The Irrationals

Julian Havil

With a new foreword by Andrew Granville

PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD

Published by Princeton University Press,
41 William Street, Princeton, New Jersey 08540

In the United Kingdom: Princeton University Press,
99 Banbury Road, Oxford OX2 6JX

Princeton Science Library paperback edition, 2023
Paperback ISBN 9780691247663
E-book ISBN 9780691247670

British Library Cataloguing-in-Publication Data is available

This book has been composed in LucidaBright

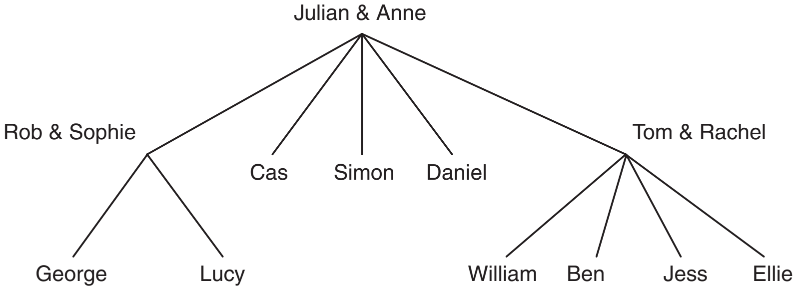
Typeset by T&T Productions Ltd, London

Cover image: Pi number spiral representation, illustration.
MARTIN KRZYWINSKI / Science Source

press.princeton.edu

Printed in the United States of America

This book is dedicated to our world's most significant tree



It was difficult enough being a mathematician, this being the frightening subject of which even educated people knew nothing, not even what it was, and of which they might proudly boast ignorance.

Andrew Hodges (of Alan Turing)

A mathematician learns more and more about less and less, until he knows everything about nothing; whereas a philosopher learns less and less about more and more, until he knows nothing about everything.

Anonymous

Contents

Foreword to the Princeton Science Library Edition ix

Acknowledgments xv

Introduction 1

CHAPTER ONE
Greek Beginnings 9

CHAPTER TWO
The Route to Germany 52

CHAPTER THREE
Two New Irrationals 92

CHAPTER FOUR
Irrationals, Old and New 109

CHAPTER FIVE
A Very Special Irrational 137

CHAPTER SIX
From the Rational to the Transcendental 154

CHAPTER SEVEN
Transcendentals 182

CHAPTER EIGHT
Continued Fractions Revisited 211

CHAPTER NINE
The Question and Problem of Randomness 225

CHAPTER TEN
One Question, Three Answers 235

CHAPTER ELEVEN
Does Irrationality Matter? 252

APPENDIX A
The Spiral of Theodorus 272

APPENDIX B
Rational Parameterizations of the Circle 278

APPENDIX C
Two Properties of Continued Fractions 281

APPENDIX D
Finding the Tomb of Roger Apéry 286

APPENDIX E
Equivalence Relations 289

APPENDIX F
The Mean Value Theorem 294

Index 295

Foreword to the Princeton Science Library Edition

Although only ten years old, Julian Havil's book on the history and understanding of irrational numbers is already, quite rightly, being treated as a "classic" by Princeton University Press. Havil leads us through the maelstrom of ideas used to understand irrationality through the ages, while confronting the contemporary mathematical issues. He likes a good story, especially those that beautifully frame the mathematics.

This contrasts with many mathematics books, even supposedly historical tomes, which benefit from hindsight to tell a consistent narrative that inevitably leads to where thinking is today (for example, the beautiful book of the same title by Ivan Niven from 1955). On the other hand, Havil explores the many byways that thinking went down, the subtleties that arose, the choices that were made and why (always taking care to identify what sources are far from contemporary and what stories are not verifiable).

To give a sense of the richness of history, one arguably needs to see all these different ideas that had importance at different times, and their interplay, even ideas that are not so important in the modern theory. Too much written history condenses all from a narrow perspective and misses the wealth of thoughts; Havil masterfully weaves many streams of ideas together, and there are digressions, and the book is better for this breadth as well its focus and depth.

It begins and ends on philosophical questions, with a lot of mathematics packed in-between. Havil begins with a tour of how Greek thoughts evolved, particularly how their geometric rather than algebraic insight led them to object to the possibility of "incommensurable" geometric bodies.[1] Indeed, an irrational root of

---

[1] A geometric body is "incommensurable" if it cannot be scaled so that all sides are integers.

an equation feels as if it arrives more naturally when solving an equation than when constructing an equation to determine the length of a side of a right-angled triangle.

Havil posits that since Euclid's book was written at least 200 years after the heyday of the Pythagorean school, it may be that their original approach was more geometric and was subsequently modified and developed in a more algebraic direction. As evidence, he studies two of Plato's Socratic dialogues (from the period in-between Pythagoras and Euclid) and suggests that a later dialogue indicates that the early Pythagoreans did not have an easily generalizable method for proving the irrationality of $\sqrt{n}$ as n grows. By contrast, when eighth-century Arab mathematicians explored algebra, they saw numbers as roots of equations, and so irrationals came naturally to them and their successors. Indeed, the great mathematician and poet Omar Khayyam (1048–1122 C.E.) went so far as to derive a cubic equation from geometric considerations, solving it through a sequence of ever-better rational approximations (using continued fractions), and noting that it "cannot be solved by plane geometry." Such ideas of number really only came to full fruition 800 years later.

Havil takes joy in side stories, like the following one about Duns Scotus (1266–1308 C.E.):[2] The idea that the universe eventually reverts cyclically over time to its original state, the celestial bodes all aligning in the same positions relative to each other, has its roots in ancient Greece. Scotus made the wonderful mathematical observation that if any celestial motion is incommensurate with any other, then this cannot be true, that all the bodies can never simultaneously return to exactly the same place!

Havil's historical digressions give him every opportunity to discuss the many approaches to surds, in particular the Golden Ratio, and subsequently our understanding of the locus of points on algebraic curves, and then the creation of calculus and surprising (but often irrelevant) connections between a sequence of irrationals and a limit point that is rational.

Any real number with an infinite continued fraction is irrational (and quadratics have periodic infinite continued fractions). By determining the continued fraction of e (as well as various

---

[2] Duns's name led unfairly to the word "dunce."

surds involving e), Euler deduced in 1737 that e is irrational (and even non-quadratic). It required more ingenuity to establish that π is irrational: In 1761, Lambert determined a continued fraction type representation for tan x, which allowed him to show that tan r is irrational whenever r is rational. Hence π is irrational as $\tan\frac{\pi}{4}=1$

By 1815 Fourier observed that it is far easier to prove the irrationality of e using the representation $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$ yet, Havil notes, there are many seemingly as simple series, like $\frac{1}{1!+1}+\frac{1}{2!+1}+\frac{1}{3!+1}+\cdots,$ whose irrationality remains an open question.

This led to a broadening of perspective in the nineteenth century answering such questions as: What classes of numbers are irrationals? What functions of a given irrational number are irrational? and When can we add irrational numbers to get more irrational numbers? The fundamental breakthrough came from Hermite's cooking up of polynomials with special properties, so that integrals involving those polynomials could be evaluated in several ways (using integration by parts) and the special properties allow control of endpoints. Hermite's idea had a large impact on the irrationality of values of special functions (like trig functions, exponentials, and logarithms). At a similar time, precursors to Galois Theory (such as the theory of symmetric functions) allowed authors to add surds and determine irrationality, and a deeper understanding of irrationality of polynomials (such as Gauss's "rational root theorem") made it easy to determine all rational roots by hand.

One of the greatest changes in perspective came with Abel's result that *there are algebraic numbers which are not expressible by radicals*, and developed by Galois into his eponymous theory (based on ideas of Gauss and Ruffini). Havil calls Gauss "inimitable," Abel "tragic," and Galois "equally tragic, political, love-struck and outspoken"! These personal descriptions tell us that there is much more to explore, not all of it mathematical.

Havil makes space for a careful evaluation of Apéry's 1978 proof of the irrationality of $\zeta(3)$, proudly quoting William Blake's “what is now proved was once only imagined” and even sharing the inscription on Apéry’s Parisian tomb. This careful examination leads Havil to pursue the relevance of approximating irrationals by a sequence of rationals, to better understand them.[3] Approximations to π are useful (but difficult), whereas at the end of the nineteenth century, Hurwitz, Markov, Perron, and others realized that a lot could be said about the quality of approximations to quadratic surds. Havil revels in these discoveries, especially the Lagrange spectrum.

This exploration into the quality of rational approximations culminated in Klaus Roth’s 1955 results showing that one can explicitly bound the quality of rational approximations to any real algebraic integer α: It is known from the theory of continued fractions that there exists an infinite sequence of rationals p/q with $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$ *but* this upper bound is roughly as good as it gets: For any $\varepsilon > 0$ we have $\left|\alpha - \frac{p}{q}\right| > \frac{1}{q^{2+\varepsilon}}$ for all sufficiently large q. Roth’s great theorem opens the door to searching for *transcendental numbers*, numbers that are not the root of an algebraic equation. If α can be *too well approximated* infinitely often, then it can’t be algebraic, so it must be transcendental.

Back in 1844, Liouville had identified a sequence of transcendental numbers, using a weaker theory of approximations, but he was frustrated in his aim to prove e is transcendental. It was not until 1873 that Hermite used one of his brilliant polynomial constructions in a cleverly constructed integral to prove that e is transcendental, a tour-de-force of calculus and cleverness. Surely the door was now open to Hermite proving that π transcendental? However, Hermite was exhausted, writing, “If others undertake this enterprise, no one will be happier than I in their success.” Indeed, in 1881, Lindemann did produce such a proof, which clearly had its genesis in Hermite’s work.

Just one year after Hermite’s paper, Cantor began his revolution of mathematical logic, sets, and infinities, with a paper in

---

[3] How well this can be done for a typical irrational is the subject of the celebrated 2020 proof, by Koukoulopoulos and Maynard, of the Duffin-Schaefer conjecture.

which he proved that the set of transcendental numbers is far larger than the set of algebraic numbers (an *uncountable infinity* versus a countable one) yet his methods left him unable to identify any particular transcendental. At the time this approach to mathematics seemed so counterintuitive that Kronecker called Cantor “a corruptor of youth,” and Poincaré declared that Cantor’s theories were “a malady that would be cured one day” (but they haven’t been).

There is much more in this book besides, and it ends with some philosophy in the chapter “Does Irrationality Matter?,” finding that in most engineering examples a good rational approximation (say to π) will do, but when taking certain limits it can make all the difference, ending up with considering tiling squares with unequal squares.

In this foreword I have attempted to give a sense of many of the themes embraced by Havil. *The Irrationals* will continue to be an inspiration to experienced mathematicians and novices alike. There is a lot of content, so one cannot expect to digest too much too fast! *The Irrationals* has a clear narrative theme and a consistent perspective on what to expect of the reader. It is a good read as is, but since it is chock-full of ideas and gems from the history of mathematics, it is also a pleasure to pick up, open to a random page, and rediscover what is there. Enjoy!

—Andrew Granville