BIPARTITE PLANES

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ABSTRACT

It is well-known that the biclique partition number of the complete graph on n vertices (i.e. the smallest number of complete bipartite graphs required to partition the edge set of K_n) is n-1.

In this paper we address the following problem: For which integers s, t and n with $st=n/2$ does the complete graph K_n admit a decomposition into $(n-1)$ $K_{s,t}$'s ?

1. INTRODUCTION

Let $n>0$ be an integer, K_n denote the complete graph on n vertices and $\mathfrak{G}$ be a class of graphs where $K_n \notin \mathfrak{G}$. By a *$\mathfrak{G}$-plane of size n* we will mean a decomposition D of the edge set of K_n into copies of a fixed graph $G \in \mathfrak{G}$ with the property that, for any decomposition D' of K_n into (not necessarily isomorphic) graphs from $\mathfrak{G}$, $|D| \leq |D'|$.

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For example, if n is of the form k^2+k+1 and $\mathcal{G}$ is the class of all complete graphs except K_n, then a *$\mathcal{G}$-plane of size n* is equivalent to a projective plane of order k (whenever this exists) since the smallest non-trivial decomposition of K_n into complete subgraphs always consists of exactly n graphs (the minimum decompositions are either near-pencils or projective planes). We will herein be concerned with the class $\mathcal{G}$ of all complete bipartite graphs (bicliques).

It is well-known (see [3], [5] and [6]) that the biclique partition number of K_n is n-1, and we will denote by B(s,t;n) a decomposition of K_n into n-1 copies of $K_{s,t}$. Thus we pose the following

PROBLEM: For which s, t *and* n *does there exist a* B(s,t;n) *?*

The above problem was motivated by the following similar question posed by D. de Caen [1]: For which s and t with st=n-1 can the complete symmetric directed graph $\vec{K}_n$ be decomposed into n directed $K_{s,t}$'s (i.e. all arcs oriented from one bipartition to the other) ? (This has some interesting applications to matrix decompositions, see D. de Caen and D. Gregory [2]). This latter problem admits to a simple solution.

THEOREM 1.1: Given any positive integers s, t *and* n *with* st=n-1 *there is a (cyclic) decomposition of* $\vec{K}_n$ *into directed* $K_{s,t}$*'s.*

PROOF: Label the vertices of $\vec{K}_n$ with the elements of $\mathbb{Z}_n$. Develop the following directed biclique (S,T) modulo n:

S={ t, 2t,..., st }   and   T={ 0, 1,..., t-1 }. ■

The undirected analogue, which is the problem that we are herein addressing, appears to be much more difficult. It is easy to see that in a B(s,t;n), n must be even. Furthermore, by considering the bicliques containing a given vertex $x \in V(K_n)$, we see that the g.c.d.(s,t) must divide n-1. On the other hand since $st=n/2$ we clearly have that the g.c.d.(s,t) divides n. This means that s and t must be relatively prime. We record these simple observations as:

LEMMA 1.2: If there exists a B(s,t;n) *then*

(i) n *is even, and*

(ii) s *and* t *are relatively prime.*

Notwithstanding the trivial design B(1,1;2) we can therefore assume that $0<s<t<n$ in our notation B(s,t;n). A B(s,t;n) with $s=1$ will be called a *claw plane*. We will show that for each (even) n there exists a *claw plane* of size n. We will also prove the somewhat surprising result that, there *does not* exist a B(s,t;n) with $s=2$, for any n.

2. THE RESULTS

THEOREM 2.1: For each even integer $n>0$ *there exists a claw plane of size n.*

PROOF: A *claw plane* of size n is a B(1,n/2;n). Label the vertices of $\mathbb{Z}_n$ with $\{\infty\}\cup\mathbb{Z}_{n-1}$. Develop the following biclique (X,Y) modulo (n-1): $X = \{ 0 \}$ and $Y = \{ \infty, 1, 2, \ldots, (n/2)-1 \}$. ■

Before proceeding we shall have to look a little more carefully at the relationships between the vertices and bicliques in a B(s,t;n). We will assume from here on that $s \geq 2$. For each vertex x in $V(K_n)$ and each i=s,t let x_i denote the number of $K_{s,t}$'s whose bipartition of size i contains x; we will then say that vertex x has *type* (x_s, x_t). Now we clearly have

$$sx_t + tx_s = n-1 = 2st-1 \tag{2.1}$$

whence

$$\left.\begin{array}{ll} x_s \equiv -(1/t) & \text{modulo } s \\ x_t \equiv -(1/s) & \text{modulo } t \end{array}\right\} \tag{2.2}$$

For ease of expression let $\alpha(a,b)$ denote the least positive residue of $-(1/a)$ modulo b, where a and b are relatively prime. Then:

LEMMA 2.2: For any relatively prime integers a *and* b, *where* $a,b>1$, *we have that* $a\alpha(a,b)+b\alpha(b,a) = ab-1$.

PROOF: Consider the expression $\left[1+b\alpha(b,a)\right]/a$. From the definition of $\alpha(b,a)$ it follows immediately that this expression is an integer between 1 and b-1, whence so is $\left[ab-1-b\alpha(b,a)\right]/a$. But this latter expression is clearly congruent to $-(1/a)$ modulo b; that is, $\left[ab-1-b\alpha(b,a)\right]/a = \alpha(a,b)$. Rearranging we get $a\alpha(a,b)+b\alpha(b,a) = ab-1$, as desired. ∎

LEMMA 2.3: In a B(s,t;n) *with* $s \geq 2$ *there are exactly* $2s\alpha(s,t)+1$ *vertices of type* $(s+\alpha(t,s),\alpha(s,t))$ *and* $n-1-2s\alpha(s,t)$ *vertices of type* $(\alpha(t,s),t+\alpha(s,t))$.

PROOF: From expressions (2.1), (2.2) and Lemma 2.2 it follows that for any vertex x, either

(i) $x_s = \alpha(t,s)$ and $x_t = t+\alpha(s,t)$, or

(ii) $x_s = s+\alpha(t,s)$ and $x_t = \alpha(s,t)$.

Let y be the number of vertices of type (i) and z be the number of vertices of type (ii). By noting that

$$\sum_{x\in V(K_n)} x_t = t(n-1)$$

we obtain the system

$$\begin{aligned}\left(t+\alpha(s,t)\right)y+\alpha(s,t)z &= t(n-1)\\ y+z = n &= 2st\end{aligned}$$

which yields $y = n-1-2s\alpha(s,t)$ and $z = 2s\alpha(s,t)+1$ as asserted. ■

REMARK: By using Lemma 2.2 we can rewrite $y = n-1-2s\alpha(s,t)$ as $y = 2t\alpha(t,s)+1$. In particular there are vertices of both types represented; since $s\neq t$ a B(s,t;n) can therefore never be *balanced* (in the sense of Huang and Rosa [4]).

We are now ready to prove the following.

THEOREM 2.4: There does not exist a B(s,t;n), *with* s = 2, *for any* n.

PROOF: Suppose if possible that we have a B(2,n/4;n). From Lemma 2.3 there are $4\left(((n/4)-1)/2\right)+1 = (n/2)-1$ vertices of *type* (3,(n-4)/8) and (n/2)+1 vertices of *type* (1,(3n-4)/8).

Let H denote the set of vertices of the former *type* and J the s of vertices of the latter *type*. For each vertex $j \in J$ there is a uniq biclique B_j whose bipartition of size 2 contains j. Since the set $\{ B_j : j \in J \}$ must pick up all edges joining pairs of vertices in J it folows that:

(i) If $j_1 \neq j_2$ then $B_{j_1} \neq B_{j_2}$, else the edge joining j_1 to j_2 could not be covered, and

(ii) For each $j \in J$ the bipartition of size n/4 in B_j is a subset of J, because $\binom{|J|}{2} = \left((n/2)+1\right)(n/4) = |\{ B_j : j \in J \}|(n/4)$.

From (ii) we see that a vertex in H can be contained in the bipartition of size 2 in at most two B_j's. On the other hand, since $|J|>|H|$, (i) implies that there is a vertex $h \in H$ which is contained i the bipartition of size 2 in exactly two B_j's.

Let G be that subgraph of K_n obtained by removing all edges covered by the B_j's. Then the edges of G are being partitioned by th remaining (n/2)-2 bicliques $C_1, \ldots, C_{(n/2)-2}$ in the B(2,n/4;n). But G contains all the edges joining pairs of vertices in H, so that by the Graham-Pollack theorem,

(iii) Each biclique $C_1, \ldots, C_{(n/2)-2}$ contains at least one edge joining a pair of vertices in H.

Now, in G, h is adjacent to exactly one vertex $j_0 \in J$. Without loss of generality let C_1 be the biclique containing the edge hj and let {h,h'} be the bipartition of size 2 in C_1, with $h' \in H$. Note that $\{h, j_0\}$ must have been the bipartition of size 2 in some B_j, so the same cannot be true of $\{h', j_0\}$ as j_0 has *type* (1,(3n-4)/8). This means that h' is adjacent to either (n/4)+1 or (n/2)+1 vertices c J in G, depending on whether it was contained in the bipartition of size 2 in one or no B_j's. In the first case the n/4 edges joining h' to vertices in J which remain after removing C_1 from G must be covered by bicliques from $C_2,\ldots,C_{(n/2)-2}$, each with the property that its bipartition of size 2 contains h'. But h' has *type* (3,(n-4)/8) so that there can be only one such biclique, say C_2. Then the bipartition of size n/4 in C_2 must consist of the n/4 vertices in J to which h' is still adjacent and this means that C_2 contains no edges joining pairs of points in H, contradicting (iii). A similar argument rules out the second case. Thus no B(2,n/4;n) can exist. ■

Finally, an immediate consequence of Lemma 1.2 and Theorem 2.4 is

COROLLARY 2.5: Let n = 2q *or* 4q *where* q *is a prime power. Then the only bipartite planes of size* n *are the claw planes.*

3. SUMMARY

We do not at present know of a single example of a bipartite plane that is not a *claw plane*. From Corollary 2.5 the smallest possible example would be a B(3,4;24).

We would also like to mention a similar problem, posed by D. de Caen.

PROBLEM: For which integers k *can the complete graph* K_n, *with* $n = \binom{k}{2}+1$, *be decomposed into* n-1 *complete bipartite subgraphs, each containing a total of* k *vertices ?*

It so happens that the existence of such a decomposition is a necessary one in order that a *signed symmetric* (n,k,2)-BIBD exists.

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