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CORRIGENDUM TO "REFINEMENTS OF GOLDBACH'S CONJECTURE, AND THE GENERALIZED RIEMANN HYPOTHESIS"

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Abstract: Karin Halupczok [4] pointed out that we have stated an estimate in [3] that does not follow as easily as claimed. Although we are unable to obtain the claimed estimate, we prove a good enough estimate to (mostly) recover the theorems claimed in [3].

An explicit version of the prime number theorem states that if x is an integer and $1 \le T \le x$ then

$$\sum_{p \le x} \log p = x - \sum_{\substack{\rho \\ |\mathrm{Im}\rho| \le T}} \frac{x^\rho}{\rho} + O\left(\frac{x(\log x)^2}{T}\right), \tag{1}$$

where the sum is over zeros ρ of $\zeta(\rho) = 0$ with $\mathrm{Re}(\rho) > 0$. Let $B = \sup\{\mathrm{Re}\,\rho \colon \zeta(\rho) = 0\}$ (note that $1 \ge B \ge 1/2$). We claimed [3, (5.1)] that by partial summation with $T = x$ it is not hard to show that

$$\sum_{2N \le x} G(2N) = \sum_{p+q \le x} \log p \log q = \frac{x^2}{2} - 2 \sum_{\substack{\rho \\ |\mathrm{Im}\rho| \le x}} \frac{x^{1+\rho}}{\rho(1+\rho)} + O(x^{2B+o(1)});$$

however we have not been able to repeat the argument and Karen Halupczok [4] pointed out the references [1,2] where this issue has been investigated in some detail for $B = \frac{1}{2}$, and nothing so strong has been proved. Here we sketch a simple argument to prove that

$$\sum_{2N \le x} G(2N) = \sum_{p+q \le x} \log p \log q = \frac{x^2}{2} - 2 \sum_{\substack{\rho \\ |\mathrm{Im}\rho| \le x}} \frac{x^{1+\rho}}{\rho(1+\rho)} + O(x^{\frac{2+4B}{3}}(\log x)^2). \tag{2}$$

Thanks are due to Karin Halupczok for finding the mistake and for discussing this correction.

Note that $2B < \frac{2+4B}{3} < 1+B$ as $B<1$, so our error term is not quite as good as was claimed in [3], but it is comfortably strong enough to recover Theorem 1A of [3]. Using a zero-density estimate one can improve the error term to $\ll x^{2B}(\log x)^{O(1)}$, as claimed, when $B \geq \frac{3}{4}$, and to an exponent between $2B$ and $\frac{2+4B}{3}$ when $\frac{1}{2} \leq B \leq \frac{3}{4}$.

This mistake is repeated in all four parts of Theorem 1 in [3], so corrections are needed throughout: Replacing $2B$ by $\frac{2+4B}{3}$ on the fourth line of page 171 allows us to recover Theorem 1B. Similarly replacing $2B$ by $\frac{2+4B}{3}$ in (1.3) allows us to recover Theorem 1D. There is a mistake in the proof of Theorem 1C, two lines above (5.5), where complex variables ρ and σ are treated as if they are real variables. If a similar correction is made there we do not quite recover Theorem 1C. Instead we can prove that if (1.2) holds then the Riemann Hypothesis for Dirichlet L-functions mod q holds; and if the Riemann Hypothesis for Dirichlet L-functions mod q holds then we obtain (1.2) with error term $O(x^{\frac{4}{3}}(\log x)^2)$.

Sketch of proof of (2). What does follow from (1) by partial summation (and noting that $\sum_{T<|\mathrm{Im}\rho|\leq x} |x^{1+\rho}/\rho(1+\rho)| \ll x^2 \log T/T$), is

$$\sum_{p+q\leq x} \log p \log q = \frac{x^2}{2} - 2 \sum_{\substack{\rho \\ |\mathrm{Im}\rho|\leq x}} \frac{x^{1+\rho}}{\rho(1+\rho)} + \sum_{\substack{\rho,\rho' \\ |\mathrm{Im}\rho|,|\mathrm{Im}\rho'|\leq T}} \frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(\rho+\rho')} \cdot \frac{x^{\rho+\rho'}}{\rho+\rho'} + O\left(\frac{x^2(\log x)^2}{T}\right).$$

Stirling's formula implies that $|e^\rho\Gamma(\rho)| \asymp |\rho^{\rho-1/2}| = |\rho|^{\mathrm{Re}(\rho)-\frac{1}{2}} e^{-\arg(\rho)\mathrm{Im}(\rho)}$ so that if $\rho = \beta+i\gamma$ with $\beta\in(0,1)$ and $|\gamma|\gg 1$ then $|e^\rho\Gamma(\rho)| \asymp |\gamma|^{\beta-\frac{1}{2}} e^{-\frac{\pi}{2}|\gamma|}$, since $\arg(\rho) = \pm(\frac{\pi}{2} + O(\frac{1}{|\gamma|}))$ when $\mathrm{Im}(\rho) = \pm|\gamma|$. Let $\rho' = \beta' + i\gamma'$ with $|\gamma| \geq |\gamma'|$. Therefore if γ and γ' have the same sign then

$$\Gamma(\rho)\Gamma(\rho')/(\rho+\rho')\Gamma(\rho+\rho') \asymp |\gamma|^{\beta-\frac{1}{2}}|\gamma'|^{\beta'-\frac{1}{2}}/|\gamma+\gamma'|^{\beta+\beta'+\frac{1}{2}} \asymp |\gamma'|^{\beta'-\frac{1}{2}}/|\gamma|^{\beta'+1}.$$

If γ and γ' have opposite signs then

$$\begin{aligned}\Gamma(\rho)\Gamma(\rho')/(\rho+\rho')\Gamma(\rho+\rho') &\asymp |\gamma|^{\beta-\frac{1}{2}}|\gamma'|^{\beta'-\frac{1}{2}}e^{-\pi|\gamma'|}/(1+|\gamma+\gamma'|)^{\beta+\beta'+\frac{1}{2}} \\ &\ll |\gamma'|^{\beta'-\frac{1}{2}}/|\gamma|^{\beta'+1},\end{aligned}$$

since $|\gamma|^{\beta+\beta'+\frac{1}{2}}e^{-\pi|\gamma'|} \ll (1+|\gamma+\gamma'|)^{\beta+\beta'+\frac{1}{2}}$. We have $(|\gamma'|/|\gamma|)^{\beta'} \leq 1$ which implies that $|\gamma'|^{\beta'-\frac{1}{2}}/|\gamma|^{\beta'+1} \leq 1/|\gamma'|^{\frac{1}{2}}|\gamma|$. Hence the final sum in the last displayed equation is $\ll x^{2B}\sum_{|\gamma'|\leq|\gamma|\leq T} 1/|\gamma'|^{\frac{1}{2}}|\gamma| \ll x^{2B}\sum_{|\gamma|\leq T}(\log|\gamma|)/|\gamma|^{\frac{1}{2}} \ll x^{2B}T^{1/2}(\log T)^2$; and (2) follows by selecting $T = x^{\frac{4}{3}(1-B)}$.

Improvement using a zero-density estimate. In the bound above the contribution is majorized by those terms with $\beta,\beta' \geq \frac{1}{2}$ and $\gamma,\gamma' \geq 0$ (using the symmetries of the zeros). By using Carlson's zero-density estimate $\#\{\rho : \zeta(\rho) = 0 \text{ and } \beta \geq \sigma, |\gamma| \leq T\} \ll T^{4\sigma(1-\sigma)}(\log T)^{O(1)}$, we can improve our bound (we will select $T \leq x^{1/(8B-4)}$ below, which simplifies several steps, since then $B \leq \frac{1}{2} + \frac{\log x}{8 \log \gamma}$): throughout we sum over the zeros arranged by height, in dyadic intervals, and obtain that the final sum in the displayed equation is

$$
\begin{aligned}
&\ll \sum_{1 \leq \gamma' \leq \gamma \leq T} \frac{(\gamma')^{\beta' - \frac{1}{2}}}{\gamma^{\beta'+1}} \cdot x^{\beta+\beta'} \ll \mathcal{L} \sum_{\gamma \leq T} \max_{1 \leq t \leq \gamma} \int_{\sigma=1/2}^{B} \frac{t^{\sigma-\frac{1}{2}}}{\gamma^{\sigma+1}} \cdot x^{\beta+\sigma} t^{4\sigma(1-\sigma)} d\sigma \\
&\ll \mathcal{L} \sum_{\gamma \leq T} \max_{1/2 \leq \sigma \leq B} x^{\beta+\sigma} \gamma^{4\sigma(1-\sigma)-\frac{3}{2}} \ll \mathcal{L} \sum_{\gamma \leq T} x^{\beta+B} \gamma^{4B(1-B)-\frac{3}{2}} \\
&\ll \mathcal{L} \max_{u \leq T} \max_{1/2 \leq \tau \leq B} x^{\tau+B} u^{4B(1-B)+4\tau(1-\tau)-\frac{3}{2}} \ll \mathcal{L} \max_{u \leq T} x^{2B} u^{8B(1-B)-\frac{3}{2}} \\
&\ll x^{2B}(1 + T^{8B(1-B)-\frac{3}{2}})(\log x)^{O(1)}
\end{aligned}
$$

where $\mathcal{L} = (\log x)^{O(1)}$. If $B \geq \frac{3}{4}$ then this is $\ll \mathcal{L}x^{2B}$; selecting $T = x^{1/(8B-4)}$ we get an error term $\ll x^{2B}(\log x)^{O(1)}$, which is as good as can be hoped for. If $B \leq \frac{3}{4}$ then the above error term is $\ll \mathcal{L}x^{2B}T^{8B(1-B)-\frac{3}{2}}$; to minimize we select $T = x^{4(1-B)/(16B(1-B)-1)}$, which leads to an error term of $x^{\frac{2+4B}{3}-\theta_B}(\log x)^{O(1)}$ where $\theta_B = \frac{16(1-B)(1-2B)^2}{3(16B(1-B)-1)}$.

Bibliography

[1] Gautami Bhowmik and Jan-Christoph Schlage-Puchta, *Distribution of Goldbach numbers* (to appear).

[2] A. Fujii, *An additive problem of prime numbers, II* Proc. Japan Acad **67A** (1991), 248–252

[3] Andrew Granville, *Refinements of Goldbach's conjecture, and the Generalized Riemann Hypothesis* Functiones et Approximatio **37** (2007), 159–173

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