CORRIGENDUM TO “REFINEMENTS OF GOLDBACH’S CONJECTURE, AND THE GENERALIZED RIEMANN HYPOTHESIS”

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Abstract: Karin Halupczok [4] pointed out that we have stated an estimate in [3] that does not follow as easily as claimed. Although we are unable to obtain the claimed estimate, we prove a good enough estimate to (mostly) recover the theorems claimed in [3].

An explicit version of the prime number theorem states that if $x$ is an integer and $1 \leq T \leq x$ then

$$
\sum_{p \leq x} \log p = x - \sum_{\rho \atop |\text{Im}\rho| \leq T} \frac{x^\rho}{\rho} + O \left( \frac{x(\log x)^2}{T} \right),
$$

where the sum is over zeros $\rho$ of $\zeta(\rho) = 0$ with $\text{Re}(\rho) > 0$. Let $B = \sup \{\text{Re} \rho; \zeta(\rho) = 0\}$ (note that $1 \geq B \geq 1/2$). We claimed [3, (5.1)] that by partial summation with $T = x$ it is not hard to show that

$$
\sum_{2N \leq x} G(2N) = \sum_{p+q \leq x} \log p \log q = \frac{x^2}{2} - 2 \sum_{\rho \atop |\text{Im}\rho| \leq x} \frac{x^{1+\rho}}{\rho(1+\rho)} + O(x^{2B+o(1)});
$$

however we have not been able to repeat the argument and Karen Halupczok [4] pointed out the references [1,2] where this issue has been investigated in some detail for $B = \frac{1}{2}$, and nothing so strong has been proved. Here we sketch a simple argument to prove that

$$
\sum_{2N \leq x} G(2N) = \sum_{p+q \leq x} \log p \log q = \frac{x^2}{2} - 2 \sum_{\rho \atop |\text{Im}\rho| \leq x} \frac{x^{1+\rho}}{\rho(1+\rho)} + O(x^{2+4B} (\log x)^2). \hspace{1cm} (2)
$$

Thanks are due to Karin Halupczok for finding the mistake and for discussing this correction.
Note that $2B < \frac{2+4B}{3} < 1 + B$ as $B < 1$, so our error term is not quite as good as was claimed in [3], but it is comfortably strong enough to recover Theorem 1A of [3]. Using a zero-density estimate one can improve the error term to $\ll x^2B(\log x)^O(1)$, as claimed, when $B \geq \frac{2}{3}$, and to an exponent between $2B$ and $\frac{2+4B}{3}$ when $\frac{1}{2} \leq B \leq \frac{3}{4}$.

This mistake is repeated in all four parts of Theorem 1 in [3], so corrections are needed throughout: Replacing $2B$ by $\frac{2+4B}{3}$ on the fourth line of page 171 allows us to recover Theorem 1B. Similarly replacing $2B$ by $\frac{2+4B}{3}$ in (1.3) allows us to recover Theorem 1D. There is a mistake in the proof of Theorem 1C, two lines above (5.5), where complex variables $\rho$ and $\sigma$ are treated as if they are real variables. If a similar correction is made there we do not quite recover Theorem 1C. Instead we can prove that if (1.2) holds then the Riemann Hypothesis for Dirichlet $L$-functions mod $q$ holds; and if the Riemann Hypothesis for Dirichlet $L$-functions mod $q$ holds then we obtain (1.2) with error term $O(x^{\frac{3}{2}}(\log x)^3)$.

**Sketch of proof of (2).** What does follow from (1) by partial summation (and noting that $\sum_{T < |\text{Im}\rho| \leq x} |x^{1+\rho}/\rho(1+\rho)| \ll x^2 \log T/T$, is

$$\sum_{p+q \leq x} \log p \log q = \frac{x^2}{2} - 2 \sum_{|\text{Im}\rho| \leq x} \frac{x^{1+\rho}}{\rho(1+\rho)} + \sum_{\rho, \rho' \neq 0 \text{ with } |\text{Im}\rho|, |\text{Im}\rho'| \leq T} \frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(\rho + \rho')} \cdot \frac{x^{\rho + \rho'}}{\rho + \rho'} + O\left(\frac{x^2(\log x)^2}{T}\right).$$

Stirling’s formula implies that $|e^{\rho}\Gamma(\rho)| \asymp |\rho|^{\rho - \frac{1}{2}} = |\rho|^{\Re(\rho) - \frac{1}{2}}e^{-\arg(\rho)\text{Im}(\rho)}$ so that if $\rho = \beta + i\gamma$ with $\beta \in (0, 1)$ and $|\gamma| \gg 1$ then $|e^{\rho}\Gamma(\rho)| \asymp |\gamma|^{\beta - \frac{1}{2}}e^{-\frac{\gamma}{2}|\gamma|}$, since $\arg(\rho) = \pm\left(\frac{\pi}{2} + O\left(\frac{1}{|\gamma|}\right)\right)$ when $\text{Im}(\rho) = \pm|\gamma|$. Let $\rho' = \beta' + i\gamma'$ with $|\gamma| \geq |\gamma'|$. Therefore if $\gamma$ and $\gamma'$ have the same sign then

$$\Gamma(\rho)\Gamma(\rho')/(\rho + \rho')\Gamma(\rho + \rho') \asymp |\gamma|^{\beta - \frac{1}{2}}|\gamma'|^{\beta' - \frac{1}{2}}/\gamma + \gamma'|^{\beta + \beta' + \frac{1}{2}} \asymp |\gamma'|^{\beta' - \frac{1}{2}}/|\gamma|^{\beta' + 1}.$$

If $\gamma$ and $\gamma'$ have opposite signs then

$$\Gamma(\rho)\Gamma(\rho')/(\rho + \rho')\Gamma(\rho + \rho') \asymp |\gamma|^{\beta - \frac{1}{2}}|\gamma'|^{\beta' - \frac{1}{2}}e^{-\pi|\gamma'|}/(1 + |\gamma + \gamma'|)^{\beta + \beta' + \frac{1}{2}} \ll |\gamma'|^{\beta' - \frac{1}{2}}/|\gamma|^{\beta' + 1},$$

since $|\gamma|^{\beta + \beta' + \frac{1}{2}}e^{-\pi|\gamma'|} \ll (1 + |\gamma + \gamma'|)^{\beta + \beta' + \frac{1}{2}}$. We have $(|\gamma'|/|\gamma|)^{\beta'} \leq 1$ which implies that $|\gamma'|^{\beta' - \frac{1}{2}}/|\gamma|^{\beta' + 1} \leq 1/|\gamma'|^{\frac{1}{2}}|\gamma|$. Hence the final sum in the last displayed equation is $\ll x^{2B}\sum_{|\gamma| \leq |\gamma'| \leq T} 1/|\gamma'|^{\frac{1}{2}}|\gamma| \ll x^{2B}\sum_{|\gamma| \leq T} (\log |\gamma|)/|\gamma|^{\frac{1}{2}} \ll x^{2B}T^{1/2}(\log T)^2$; and (2) follows by selecting $T = x^{\frac{3}{4}(1-B)}$.

**Improvement using a zero-density estimate.** In the bound above the contribution is majorized by those terms with $\beta, \beta' \geq \frac{1}{2}$ and $\gamma, \gamma' \geq 0$ (using the
Goldbach’s conjecture

By using Carlson’s zero-density estimate \( \#\{\rho: \zeta(\rho) = 0 \text{ and } \beta \geq \sigma, \gamma| \leq T\} \ll T^{4\sigma(1-\sigma)}(\log T)^{O(1)} \), we can improve our bound (we will select \( T \leq x^{1/(8B-4)} \) below, which simplifies several steps, since then \( B \leq \frac{1}{2} + \frac{\log x}{8\log \gamma} \)): throughout we sum over the zeros arranged by height, in dyadic intervals, and obtain that the final sum in the displayed equation is

\[
\ll \sum_{1 \leq \gamma' \leq \gamma \leq T} \frac{(\gamma')^{\beta-1}}{\gamma' + 1} \cdot x^{\beta + \gamma'} \ll \mathcal{L} \sum_{\gamma \leq T} \max_{1 \leq t \leq \gamma} \int_{\sigma = 1/2}^{B} \frac{t^{\sigma - \frac{1}{2}}}{\gamma^{\sigma + 1}} \cdot x^{\beta + \sigma t^{4\sigma(1-\sigma)}} d\sigma
\]

\[
\ll \mathcal{L} \sum_{\gamma \leq T} \max_{1/2 \leq \sigma \leq B} x^{\beta + \sigma \gamma^{4\sigma(1-\sigma) - \frac{1}{2}}} \ll \mathcal{L} \sum_{\gamma \leq T} x^{\beta + B \gamma^{4B(1-B) - \frac{3}{2}}}
\]

\[
\ll \mathcal{L} \max_{u \leq T} \max_{1/2 \leq \tau \leq B} x^{\tau + B \max_{1/2 \leq \tau \leq B} u^4 B(1-B) + 4 \tau(1-\tau) - \frac{3}{2}} \ll \mathcal{L} \max_{u \leq T} x^{2B} u^{8B(1-B) - \frac{3}{2}}
\]

\[
\ll x^{2B(1 + T^{8B(1-B) - \frac{3}{2}})} (\log x)^{O(1)}
\]

where \( \mathcal{L} = (\log x)^{O(1)} \). If \( B \geq \frac{3}{4} \) then this is \( \ll \mathcal{L} x^{2B} \); selecting \( T = x^{1/(8B-4)} \) we get an error term \( \ll x^{2B} (\log x)^{O(1)} \), which is as good as can be hoped for. If \( B \leq \frac{3}{4} \) then the above error term is \( \ll \mathcal{L} x^{2B} T^{8B(1-B) - \frac{3}{2}} \); to minimize we select \( T = x^{4(1-B)/(16B(1-B) - 1)} \), which leads to an error term of \( x^{\frac{2+4B}{3(16B(1-B) - 1)}} \theta_B (\log x)^{O(1)} \) where \( \theta_B = \frac{16(1-B)(1-2B)^2}{3(16B(1-B) - 1)} \).

Bibliography

[4] Karin Halupczok, Email communication

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