\pi})Cov(V_1, V_2)^2}{\sqrt{\text{Var}(V_1) \cdot \text{Var}(V_2)}} \ll Cov(V_1, V_2)^2.$$

All in all, we obtain the estimate

$$J_{3,1} \ll \int_{\frac{\pi R}{2}}^{\pi m} Cov(V_1, V_2)^2 dx,$$

which (this time, using the off-diagonal elements of (76)) is

$$\begin{aligned}
(113) \quad J_{3,1} &\ll \int_{\frac{\pi R}{2}}^{\pi m} \left[ -r_N^{M,0''}(x) - \frac{r_N^{M,0}(x) \cdot r_N^{M,0'}(x)^2}{1 - r_N^{M,0}(x)^2} \right]^2 dx \ll \int_{\frac{\pi R}{2}}^{\pi m} (r_N^{M,0''}(x)^2 + r_N^{M,0}(x)^2) dx \\
&\ll \|r_N^{M,0} - r_N\|_{L^2(I_N)}^2 + \|r_N^{M,0''} - r_N''\|_{L^2(I_N)}^2 + \int_{\frac{\pi R}{2}}^{\pi m} (r_N''(x)^2 + r_N(x)^2) dx \\
&\ll \frac{1}{\sqrt{M}} + \frac{1}{R},
\end{aligned}$$

by the triangle inequality, Lemma 5.1, and the decay

$$r_N(x), r_N''(x) \ll \frac{1}{x}.$$

To bound  $J_{3,2}$ , we note that for  $\frac{\pi R}{2} < x < \pi m$ , we may expand

$$\frac{1}{\sqrt{1 - r_N^{M,0}(x)^2}} = 1 + O(r_N^{M,0}(x)^2),$$

with the constant involved in the ' $O$ '-notation being uniform, since  $r_N^{M,0}$  is bounded away from 1 (by Corollary 5.4, say). Thus we may use the triangle inequality to write

$$\begin{aligned}
(114) \quad J_{3,2} &\ll \int_{\frac{\pi R}{2}}^{\pi m} \left| \mathbb{E}|V_1| \cdot \mathbb{E}|V_2| - \mathbb{E}|Y_N'(0)| \mathbb{E}|Y_N^{M'}(x)| \right| dx + \int_{\frac{\pi R}{2}}^{\pi m} r_N^{M,0}(x)^2 \cdot \mathbb{E}|V_1| \mathbb{E}|V_2| dx \\
&\leq \int_{\frac{\pi R}{2}}^{\pi m} \mathbb{E}|V_1| (|\mathbb{E}|V_2| - \mathbb{E}|Y_N^{M'}(x)||) dx + \int_{\frac{\pi R}{2}}^{\pi m} \mathbb{E}|Y_N^{M'}(x)| (|\mathbb{E}|V_1| - \mathbb{E}|Y_N'(0)||) dx \\
&\quad + \int_{\frac{\pi R}{2}}^{\pi m} r_N^{M,0}(x)^2 \cdot \mathbb{E}|V_1| \mathbb{E}|V_2| dx.
\end{aligned}$$

Now, (111) allows us to compute  $\mathbb{E}|V_1|$ ,  $\mathbb{E}|V_2|$ ,  $\mathbb{E}|Y_N'(0)|$  and  $\mathbb{E}|Y_N^{M'}(x)|$ ; (76) implies that all of the expectations above are uniformly bounded for  $\frac{\pi R}{2} < x < \pi m$ . Thus the third term of (114) is bounded by

$$\ll \int_{\frac{\pi R}{2}}^{\pi m} r_N^{M,0}(x)^2 dx \ll \frac{1}{R} + \frac{1}{\sqrt{M}},$$

as before. We bound the first summand of (114) as

$$\begin{aligned} &\ll \int_{\frac{\pi R}{2}}^{\pi m} |\mathbb{E}|V_2| - \mathbb{E}|Y_N^{M'}(x)|| dx \ll \int_{\frac{\pi R}{2}}^{\pi m} \left[ \sqrt{\lambda_2^{M'}} - \left( \lambda_2^{M'} - \frac{r_N^{M,0'}(x)^2}{1 - r_N^{M,0}(x)^2} \right)^{1/2} \right] dx \\ &\ll \frac{1}{\sqrt{\lambda_2^{M'}}} \int_{\frac{\pi R}{2}}^{\pi m} \frac{r_N^{M,0'}(x)^2}{1 - r_N^{M,0}(x)^2} \ll \int_{\frac{\pi R}{2}}^{\pi m} r_N^{M,0'}(x)^2 dx \ll \frac{1}{\sqrt{M}} + \frac{1}{R}, \end{aligned}$$

as earlier, since  $\lambda_{2,N}^{M'}$  is bounded away from 0, and  $r_N^{M,0}$  is bounded away from 1 on the domain of the integration. The second summand of (114) is bounded similarly, resulting in the same bound. Therefore

$$(115) \quad J_{3,2} \ll \frac{1}{\sqrt{M}} + \frac{1}{R}.$$

Recalling (110), the estimates (113) and (115) imply

$$\frac{E_{3,2}}{N} \ll \frac{1}{\sqrt{M}} + \frac{1}{R}$$

by (110). Bounding  $E_{3,1}$  and  $E_{3,3}$  in a similar (but easier) way, we get (see (108))

$$(116) \quad \frac{E_3}{N} \ll \frac{1}{\sqrt{M}} + \frac{1}{R}$$

5.4.4. *Collecting all the estimates.* Collecting the estimates (92), (107) and (116), we see that

$$\begin{aligned} \frac{\text{Var}(Z_N^+ - Z_N^{M,+})}{N} &= \frac{E_1}{N} + \frac{E_2}{N} + \frac{E_3}{N} \\ &= O\left(\epsilon S + \frac{S}{\sqrt{M}\epsilon^2} + \frac{1}{S}\right) + O\left(\frac{RS^{2A}T^6}{M^{1/6}} + \frac{RS}{T}\right) + O\left(\frac{1}{\sqrt{M}} + \frac{1}{R}\right), \end{aligned}$$

which could be made arbitrarily small, upon making an appropriate choice for the parameters  $\epsilon$ ,  $S$ ,  $R$  and  $T$ . Proposition 4.3 is now proved.  $\square$

#### APPENDIX A. THE THIRD MOMENT OF $Z^M$ ON SHORT INTERVALS IS BOUNDED

**Proposition A.1.** *Let  $L$  be a constant,  $K = NL$  and for  $1 \leq k \leq K$  let  $Z_{N,k}^M$  be the number of zeros of  $Y_N^M$  on  $[(k-1)\frac{\pi m}{K}, k\frac{\pi m}{K}]$ . Then for  $L$  sufficiently large, all the third moments  $\mathbb{E}(Z_{N,k}^M)^3$  are uniformly bounded by a constant, independent of  $N$  and  $k$ .*

*Proof.* By stationarity, we may assume that  $k = 1$ . For any  $\tau > 0$  let  $Z = Z_N^M(\tau)$  be the number of zeros of  $Y = Y_N^M$  on  $[0, \tau]$ . Since  $L$  is arbitrarily large, we may reduce the statement of the present Proposition to bounding  $\mathbb{E}Z_N^M(\tau)^3$ , for  $\tau > 0$  sufficiently small. It will be convenient to use the shortcut  $r = r_N^M$ .



Using the formula for the high combinatorial moments of the number of crossings of stationary processes [CL] (see remarks 2.3 and 2.4), we obtain the bound

$$\mathbb{E}[Z(Z-1)(Z-2)] \ll \iint_{[0,\tau]^2} P(x,y) dx dy$$

for the third combinatorial moment (the number of triples) of  $Z(\tau)$ , where  $P = P_N^M$  is given by

$$(117) \quad P(x,y) = \frac{\mathbb{E}[|Y'(0)Y'(x)Y'(y)| | Y(0) = Y(x) = Y(y) = 0]}{(2\pi)^{3/2} \sqrt{f(x,y)}}$$

with  $f(x,y) = f_N^M(x,y) = \det A$  with

$$A = \begin{pmatrix} 1 & r(x) & r(y) \\ r(x) & 1 & r(y-x) \\ r(y) & r(y-x) & 1 \end{pmatrix}.$$

It is easy to compute  $f$  explicitly as

$$(118) \quad f(x,y) = 1 - r(x)^2 - r(y)^2 - r(y-x)^2 + 2r(x)r(y)r(y-x).$$

Since

$$\mathbb{E}Z = \frac{\tau}{\pi} \sqrt{-r''(0)} = O(1),$$

and we proved that the second moments  $\mathbb{E}Z^2$  are uniformly bounded, while proving Proposition 4.4 (see condition 2 of Theorem 4.1), it is then sufficient to prove that the function  $P(x,y)$  is uniformly bounded near the origin. Denote the random vector

$$(V_1, V_2, V_3) = (Y'(0), Y'(x), Y'(y))$$

conditioned upon  $Y(0) = Y(x) = Y(y) = 0$ . The random vector  $(V_1, V_2, V_3)$  has a mean zero multivariate Gaussian distribution and we have

$$(119) \quad P(x,y) = \frac{\mathbb{E}[|V_1 V_2 V_3|]}{\sqrt{f(x,y)}}.$$

by the definition (117).

Applying the Cauchy-Schwartz inequality twice implies the bound

$$\mathbb{E}[|V_1 V_2 V_3|] \leq (\mathbb{E}V_1^2)^{1/2} (\mathbb{E}V_2^4)^{1/4} (\mathbb{E}V_3^4)^{1/4}.$$

Let  $\mathcal{V}_i = \mathcal{V}_i(x,y)$  be the variance of  $V_i$  for  $i = 1, 2, 3$ . Computing explicitly, we have  $\mathcal{V}_i = \frac{1}{f(x,y)} \mathcal{R}_i(x,y)$ , where

$$\begin{aligned} \mathcal{R}_1(x,y) &:= \lambda'_2 \det A - r'(x)^2(1 - r(y)^2) - r'(y)^2(1 - r(x)^2) \\ &\quad - 2r'(x)r'(y)(r(x)r(y) - r(y-x)), \end{aligned}$$

$$\begin{aligned} \mathcal{R}_2(x,y) &:= \lambda'_2 \det A - r'(x)^2(1 - r(y-x)^2) - r'(x-y)^2(1 - r(x)^2) \\ &\quad - 2r'(x)r'(x-y)(r(x)r(y-x) - r(y)), \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_3(x,y) &:= \lambda'_2 \det A - r'(y)^2(1 - r(y-x)^2) - r'(y-x)^2(1 - r(y)^2) \\ &\quad - 2r'(y)r'(y-x)(r(y)r(y-x) - r(x)), \end{aligned}$$

where, as usual, we denote  $\lambda'_2 := -r''(0)$ .

We then have

$$(120) \quad \frac{\mathbb{E}[|V_1 V_2 V_3|]}{\sqrt{f(x, y)}} \ll \frac{\sqrt{\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3}}{f(x, y)^2}.$$

The uniform boundedness of  $P(x, y)$  around the origin  $(x, y) = (0, 0)$  then follows from applying Lemmas A.2 and A.4 on (120), bearing in mind (119).  $\square$

**Lemma A.2.** *Let  $f(x, y)$  be defined by (118) with  $r = r_N^M$ . Then*

$$f(x, y) \gg x^2 y^2 (y - x)^2$$

*uniformly w.r.t.  $N$  in some (fixed) neighbourhood of the origin.*

*Proof.* Recall that  $r_N^M(x) = r_N(x) \cdot S_M(x)$ , and we assume that the neighbourhood is sufficiently small so that  $S_M$  is given by a single polynomial (52) of degree 7 in  $\frac{|x|}{M}$ . We assume with no loss of generality, that  $x, y > 0$ , and furthermore, that  $y > x$ . Let

$$\theta_m^M(x) := r_N^M(x) - 1,$$

and

$$\theta_m(x) := r_N(x) - 1.$$

Let also

$$\theta_\infty(x) = \frac{\sin x}{x} - 1$$

be the limiting function. We will omit the parameters  $m$  and  $M$ , whenever there is no ambiguity.

We rewrite the definition of  $f(x, y)$  as

$$\begin{aligned} f(x, y) = & -(\theta(x)^2 + \theta(y)^2 + \theta(y - x)^2) \\ & + 2(\theta(x)\theta(y) + \theta(y)\theta(y - x) + \theta(x)\theta(y - x)) + 2\theta(x)\theta(y)\theta(y - x). \end{aligned}$$

It is easy to Taylor expand  $\theta = \theta_m^M(x)$  as

$$\theta(x) = a_{2,m}^M x^2 + O(x^4),$$

where the constant in the ‘ $O$ ’-notation is universal, and

$$a_{2,m}^M = a_2 \left( 1 + O\left( \frac{1}{m} + \frac{1}{M^2} \right) \right),$$

where  $a_2 = -\frac{1}{6}$  is the corresponding Taylor coefficient of the limiting function  $\theta_\infty$ . We rewrite it as

$$\theta(x) = -\frac{1}{6} x^2 \left( 1 + O\left( x^2 + \frac{1}{m} + \frac{1}{M^2} \right) \right),$$

so that

$$(121) \quad \theta(x)\theta(y)\theta(y-x) = -\frac{1}{6^3} x^2 y^2 (y-x)^2 \left( 1 + O\left( x^2 + y^2 + (y-x)^2 + \frac{1}{m} + \frac{1}{M^2} \right) \right).$$

Thus, it remains to estimate

$$f_2(x, y) := 2(\theta(x)\theta(y) + \theta(y)\theta(y-x) + \theta(x)\theta(y-x)) - (\theta(x)^2 + \theta(y)^2 + \theta(y-x)^2).$$

Let

$$(122) \quad \theta_\infty(x) = \sum_{n=2}^{\infty} a_n x^n,$$

where it is easy to compute  $a_n$  to be

$$(123) \quad a_n = \begin{cases} \frac{(-1)^n}{(n+1)!}, & n \text{ even} \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, we expand  $\theta_m$  and  $\theta_m^M$  into Taylor series

$$\theta_m(x) = \sum_{n=2}^{\infty} a_{n,m} x^n,$$

and

$$\theta_m^M(x) = \sum_{n=2}^{\infty} a_{n,m}^M x^n.$$

We need the following estimates concerning the Taylor coefficients of  $\theta_m$  and  $\theta_m^M$ .

**Lemma A.3.** (1) *We have the following estimates for the coefficients of  $\theta_m$ ,*

$$a_{2n,m} = a_{2n} \left( 1 + O\left( \frac{1}{m} + \frac{n^2}{m^2} \right) \right)$$

for  $n \ll m$  and

$$a_{2n,m}, a_{2n} \ll e^{4\pi m} \left( \frac{1}{4\pi m} \right)^{2n} n^{O(1)}$$

for  $n \gg m$ . We have  $a_{2n+1,m} = 0$  for every  $n$ .

(2) *We have the following estimates for the coefficients of  $\theta_m^M$ ,*

$$a_{2n,m}^M = a_{2n} \left( 1 + O\left( \frac{1}{m} + \frac{n^2}{m^2} \right) \right) + O\left( \frac{1}{M^2(2n-5)!} \right)$$

$$(124) \quad a_{2n+1,m}^M = \begin{cases} 0, & n \leq 2 \\ O\left( \frac{1}{M^2(2n-6)!} \right), & n \geq 3 \end{cases},$$

for  $n \ll m$ , and

$$(125) \quad a_{n,m}^M \ll e^{4\pi m} \left( \frac{1}{4\pi m} \right)^{2n} n^{O(1)}$$

for  $n \gg m$ .

We postpone the proof of Lemma A.3 until after the end of the proof of Lemma (A.2).

We write

$$\begin{aligned}
f_{2,m}^M(x, y) &= \sum_{\substack{i,j=2 \\ i,j \neq 3,5}}^{\infty} a_{i,m}^M a_{j,m}^M \cdot (2(x^i y^j + y^i (y-x)^j + x^i (y-x)^j) \\
&\quad - (x^{(i+j)} + y^{(i+j)} + (y-x)^{(i+j)})) \\
&= \sum_{\substack{i,j=2 \\ i,j \neq 3,5}}^{\infty} a_{i,m}^M a_{j,m}^M \cdot (x^i y^j + x^j y^i + y^i (y-x)^j + y^j (y-x)^i \\
&\quad + x^i (y-x)^j + x^j (y-x)^i - x^{(i+j)} - y^{(i+j)} - (y-x)^{(i+j)}),
\end{aligned}$$

adding the summands corresponding to  $(i, j)$  and  $(j, i)$ . We introduce the polynomials

$$\begin{aligned}
(126) \quad F_{ij}(x, y) &:= x^i y^j + x^j y^i + y^i (y-x)^j + y^j (y-x)^i + x^i (y-x)^j \\
&\quad + x^j (y-x)^i - x^{(i+j)} - y^{(i+j)} - (y-x)^{(i+j)} \in \mathbb{Z}[x, y],
\end{aligned}$$

so that

$$f_{2,m}^M(x, y) = \sum_{\substack{i,j=2 \\ i,j \neq 3,5}}^{\infty} a_{i,m}^M a_{j,m}^M F_{i,j}(x, y)$$

and

$$f_{2,\infty}(x, y) = \sum_{\substack{i,j=2 \\ i,j \text{ even}}}^{\infty} a_i a_j F_{i,j}(x, y).$$

Note that  $F_{2,2} = 0$  and for every even tuple  $(i, j) \neq (2, 2)$ ,

$$x^2 y^2 (y-x)^2 |F_{i,j}(x, y),$$

so that in this case, we may define

$$H_{i,j}(x, y) = \frac{F_{2i,2j}(x, y)}{x^2 y^2 (y-x)^2} \in \mathbb{Z}[x, y].$$

It is easy to compute  $H_{2,4}$  and  $H_{4,2}$  to be

$$H_{2,4}(x, y) = H_{4,2}(x, y) = -6.$$

We claim that for  $|x|, |y| \leq \tau$ , if  $\tau$  is sufficiently small,

$$(127) \quad |H_{i,j}(x, y)| \ll \max\{2|x|, 2|y|\}^{2(i+j-3)}$$

To prove our claim, we write

$$\begin{aligned}
\frac{F_{2i,2j}(x, y)}{(y-x)^2} &= (y^{2i} (y-x)^{2(j-1)} + y^{2j} (y-x)^{2(i-1)} + x^{2i} (y-x)^{2(j-1)} \\
&\quad + x^{2j} (y-x)^{2(i-1)} - (y-x)^{2(i+j-1)}) - \frac{(y^{2j} - x^{2j})(y^{2i} - x^{2i})}{y-x}.
\end{aligned}$$

The sum of the coefficients of the homogeneous monomials of the last polynomial is  $\ll 2^{2(i+j-1)}$ . Since dividing by  $x^2 y^2$  does not increase the coefficients, the  $H_{i,j}$  are bounded by

$$\ll 2^{2(i+j-1)} \cdot \max\{|x|, |y|\}^{2(i+j-3)} \ll \max\{2|x|, 2|y|\}^{2(i+j-3)},$$

which is our claim (127).

We then have

$$\frac{f_{2,\infty}(x, y)}{x^2 y^2 (y-x)^2} = -12a_2 a_4 + \sum_{i+j \geq 4} a_{2i} a_{2j} H_{i,j}(x, y),$$

so that on any fixed neighbourhood of the origin,

$$\begin{aligned} \left| \frac{f_{2,\infty}(x, y)}{x^2 y^2 (y-x)^2} + 12a_2 a_4 \right| &\leq \sum_{i+j \geq 4} |a_{2i} a_{2j}| |H_{i,j}(x, y)| \\ &\ll (x^2 + y^2) \sum_{i+j \geq 4} \frac{1}{(2i+1)!} \frac{1}{(2j+1)!} 2^{2(i+j-3)} \ll (x^2 + y^2), \end{aligned}$$

by (123) and (127).

Thus to finish the proof of Lemma A.2 it is sufficient to bound

$$\left| \frac{f_{2,\infty}(x, y) - f_{2,m}^M(x, y)}{x^2 y^2 (y-x)^2} \right|.$$

We have

$$\left| \frac{f_{2,\infty}(x, y) - f_{2,m}^M(x, y)}{x^2 y^2 (y-x)^2} \right| \leq \sum_{i+j \geq 6} |a_{i,m}^M a_{j,m}^M - a_i a_j| \left| \frac{F_{i,j}(x, y)}{x^2 y^2 (y-x)^2} \right| = \Sigma^{\text{even}} + \Sigma^{\text{odd}} + 2\Sigma^{\text{mixed}},$$

where

$$\Sigma^{\text{even}} := \sum_{i,j \text{ even}} ; \quad \Sigma^{\text{odd}} := \sum_{i,j \text{ odd}} ; \quad \Sigma^{\text{mixed}} := \sum_{i \text{ odd}, j \text{ even}} .$$

We have

$$\begin{aligned} |\Sigma^{\text{even}}| &= \sum_{i+j \geq 3} |a_{2i,m}^M a_{2j,m}^M - a_{2i} a_{2j}| |H_{i,j}(x, y)| \\ &\ll \sum_{i,j \ll m} \left( \left( \frac{1}{m} + \frac{i^2 + j^2}{m^2} \right) \frac{1}{(2i+1)!(2j+1)!} + \frac{1}{M^2 \cdot (2j-5)!(2i-5)!} \right) \cdot \tau^{2(i+j-3)} \\ &\quad + \sum_{\substack{i \ll m \\ j \gg m}} \frac{1}{(2i-5)!} \cdot \frac{e^{4\pi m}}{(4\pi m)^{2j}} j^{O(1)} \tau^{2(i+j-3)} \\ &\quad + \sum_{\substack{i \gg m \\ j \gg m}} \frac{e^{8\pi m}}{(4\pi m)^{2(i+j)}} (i \cdot j)^{O(1)} \tau^{2(i+j-3)} \ll \left( \frac{1}{m} + \frac{1}{M^2} \right), \end{aligned}$$

which is sufficient.

The main problem while treating the odd and the mixed terms is that for such a tuple  $(i, j)$ ,  $F_{i,j}(x, y)$  is not necessarily divisible by  $x^2 y^2 (y-x)^2$ . However, in any case, it is divisible by  $x^2 (y-x)^2$ . We recall then our assumption  $y > x$ , and thus

$$\left| \frac{F_{i,j}}{x^2 y^2 (y-x)^2} \right| = \left| \frac{1}{y^2} \cdot \frac{F_{i,j}}{x^2 (y-x)^2} \right|.$$

We then define for an odd  $i \geq 7$  and any  $j$  (i.e.  $j \geq 2$  and  $j \neq 3, 5$ ) the polynomial

$$G_{i,j}(x, y) = \frac{F_{i,j}}{x^2 (y-x)^2}.$$

The polynomial  $G_{i,j}$  is a homogeneous polynomial of degree  $i + j - 4$  and we have

$$(128) \quad G_{i,j}(x, y) \ll \max\{2|x|, 2|y|\}^{i+j-4},$$

similarly to (127). Thus, one has

$$(129) \quad \left| \frac{1}{y^2} G_{i,j}(x, y) \right| \ll \max\{2|x|, 2|y|\}^{i+j-6}$$

(here we use  $|x| \leq |y|$ ).

We then write

$$\Sigma^{odd} = \sum a_{i,m}^M a_{j,m}^M \frac{G_{i,j}(x, y)}{y^2} \ll \frac{1}{M^{14}} + \frac{1}{m^2},$$

using the same approach as in case of  $\Sigma^{even}$ , this time plugging (129). Similarly, one obtains the estimate

$$\Sigma^{mixed} = \frac{1}{M^7} + \frac{1}{m^2}.$$

Combining (121) with the estimates on  $f_{2,m}^M$  shows that

$$\frac{f(x, y)}{x^2 y^2 (y - x)^2} = a + O\left(x^2 + y^2 + \frac{1}{m} + \frac{1}{M^2}\right),$$

where

$$a = -12a_2 a_4 - 2/6^3 = \frac{1}{135}.$$

This concludes the proof of Lemma A.2 assuming Lemma A.3.  $\square$

*Proof of Lemma A.3.* First, it is clear that part (2) of Lemma A.3 follows from part (1) by (52), which holds for  $x > 0$  sufficiently small.

We write

$$\theta_m(x) = \frac{2m}{2m-1} \left( \frac{\sin x}{x} h\left(\frac{x}{2m}\right) - 1 \right),$$

where  $h(x) = \frac{x}{\sin x}$ . The multiplication by  $\frac{2m}{2m-1}$  poses no problem here. Now, the Taylor expansion of  $h(x)$  is well-known to be

$$h(x) = \sum_{n \geq 0} (-1)^{n+1} (2^{2n} - 2) B_{2n} \frac{x^{2n}}{(2n)!},$$

where  $B_n$  are the Bernoulli numbers. Recalling the Taylor expansion

$$\frac{\sin x}{x} = \sum_{j \geq 0} (-1)^j \frac{x^{2j}}{(2j+1)!},$$

we obtain, after a little rearranging,

$$\left( \frac{\sin x}{x} h\left(\frac{x}{2m}\right) - 1 \right) = - \sum_{k \geq 1} \left\{ \sum_{\substack{j, n \geq 0 \\ j+n=k}} \binom{2k+1}{2n} (2^{2n} - 2) \frac{B_{2n}}{(2m)^{2n}} \right\} \frac{(-1)^k x^{2k}}{(2k+1)!}.$$

Now,

$$B_{2n} \sim (-1)^{n-1} \cdot 2 \frac{(2n)!}{(2\pi)^{2n}},$$

and therefore the coefficient of  $\frac{(-1)^k x^{2k}}{(2k+1)!}$  is

$$1 + O\left(\sum_{n \geq 1} \binom{2k+1}{2n} \frac{(2n)!}{(4\pi m)^{2n}}\right).$$

By comparing consecutive terms in this series, we find that this is  $1 + O(k^2/m^2)$  provided that  $k \ll m$ , and it is

$$\ll e^{4\pi m} \left(\frac{2k+1}{4e\pi m}\right)^{2k} k^{O(1)}$$

if  $k \gg m$ , using Stirling's formula, since the maximal term occurs when

$$2n \approx 2k + 1 - 4\pi m.$$

Hence we have

$$a_{2n,m} = a_{2n} \left(1 + O\left(\frac{1}{m} + \frac{n^2}{m^2}\right)\right)$$

for  $n \ll m$ , and

$$a_{2n,m}, a_{2n} \ll e^{4\pi m} \left(\frac{1}{4\pi m}\right)^{2n} n^{O(1)}$$

for  $n \gg m$ .

□

**Lemma A.4.** *Let*

$$\begin{aligned} \mathcal{R}_1(x, y) &:= \lambda'_2 f(x, y) - r'(x)^2(1 - r(y)^2) - r'(y)^2(1 - r(x)^2) \\ &\quad - 2r'(x)r'(y)(r(x)r(y) - r(y-x)), \end{aligned}$$

$$\begin{aligned} \mathcal{R}_2(x, y) &:= \lambda'_2 f(x, y) - r'(x)^2(1 - r(y-x)^2) - r'(x-y)^2(1 - r(x)^2) \\ &\quad - 2r'(x)r'(x-y)(r(x)r(y-x) - r(y)), \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_3(x, y) &:= \lambda'_2 f(x, y) - r'(y)^2(1 - r(y-x)^2) - r'(y-x)^2(1 - r(y)^2) \\ &\quad - 2r'(y)r'(y-x)(r(y)r(y-x) - r(x)), \end{aligned}$$

where  $r = r_N^M$  and  $f$  is defined by (118). Then

$$\mathcal{R}_1(x, y) = O(x^4 y^4 (y-x)^2),$$

$$\mathcal{R}_2(x, y) = O(x^4 y^2 (y-x)^4),$$

and

$$\mathcal{R}_3(x, y) = O(x^2 y^4 (y-x)^4).$$

*Proof.* We prove the first statement only, the other ones being symmetrical. We will assume that  $x, y > 0$  and moreover  $y > x$ , so that  $S_M$  is given by a polynomial (52) of degree 7 in  $\frac{x}{M}$ . For brevity, we denote  $h(x, y) = h_m^M(x, y) = \mathcal{R}_1(x, y)$ . Similarly, we denote  $h_\infty(x, y)$ , defined the same way

as  $h_m^M(x, y)$ , where we use  $r(x) = r_\infty(x) = \frac{\sin x}{x}$  rather than  $r_N^M$ . We rewrite  $h(x, y)$  in terms of  $\theta := r - 1$  as

$$\begin{aligned} h(x, y) &:= \lambda_2' f(x, y) + 2 \left( \theta'(x)^2 \theta(y) + \theta'(y)^2 \theta(x) \right. \\ &\quad \left. - \theta'(x) \theta'(y) (\theta(x) + \theta(y) - \theta(y - x)) \right) \\ &\quad + (\theta(x) \theta'(y) - \theta(y) \theta'(x))^2. \end{aligned}$$

Expanding  $\theta$  and  $\theta'$  into the Taylor series as earlier and using  $\lambda_2' = -2a_2$ , we have

$$\begin{aligned} (130) \quad h_\infty(x, y) &= \sum_{i_1, j_1, i_2, j_2 \in S} a_{2i_1} a_{2j_1} a_{2i_2} a_{2j_2} A_{2i_1, 2j_1, 2i_2, 2j_2}(x, y) \\ &\quad + \sum_{i, j, k \geq 2} a_{2i} a_{2j} a_{2k} B_{2i, 2j, 2k}(x, y) + \sum_{i, j \geq 2, k \geq 1} a_2 a_{2i} a_{2j} a_{2k} C_{2i, 2j, 2k}(x, y) \\ &\quad + \sum_{i, j \geq 2} a_2^2 a_{2i} a_{2j} D_{2i, 2j}(x, y) + \sum_{i, j \geq 2} a_2 a_{2i} a_{2j} E_{2i, 2j}(x, y) + \sum_{i \geq 2} a_2^2 a_{2i} I_{2i}(x, y) \\ &\quad + \sum_{i \geq 2} a_2^3 a_{2i} J_{2i}(x, y), \end{aligned}$$

where

$$A_{i_1, j_1, i_2, j_2}(x, y) = j_1 j_2 \cdot (x^{i_1} y^{j_1 - 1} - x^{j_1 - 1} y^{i_1}) \cdot (x^{i_2} y^{j_2 - 1} - x^{j_2 - 1} y^{i_2}),$$

$$B_{i, j, k}(x, y) = 2ij(x^{i+j-2} y^k + y^{i+j-2} x^k - x^{i-1} y^{j-1} (x^k + y^k - (y-x)^k)),$$

$$C_{i, j, k} = -4x^i y^j (y-x)^k,$$

$$\begin{aligned} D_{i, j}(x, y) &= -4(y-x)^j (x^2 y^i + x^i y^2) + 4(yx^i - xy^i)(yx^j - xy^j) \\ &\quad + 4j(x^i y - xy^i)(x^2 y^{j-1} - x^{j-1} y^2), \end{aligned}$$

$$\begin{aligned} E_{i, j}(x, y) &= -2F_{i, j} + 4i(2(x^i y^j + x^j y^i) - (x^{i-1} y + y^{i-1} x)(x^j + y^j - (y-x)^j)) \\ &\quad + 2ij(x^{i+j-2} y^2 + y^{i+j-2} x^2 - 2x^i y^j), \end{aligned}$$

(here the polynomial  $F_{i, j}$  is defined as in (126)),

$$I_i(x, y) = -4F_{i, 2} + 8(x^2 y^i + x^i y^2 - xy(x^i + y^i - (y-x)^i)) = 0,$$

so that we may disregard  $I_i$  altogether,

$$\begin{aligned} J_i(x, y) &= -4x^2 y^2 (y-x)^i - 4(y-x)^2 (x^2 y^i + x^i y^2) \\ &\quad + 8xy(yx^i - xy^i)(x-y) + 4i(x-y)(x^3 y^i - y^3 x^i), \end{aligned}$$

and

$$S = \mathbb{Z}^4 \setminus \left( (\{(i_1, 1)\} \times \{(i_2, 1)\}) \cup (\{(i_1, 1)\} \times \{(1, j_2)\}) \cup (\{(1, j_1)\} \times \{(i_2, 1)\}) \right).$$

From all the above, it is easy to check that for all the (even) indexes within the frame,  $A, C, D$  and  $J$  are divisible by  $P(x, y) := x^4 y^4 (y-x)^2$ , and, moreover,  $B_{i, j, k}(x, y) + B_{j, i, k}(x, y)$  and  $E_{i, j}(x, y) + E_{j, i}(x, y)$  are divisible by  $P(x, y)$  (in particular,  $B_{i, i, k}$  and  $E_{i, i}$  are). It follows then that all the polynomials above vanish, unless their degree is  $\geq 10$ . In addition, we



have the following estimates, which follow from the same reasoning as while proving (128):

$$\begin{aligned} & \left| \frac{A_{i_1, j_1, i_2, j_2}(x, y)}{x^4 y^4 (y-x)^2} \right|, \left| \frac{B_{i, j, k}(x, y)}{x^4 y^4 (y-x)^2} \right|, \left| \frac{C_{i, j, k}}{x^4 y^4 (y-x)^2} \right|, \left| \frac{D_{i, j}(x, y)}{x^4 y^4 (y-x)^2} \right|, \\ & \left| \frac{(E_{i, j} + E_{j, i})(x, y)}{x^4 y^4 (y-x)^2} \right|, \left| \frac{J_i(x, y)}{x^4 y^4 (y-x)^2} \right| \ll \max\{2^{1+\epsilon}|x|, 2^{1+\epsilon}|y|\}^d, \end{aligned}$$

(say), for some  $\epsilon > 0$ , where  $d$  is the degree of the corresponding polynomial on the LHS.

Thus, we have

$$(131) \quad \left| \frac{h_\infty(x, y)}{x^4 y^4 (y-x)^2} - c \right| = O(x^2 + y^2),$$

by the rapid decay (123) of the Taylor coefficients of  $\theta_\infty$ . The constant  $c$  may be computed explicitly to be

$$c = \frac{1}{212625}.$$

To conclude the proof of Lemma A.4 in this case, we need to bound

$$\left| \frac{h_\infty(x, y) - h_m^M(x, y)}{x^4 y^4 (y-x)^2} \right|.$$

Similarly to (130), we have

$$\begin{aligned} h_m^M(x, y) &= \sum_{i_1, j_1, i_2, j_2 \in S'} 'a_{i_1, m}^M a_{j_1, m}^M a_{i_2, m}^M a_{j_2, m}^M A_{i_1, j_1, i_2, j_2}(x, y) \\ &+ \sum_{i, j, k \geq 4} 'a_{i, m}^M a_{j, m}^M a_{k, m}^M B_{i, j, k}(x, y) + \sum_{i, j \geq 4, k \geq 2} 'a_{2, m}^M a_{i, m}^M a_{j, m}^M a_{k, m}^M C_{i, j, k}(x, y) \\ &+ \sum_{i, j \geq 4} '(a_{2, m}^M)^2 a_{i, m}^M a_{j, m}^M D_{i, j}(x, y) + \sum_{i, j \geq 4} 'a_{2, m}^M a_{i, m}^M a_{j, m}^M E_{i, j}(x, y) \\ &+ \sum_{i \geq 4} '(a_{2, m}^M)^3 a_{i, m}^M J_i(x, y), \end{aligned}$$

where the  $'$  in the  $\sum'$  stands for the fact that all of the indexes involved in the summations above are  $\neq 3, 5$ , and

$$S = \mathbb{Z}^4 \setminus \left( \{(i_1, 2)\} \times \{(i_2, 2)\} \cup \{(i_1, 2)\} \times \{(2, j_2)\} \cup \{(2, j_1)\} \times \{(i_2, 2)\} \right).$$

As in the proof of Lemma A.2, the tricky part here is that for odd indexes, the polynomials are no longer divisible by  $P(x, y) = x^4 y^4 (y-x)^2$ . However, we notice that, by our assumption  $y \geq x$ , it is sufficient that they are divisible by  $x^4 (y-x)^2$ , and their degree is  $\geq 10$  (see the end of the proof of Lemma A.2). To check this, we note that there is certainly no problem with  $A_{i_1, j_1, i_2, j_2}(x, y)$ ,  $(B_{i, j, k} + B_{j, i, k})(x, y)$  and  $C_{i, j, k}(x, y)$ . Moreover, one can easily check that  $D_{i, j}$  and  $E_{i, j} + E_{j, i}$  are divisible by  $x^4 (y-x)^2$  for any  $i, j \geq 2$ , and for any  $i \geq 5$ ,  $J_i$  is divisible by  $x^4 y^2 (y-x)^2$ . Finally, we check that all the relevant polynomials have degree  $\geq 10$ .

This, together with the rapid decay (124) for  $n \ll m$  and (125) for  $n \gg m$ , of the Taylor coefficients, imply that

$$\left| \frac{h_\infty(x, y) - h_m^M(x, y)}{x^4 y^4 (y - x)^2} \right| \ll \frac{1}{m} + \frac{1}{M^2}.$$

Combining the last estimate with (131) concludes the proof of the present Lemma. □

#### REFERENCES

- [BBL] Bogomolny, E.; Bohigas, O.; Leboeuf, P. Quantum chaotic dynamics and random polynomials. *J. Statist. Phys.* 85 (1996), no. 5-6, 639–679.
- [BD] Bleher, Pavel; Di, Xiaojun Correlations between zeros of a random polynomial. *J. Statist. Phys.* 88 (1997), no. 1-2, 269–305.
- [BR] Berk, Kenneth N. A central limit theorem for  $m$ -dependent random variables with unbounded  $m$ . *Ann. Probability* 1 (1973), 352–354.
- [CL] Cramer, Harald; Leadbetter, M. R. The moments of the number of crossings of a level by a stationary normal process. *Ann. Math. Statist.* 36 1965 1656–1663.
- [CZ] Cuzick, Jack A central limit theorem for the number of zeros of a stationary Gaussian process. *Ann. Probability* 4 (1976), no. 4, 547–556
- [D] Dunnage, J. E. A. The number of real zeros of a random trigonometric polynomial. *Proc. London Math. Soc.* (3) 16 1966 53–84.
- [DN] Diananda, P. H. The central limit theorem for  $m$ -dependent variables. *Proc. Cambridge Philos. Soc.* 51, (1955). 92–95.
- [EO] Erdos, Paul; Offord, A. C. On the number of real roots of a random algebraic equation. *Proc. London Math. Soc.* (3) 6 (1956), 139–160.
- [F1] Farahmand, K. On the variance of the number of real roots of a random trigonometric polynomial. *J. Appl. Math. Stochastic Anal.* 3 (1990), no. 4, 253–261.
- [F2] Farahmand, K. On the variance of the number of real zeros of a random trigonometric polynomial. *J. Appl. Math. Stochastic Anal.* 10 (1997), no. 1, 57–66.
- [GS] Gikhman, I. I.; Skorokhod, A. V. Introduction to the theory of random processes. Translated from the 1965 Russian original. Reprint of the 1969 English translation. With a preface by Warren M. Hirsch. Dover Publications, Inc., Mineola, NY, 1996. xiv+516 pp.
- [IM1] Ibragimov, I. A.; Maslova, N. B. The mean number of real zeros of random polynomials. I. Coefficients with zero mean. (Russian) *Teor. Veroyatnost. i Primenen.* 16 1971 229–248.
- [IM2] Ibragimov, I. A.; Maslova, N. B. The mean number of real zeros of random polynomials. II. Coefficients with a nonzero mean. (Russian) *Teor. Veroyatnost. i Primenen.* 16 1971 495–503.
- [K] Kac, M; On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.* 49, (1943). 314–320.
- [KL] Kratz, Marie F.; León, José R. Central limit theorems for level functionals of stationary Gaussian processes and fields. *J. Theoret. Probab.* 14 (2001), no. 3, 639–672.
- [LO1] Littlewood, J. E.; Offord, A. C. On the distribution of the zeros and  $a$ -values of a random integral function. I. *J. London Math. Soc.* 20, (1945). 130–136.
- [LO2] Littlewood, J. E.; Offord, A. C. On the distribution of zeros and  $a$ -values of a random integral function. II. *Ann. of Math.* (2) 49, (1948) 885–952; errata 50, 990–991 (1949).
- [LO3] Littlewood, J. E.; Offord, A. C. On the number of real roots of a random algebraic equation. III. *Rec. Math. [Mat. Sbornik] N.S.* 12(54), (1943). 277–286.
- [M1] Maslova, N. B. The variance of the number of real roots of random polynomials. (Russian) *Teor. Veroyatnost. i Primenen.* 19 (1974), 36–51.
- [M2] Maslova, N. B. The distribution of the number of real roots of random polynomials. (Russian) *Teor. Veroyatnost. i Primenen.* 19 (1974), 488–500.

- [ML] Malevič, T. L. Asymptotic normality of the number of crossings of the zero level by a Gaussian process. (Russian) *Teor. Veroyatnost. i Primenen* 14 1969 292–301.
- [SL] Slud, Eric Multiple Wiener-Ito integral expansions for level-crossing-count functionals. *Probab. Theory Related Fields* 87 (1991), no. 3, 349–364.
- [SSWZ] Steinberg, H; Schultheiss, P. M; Wogrin, C. A; Zweig, F.; Short-Time Frequency Measurement of Narrow-Band Random Signals by Means of a Zero Counting Process. *Journal of Applied Physics – February 1955 – Volume 26, Issue 2*, pp. 195-201
- [ST] Sodin, Mikhail; Tsirelson, Boris Random complex zeroes. I. Asymptotic normality. *Israel J. Math.* 144 (2004), 125–149.
- [SZ] Shiffman, Bernard; Zelditch, Steve Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Comm. Math. Phys.* 200 (1999), no. 3, 661–683.
- [PS] Sosee, P; The average number of the roots of a random trigonometric polynomial, Summer undergraduate research final report, submitted to ISM; McGill University, 2007.
- [Q] Qualls, Clifford; On the number of zeros of a stationary Gaussian random trigonometric polynomial. *J. London Math. Soc. (2)* 2 1970 216–220.
- [RW] Rudnick, Zev; Wigman, Igor; On the volume of nodal sets for eigenfunctions of the Laplacian on the torus. *Ann. Henri Poincaré* 9 (2008), no. 1, 109–130.
- [RWL] Romano, Joseph P.; Wolf, Michael A more general central limit theorem for  $m$ -dependent random variables with unbounded  $m$ . *Statist. Probab. Lett.* 47 (2000), no. 2, 115–124.
- [Y] Ylvisaker, N. Donald The expected number of zeros of a stationary Gaussian process. *Ann. Math. Statist* 36 1965 1043–1046.

DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUES, UNIVERSITÉ DE MONTRÉAL  
C.P. 6128, SUCC. CENTRE-VILLE MONTRÉAL, QUÉBEC H3C 3J7, CANADA.

*E-mail address:* `andrew@dms.umontreal.ca`

CENTRE DE RECHERCHES MATHÉMATIQUES (CRM), UNIVERSITÉ DE MONTRÉAL C.P.  
6128, SUCC. CENTRE-VILLE MONTRÉAL, QUÉBEC H3C 3J7, CANADA

CURRENTLY AT

INSTITUTIONEN FÖR MATEMATIK, KUNGLIGA TEKNISKA HÖGSKOLAN (KTH), LINDST-  
EDTSVÄGEN 25, 10044 STOCKHOLM, SWEDEN

*E-mail address:* `wigman@kth.se`