Basics of binary quadratic forms and Gauss composition

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SMS summer school: "Counting arithmetic objects"

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Any prime $p \equiv 1 \pmod 4$ can be written as the sum of two squares

"Geometry of numbers type" proof

Since $p \equiv 1 \pmod 4 \implies \exists i \in \mathbb{Z} : i^2 \equiv -1 \pmod p$.

Idea: Find smallest non-zero integer lattice point

$$(x, y) \in \mathbb{Z}^2 : \; x \equiv iy \pmod p$$

Since $p \equiv 1 \pmod 4 \implies \exists i \in \mathbb{Z} : i^2 \equiv -1 \pmod p$.

Consider now the set of integers

$$\{m + ni : \ 0 \le m, n \le [\sqrt{p}]\}$$

pairs m, n is $([\sqrt{p}] + 1)^2 > p$, so by the pigeonhole principle, two are congruent mod p; say that

$$m + ni \equiv M + Ni \pmod p$$

where $0 \le m, n, M, N \le [\sqrt{p}]$ and $(m, n) \ne (M, n)$.

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where $0 \le m, n, M, N \le [\sqrt{p}]$ and $(m, n) \ne (M, n)$. Let $r = m - M$ and $s = N - n$ so that

$$r \equiv is \pmod p$$

where $|r|, |s| \le [\sqrt{p}] < \sqrt{p}$, and r and s are not both 0. Now

$$r^2 + s^2 \equiv (is)^2 + s^2 = s^2(i^2 + 1) \equiv 0 \pmod p$$

and $0 < r^2 + s^2 < \sqrt{p}^2 + \sqrt{p}^2 = 2p$. The only multiple of p between 0 and $2p$ is p, and therefore $r^2 + s^2 = p$.

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Gauss's view:

A *binary quadratic form* is of the shape

$$f(x,y) := ax^2 + bxy + cy^2.$$

Here we take $f(x,y) = x^2 + dy^2$ and

$$f(a,b)f(c,e) = f(ac+dbe, ae-bc)$$

The latter values in f, namely $ac+dbe$ and $ae-bc$, are *bilinear forms* in a, b, c, e.

Does this generalize to other such multiplications?

Pell's equation

Are there integer solutions x, y to

$$x^2 - dy^2 = 1?$$

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Can always be found using continued fraction for $\sqrt{d}$. (Brahmagupta, 628 A.D.; probably Archimedes, to solve his "Cattle Problem" one needs to find a solution to

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Solution to Pell's Equation ***Let $d \geq 2$ be a non-square integer.*** $\exists x, y \in \mathbb{Z}$ ***for which***

$$x^2 - dy^2 = 1,$$

with $y \neq 0$. ***If*** x_1, y_1 ***smallest positive solution, then all others given by***

$$x_n + \sqrt{d} y_n = (x + \sqrt{d} y)^n$$

$$x^2 - dy^2 = 1?$$

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Better to look for solutions to

$$x^2 - dy^2 = \pm 4,$$

Understanding when there is solution with "−" is a difficult question (great recent progress by Fouvry and Kluners).

Theorem Any quadratic irrational real number has a continued fraction that is eventually periodic.

Here are some examples of the continued fraction for $\sqrt{d}$:

$$\sqrt{2} = [1,\ \overline{2}],\ \sqrt{3} = [1,\ \overline{1,\ 2}],\ \sqrt{5} = [2,\ \overline{4}],$$
$$\sqrt{6} = [2,\ \overline{2,\ 4}],$$
$$\sqrt{7} = [2,\ \overline{1,\ 1,\ 1,\ 4}],$$
$$\sqrt{8} = [2,\ \overline{1,\ 4}],$$
$$\sqrt{10} = [3,\ \overline{6}],$$
$$\sqrt{11} = [3,\ \overline{3,\ 6}],$$
$$\sqrt{12} = [3,\ \overline{2,\ 6}],$$
$$\sqrt{13} = [3,\ \overline{1,\ 1,\ 1,\ 1,\ 6}], \ldots$$

If p_k/q_k are the convergents for $\sqrt{d}$ then

$$p_{n-1}^2 - dq_{n-1}^2 = (-1)^n.$$

Longest continued fractions and the largest fundamental solutions

$$
\begin{aligned}
\sqrt{2} &= [1, \overline{2}], & 1^2 - 2\cdot 1^2 &= -1 \\
\sqrt{3} &= [1, \overline{1, 2}], & 2^2 - 3\cdot 1^2 &= 1 \\
\sqrt{6} &= [2, \overline{2, 4}], & 5^2 - 6\cdot 2^2 &= 1 \\
\sqrt{7} &= [2, \overline{1, 1, 1, 4}], & 8^2 - 7\cdot 3^2 &= 1 \\
\sqrt{13} &= [3, \overline{1, 1, 1, 1, 6}], & 18^2 - 13\cdot 5^2 &= -1 \\
\sqrt{19} &= [4, \overline{2, 1, 3, 1, 2, 8}], & 170^2 - 19\cdot 39^2 &= 1 \\
\sqrt{22} &= [4, \overline{1, 2, 4, 2, 1, 8}], & 197^2 - 22\cdot 42^2 &= 1 \\
\sqrt{31} &= [5, \overline{1, 1, 3, 5, 3, 1, 1, 10}], & 1520^2 - 31\cdot 273^2 &= 1 \\
\sqrt{43} &= [6, \overline{1, 1, 3, 1, 5, 1, 3, 1, 1, 12}], & 3482^2 - 43\cdot 531^2 &= 1 \\
\sqrt{46} &= [6, \overline{1, 3, 1, 1, 2, 6, 2, 1, 1, 3, 1, 12}], & 24335^2 - 46\cdot 3588^2 &= 1 \\
\sqrt{76} &= [8, \overline{1, 2, 1, 1, 5, 4, 5, 1, 1, 2, 1, 16}], & 57799^2 - 76\cdot 6630^2 &= 1
\end{aligned}
$$

Length of longest cont fracts and fundl solutions

$$16: \quad 2143295^2 - 94 \cdot 221064^2 = 1$$
$$16: \quad 4620799^2 - 124 \cdot 414960^2 = 1$$
$$16: \quad 2588599^2 - 133 \cdot 224460^2 = 1$$
$$18: \quad 77563250^2 - 139 \cdot 6578829^2 = 1$$
$$20: \quad 1728148040^2 - 151 \cdot 140634693^2 = 1$$
$$22: \quad 1700902565^2 - 166 \cdot 132015642^2 = 1$$
$$26: \quad 278354373650^2 - 211 \cdot 19162705353^2 = 1$$
$$26: \quad 695359189925^2 - 214 \cdot 47533775646^2 = 1$$
$$26: \quad 5883392537695^2 - 301 \cdot 339113108232^2 = 1$$
$$34: \quad 2785589801443970^2 - 331 \cdot 153109862634573^2 = 1$$
$$37: \quad 4404244569682141 8^2 - 421 \cdot 2146497463530785^2 = -1$$
$$40: \quad 840560915469529337 75^2 - 526 \cdot 36650197573242955 32^2 = 1$$
$$42: \quad 181124355061630786130^2 - 571 \cdot 7579818350628982587^2 = 1$$

Length of fundamental solutions

The length of the continued fractions here are around $2\sqrt{d}$, and the size of the fundamental solutions $10^{\sqrt{d}}$.

How big is the smallest solution?

We believe that the smallest solution is typically of size $C^{\sqrt{d}}$ but not much proved.

Understanding the distribution of sizes of the smallest solutions to Pell's equation is an outstanding open question in number theory.

Descent on solutions of $x^2 - dy^2 = n,\ d > 0$

Let $\epsilon_d = x_1 + y_1\sqrt{d}$, the smallest solution x_1, y_1 in positive integers to

$$x_1^2 - dy_1^2 = 1.$$

Given a solution of

$$x^2 - dy^2 = n$$

with $x, y \geq 0$, let

$$\alpha := x + y\sqrt{d} > \sqrt{n}.$$

If $\sqrt{n}\epsilon_d^k \leq \alpha < \sqrt{n}\epsilon_d^{k+1}$ let

$$\beta := \alpha\epsilon_d^{-k} = u + \sqrt{d}v$$

so that

$$\sqrt{n} \leq \beta < \sqrt{n}\epsilon_d$$

with $u, v \geq 1$ and $u^2 - dv^2 = n$.

Representation of integers by binary quadratic forms

What integers are represented by *binary quadratic form*

$$f(x, y) := ax^2 + bxy + cy^2 \quad ?$$

That is, for what N are there coprime m, n such that

$$N = am^2 + bmn + cn^2 \ \ ?$$

WLOG $\gcd(a, b, c) = 1$. Complete the square to obtain

$$4aN = (2am + bn)^2 - dn^2$$

where *discriminant* $d := b^2 - 4ac$, so

$$d \equiv 0 \ \text{ or } \ 1 \pmod 4.$$

When $d < 0$ the right side can only take positive values ... easier than when $d > 0$.
If $a > 0$ then *positive definite* binary quadratic form.

$x^2 + y^2$ represents the same integers as $X^2 + 2XY + 2Y^2$

If $N = m^2 + n^2$ then $N = (m-n)^2 + 2(m-n)n + 2n^2$,

If $N = u^2 + 2uv + 2v^2$ then $N = (u+v)^2 + v^2$.

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$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} X \\ Y \end{pmatrix} \qquad \text{where } M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

transforms $x^2 + y^2$ into $X^2 + 2XY + 2Y^2$, and the transformation is invertible, since $\det M = 1$.

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Much more generally define

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \ \alpha, \beta, \gamma, \delta \in \mathbb{Z} \text{ and } \alpha\delta - \beta\gamma = 1 \right\}.$$

Then $ax^2 + bxy + cy^2$ represents the same integers as $AX^2 + BXY + CY^2$ whenever $\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} X \\ Y \end{pmatrix}$ with $M \in \mathrm{SL}(2,\mathbb{Z})$. These quadratic forms are *equivalent*.

Equivalence

$ax^2 + bxy + cy^2$ is *equivalent* to $AX^2 + BXY + CY^2$ if equal whenever $\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} X \\ Y \end{pmatrix}$ with $M \in \mathrm{SL}(2, \mathbb{Z})$.

This yields an equivalence relation and splits the binary quadratic forms into equivalence classes. Write

$$ax^2 + bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\mathrm{Discriminant}(f) = -\det \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$. We deduce that

$$AX^2 + BXY + CY^2 = \begin{pmatrix} X & Y \end{pmatrix} M^{\mathrm{T}} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} M \begin{pmatrix} X \\ Y \end{pmatrix},$$

so $A = a\alpha^2 + b\alpha\gamma + c\gamma^2$ and $C = a\beta^2 + b\beta\delta + c\delta^2$ as

$$\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} = M^{\mathrm{T}} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} M.$$

Hence two equivalent bqfs have same discriminant.

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If so, $|d| = 4ac - (|b|)^2 \ge 4a \cdot a - a^2 = 3a^2$ and hence

$$a \le \sqrt{|d|/3}.$$

Therefore, for given $d < 0$, finitely many a, and so b (as $|b| \le a$), and then $c = (b^2 - d)/4a$ is determined; so only finitely many ($h(d)$, the *class number*, the number of equivalence classes) reduced bqfs of discrim d.

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$$\begin{cases} x^2 - (d/4)y^2 & \text{when } d \equiv 0 \pmod 4, \\ x^2 + xy + \frac{(1-d)}{4}y^2 & \text{when } d \equiv 1 \pmod 4. \end{cases}$$

Gauss's reduction Theorem

Every positive definite binary quadratic form is properly equivalent to a reduced form.

Proof. A sequence of equivalent forms; algorithm terminates when we reach one that is reduced. Given (a, b, c):

i) If $c < a$ the transformation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$, yields $(c, -b, a)$ which is properly equivalent to (a, b, c).

$\square$

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ii) If $b > a$ or $b \leq -a$ let b' be the least residue, in absolute value, of $b \pmod{2a}$, so $-a < b' \leq a$, say $b' = b - 2ka$. Then let $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$. The resulting form (a, b', c') is properly equivalent to (a, b, c).

iii) If $c = a$ and $-a < b < 0$ then we use the transformation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$ yielding the form $(a, -b, a)$.

If resulting form not reduced, **repeat**

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ii) If $b > a$ or $b \leq -a$ then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$.

iii) If $c = a$ and $-a < b < 0$ then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$

If resulting form not reduced, **repeat**

The algorithm terminates after (iii), and since (ii) is followed by (i) or (iii), and since (i) reduces the size of a.

Gauss's reduction Theorem; examples

$(76, 217, 155)$ of discriminant -31, The sequence of forms is

$$(76, 65, 14), (14, -65, 76), (14, -9, 2), (2, 9, 14), (2, 1, 4),$$

the sought after reduced form.

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$(11, 49, 55)$ of discriminant -19, gives the sequence of forms

$$(11, 5, 1), (1, -5, 11), (1, 1, 5).$$

Restriction on values taken by a bqf

Suppose $d = b^2 - 4ac$ with $(a, b, c) = 1$, and p is a prime.

- (i) If $p = am^2 + bmn + cn^2$ for some integers m, n then d is a square mod $4p$.
- (ii) If d is a square mod $4p$ then there exists a binary quadratic form of discriminant d that represents p.

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- (ii) If d is a square mod $4p$ then there exists a binary quadratic form of discriminant d that represents p.

Proof. (i) If $p \nmid 2ad$ and $p = am^2 + bmn + cn^2$. Therefore $4ap = (2am + bn)^2 - dn^2$ and so dn^2 is a square mod $4p$. Now $p \nmid n$ else $p | 4ap + dn^2 = (2am + bn)^2$ so that $p | 2am$ which is impossible as $p \nmid 2a$ and $(m, n) = 1$. We deduce that d is a square mod p.
(ii) If $d \equiv b^2 \pmod{4p}$ then $d = b^2 - 4pc$ for some integer c, and so $px^2 + bxy + cy^2$ is a quadratic form of discriminant d which represents $p = p \cdot 1^2 + b \cdot 1 \cdot 0 + c \cdot 0^2$. $\square$

Class number one

Theorem Suppose $h(d) = 1$. Then p is represented by the form of discrim d if and only if d is a square mod $4p$.

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(*Fundamental discriminants*: If $q^2|d$ then $q = 2$ and $d \equiv 8 \text{ or } 12 \pmod{16}$.)
The only fundamental $d < 0$ with $h(d) = 1$ are $d = -3, -4, -7, -8, -11, -19, -43, -67, -163$. (Heegner/ Baker/ Stark)

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Euler noticed that the polynomial $x^2 + x + 41$ is prime for $x = 0, 1, 2, \ldots, 39$, and some other polynomials.

Rabinowiscz's criterion We have $h(1 - 4A) = 1$ for $A \geq 2$ if and only if $x^2 + x + A$ is prime for $x = 0, 1, 2, \ldots, A - 2$.

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If $p \nmid d$ then
p is rep'd by x^2+y^2 if and only if $(-1/p)=1$,
p is rep'd by x^2+2y^2 if and only if $(-2/p)=1$,
p is rep'd by x^2+xy+y^2 if and only if $(-3/p)=1$,
p is rep'd by $x^2+xy+2y^2$ if and only if $(-7/p)=1$,
p is rep'd by $x^2+xy+3y^2$ if and only if $(-11/p)=1$,
p is rep'd by $x^2+xy+5y^2$ if and only if $(-19/p)=1$,
p is rep'd by $x^2+xy+11y^2$ if and only if $(-43/p)=1$,
p is rep'd by $x^2+xy+17y^2$ if and only if $(-67/p)=1$,
p is rep'd by $x^2+xy+41y^2$ if and only if $(-163/p)=1$.

Class number *not* one

What about when the class number is not one?
First example, $h(-20) = 2$, the two reduced forms are

$$x^2 + 5y^2 \text{ and } 2x^2 + 2xy + 3y^2.$$

p is represented by x^2+5y^2 if and only if $p = 5$, or $p \equiv 1$ or $9 \pmod{20}$;
p is represented by $2x^2 + 2xy + 3y^2$ if and only if $p = 2$, or $p \equiv 3$ or $7 \pmod{20}$.

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Cannot always distinguish which primes are represented by which bqf of discriminant d by congruence conditions. Euler found 65 such *idoneal numbers*. No more are known – at most one further idoneal number.

Ideals in quadratic fields

Any ideal I in a quadratic ring of integers:

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Claim: If $u + v\sqrt{d} \in I$ then s divides v
(else if $ks + \ell v = g := \gcd(s, v)$ then
$(kr + \ell u) + g\sqrt{d} = k(r + s\sqrt{d}) + \ell(u + v\sqrt{d}) \in I$ #)

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$(kr + \ell u) + g\sqrt{d} = k(r + s\sqrt{d}) + \ell(u + v\sqrt{d}) \in I \ \#$)
Let $v = ms$, so that $(u+v\sqrt{d}) - m(r+s\sqrt{d}) = u - mr$.

Therefore $I = \{m(r + s\sqrt{d}) + n : \ m \in \mathbb{Z}, \ n \in I \cap \mathbb{Z}\}$.

Now $I \cap \mathbb{Z}$ is an ideal in $\mathbb{Z}$ so principal, $= \langle g \rangle$ say hence

$$I = \langle r + s\sqrt{d}, g \rangle_{\mathbb{Z}}.$$

Any ideal $I \subset R := \{a + b\sqrt{d} :\ a, b \in \mathbb{Z}\}$ has the form

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More: $\sqrt{d} \in R$, so $g\sqrt{d} \in I$ and $sd + r\sqrt{d} \in I$, and so s divides both g and r.

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Therefore $r = sb$ and $g = sa$. Also

$$s(b^2 - d) = (r + s\sqrt{d})(b - \sqrt{d}) \in I \cap \mathbb{Z}$$

and so $s(b^2 - d)$ is a multiple of $g = sa$; hence a divides $b^2 - d$. Therefore

$$\boxed{I = s\langle b + \sqrt{d}, a\rangle_{\mathbb{Z}}}$$

for some integers s, a, b where a divides $b^2 - d$.

Binary quadratic forms and Ideals

$$I = s\langle a,\ b + \sqrt{d}\rangle_{\mathbb{Z}}$$

If $f(x, y) = ax^2 + bxy + cy^2$ then

$$af(x, y) = \left(ax + \frac{b + \sqrt{d}}{2}\, y\right)\left(ax + \frac{b - \sqrt{d}}{2}\, y\right)$$

so we see that $af(x, y)$ is the *Norm* of $\left(ax + \frac{b+\sqrt{d}}{2}\, y\right)$.

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so we see that $af(x,y)$ is the *Norm* of $\left(ax + \frac{b+\sqrt{d}}{2}\,y\right)$. So the set of possible values of $f(x,y)$ with $x, y \in \mathbb{Z}$ is in 1-to-1 correspondence with the elements of $\langle a, \frac{b+\sqrt{d}}{2}\rangle_{\mathbb{Z}}$.

Equivalence of ideals

Any two equivalent bqfs can be obtained from each other by a succession of two basic transformations:

$$x \to x+y,\ y \to y \text{ gives } \langle a, \frac{b+\sqrt{d}}{2}\rangle_{\mathbb{Z}} \to \langle a, \frac{2a+b+\sqrt{d}}{2}\rangle_{\mathbb{Z}}$$

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$$x \to -y,\ y \to x \text{ gives } \langle a, \frac{b+\sqrt{d}}{2}\rangle_{\mathbb{Z}} \to \langle c, \frac{-b+\sqrt{d}}{2}\rangle_{\mathbb{Z}}.$$

Since $\frac{-b+\sqrt{d}}{2} \cdot \frac{b+\sqrt{d}}{2} = \frac{d-b^2}{4} = -ac$, and therefore

$$\frac{-b+\sqrt{d}}{2} \cdot \langle a, \frac{b+\sqrt{d}}{2}\rangle_{\mathbb{Z}} = a \cdot \langle \frac{-b+\sqrt{d}}{2}, -c\rangle_{\mathbb{Z}}.$$

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So, equivalence of forms, in setting of ideals, gives: For ideals I, J of $\mathbb{Q}(\sqrt{d})$, we have that
$I \sim J$ if and only there exists $\alpha \in \mathbb{Q}(\sqrt{d})$, such that

$$J = \alpha I.$$

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This works in any number field; moreover then one has finitely many equivalence classes, and i bounds for the "smallest" element of each class.
Any ideal $I = \langle a, \frac{b+\sqrt{d}}{2} \rangle$ with $d < 0$ then we plot $\mathbb{Z}$-linear combinations on the complex plane and they form a *lattice*, $\Lambda = \langle a, \frac{b+\sqrt{d}}{2} \rangle$ — geometry of lattices.
Equivalence: Two lattices Λ, Λ' are *homothetic* if there exists $\alpha \in \mathbb{C}$ such that $\Lambda' = \alpha\Lambda$, and we write $\Lambda' \sim \Lambda$.
Divide through by a, every such lattice is homothetic to $\langle 1, \tau \rangle$ where $\tau = \frac{b+\sqrt{d}}{2a}$, in the upper half plane.

Fundamental discriminants and orders

A square class of integers, like $3, 12, 27, 48, \ldots$ gives same field $\mathbb{Q}(\sqrt{3n^2}) = \mathbb{Q}(\sqrt{3})$ — minimal one? Candidate: The only one that is squarefree? However, from theory of bqfs need discriminant $\equiv 0$ or $1 \pmod 4$. Divisibility by 4 correct price to pay. The *fundamental discriminant* of a quadratic field to be the smallest element of the square class of the discriminant which is $\equiv 0$ or $1 \pmod 4$. For d squarefree integer, the fundamental discriminant D is

$$D = \begin{cases} d & \text{if } d \equiv 1 \pmod 4 \\ 4d & \text{if } d \equiv 2 \text{ or } 3 \pmod 4 \end{cases}.$$

The ring of integers is $\mathbb{Z}\left[\frac{D+\sqrt{D}}{2}\right]$ or $\mathbb{Z}[\omega] = \langle 1, \omega \rangle_{\mathbb{Z}}$,

$$\omega := \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4 \\ \sqrt{d} = \sqrt{D}/2 & \text{if } d \equiv 2 \text{ or } 3 \pmod 4 \end{cases}.$$

Gauss's Composition Law

The product of any two values of a principal form gives a third value of that quadratic form:

$$(a^2 + db^2)(c^2 + de^2) = (ac + dbe)^2 + d(ae - bc)^2.$$

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Gauss: if f and g are bqfs discrim d, then $\exists$ bqf h of discrim d, such that any

$$f(a, b)g(c, e) = h(m, n),$$

$m = m(a, b, c, e)$, $n = n(a, b, c, e)$ are bilinear forms.

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$m = m(a, b, c, e)$, $n = n(a, b, c, e)$ are bilinear forms.
Gauss showed this explicitly via formulae;
e.g., for three bqfs of discrim -71,

$$2m^2 + mn + 9n^2 = (4a^2 + 3ab + 5b^2)(3c^2 + ce + 6e^2).$$

with $m = ac - 3ae - 2bc - 3be$ and $n = ac + ae + bc - be$.
Gauss called this *composition*.

$$2m^2 + mn + 9n^2 = (4a^2 + 3ab + 5b^2)(3c^2 + ce + 6e^2).$$

Gauss showed composition stays consistent under the equivalence relation.

Allows us to find a group structure on the classes of quadratic forms of given discriminant, the *class group*.

Gauss's proof is monstrously difficult, even in the hands of the master the algebra involved is so overwhelming that he does not include many details.

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Gauss's student Dirichlet found several ways to simplify composition. The first involved finding forms that are equivalent to f and g that are easier to compose:

Dirichlet's composition of forms

- For any given integer w there exist integers m, n with $(am^2 + bmn + cn^2, w) = 1$.
- Given quadratic forms f and g, find $f' \sim f$ such that $(f'(1,0), g(1,0)) = 1$.
- There exists $F \sim f'$ and $G \sim g$ such that $F(x,y) = ax^2 + bxy + cy^2$ and $G(x,y) = Ax^2 + bxy + Cy^2$ with $(a, A) = 1$.
- If f and g have the same discriminant then there exist h such that $F(x,y) = ax^2 + bxy + Ahy^2$ and $G(x,y) = Ax^2 + bxy + ahy^2$ with $(a, A) = 1$.
- $d = b^2 - 4aAh$. If $H(x,y) = aAx^2 + bxy + hy^2$ then

$$H(ux - hvy, auy + Avx + bvy) = F(u,v)G(x,y)$$

Dirichlet's composition of ideals

Dirichlet simplified by defining ideals: To multiply two ideals, $IJ = \{ij : \ i \in I, \ j \in J\}$.

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$(4, \frac{3+\sqrt{-71}}{2})$ corresponds to $4a^2 + 3ab + 5b^2$, and
$(3, \frac{1+\sqrt{-71}}{2})$ corresponds to $3c^2 + ce + 6e^2$.

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$$\left(4, \frac{3+\sqrt{-71}}{2}\right)\left(3, \frac{1+\sqrt{-71}}{2}\right) = \left(12, \frac{-5+\sqrt{-71}}{2}\right),$$

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which corresponds to $12x^2 - 5xy + 2y^2$, also of disc -71, but not reduced. Reduction then yields:

$$(12, -5, 2) \sim (2, 5, 12) \sim (2, 1, 9)$$

Comparing Dirichlet's compositions

If $F = ax^2 + bxy + Ahy^2$, $G = Ax^2 + bxy + ahy^2$ then

$$H(ux - hvy, auy + Avx + bvy) = F(u, v)G(x, y)$$

for $H(x, y) = aAx^2 + bxy + hy^2$.

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The two quadratic forms F and G correspond to $\left(a, \frac{-b+\sqrt{d}}{2}\right)$ and $\left(A, \frac{-b+\sqrt{d}}{2}\right)$. The product is $\left(aA, \frac{-b+\sqrt{d}}{2}\right)$, so the composition of F and G must be $aAx^2 + bxy + hy^2$.

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Identity of ideal class group: principal ideas. Inverses:

$$\left(a, \frac{b+\sqrt{d}}{2}\right)\left(a, \frac{b-\sqrt{d}}{2}\right) = \left(a^2, a\frac{b+\sqrt{d}}{2}, a\frac{b-\sqrt{d}}{2}, \frac{b^2-d}{4}\right)$$
$$\supseteq a\,(a, b, c) = (a),$$

So an ideal and its conjugate are inverses in class group.

A more general set up

Let $G(\mathbb{Z})$ be $\mathrm{SL}(2,\mathbb{Z})$, an "algebraic group";
$V(\mathbb{Z})$ the space of bqfs over $\mathbb{Z}$, a "representation".

Seen that: *The $G(\mathbb{Z})$-orbits parametrize the ideal classes in the associated quadratic rings.*

Do other such pairs exist? That is an algebraic group G and associated representation V such that $G(\mathbb{Z})\setminus V(\mathbb{Z})$ parametrizes something interesting?
Eg rings, modules etc of arithmetic interest.

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In our example there is just one orbit over $\mathbb{C}$:
A pre-homogenous vector space *is a pair (G,V) where G is an algebraic group and V is a rational vector space representation of G such that the action of $G(\mathbb{C})$ on $V(\mathbb{C})$ has just one Zariski open orbit.*

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Bhargava's program centres around study of $G(\mathbb{Z}) \diagdown V(\mathbb{Z})$ for pre-homogenous vector spaces (G, V).

There are just 36 of them (Sato-Kimura, 1977), but they have proved yo be incredibly rich in structure of interest to number theorists.

Bhargava composition

Recently Bhargava gave a new insight into the composition law.

Note: If $IJ = K$ then $IJ\overline{K}$ is principal .

Bhargava composition

We begin with a 2-by-2-by-2 cube. a, b, c, d, e, f, g, h. Six faces, can be split into three parallel pairs. To each consider pair of 2-by-2 matrices by taking the entries in each face, with corresponding entries corresponding to opposite corners of the cube, always starting with a. Hence we get the pairs

$$M_1(x, y) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} x + \begin{pmatrix} e & f \\ g & h \end{pmatrix} y,$$

$$M_2(x, y) := \begin{pmatrix} a & c \\ e & g \end{pmatrix} x + \begin{pmatrix} b & d \\ f & h \end{pmatrix} y,$$

$$M_3(x, y) := \begin{pmatrix} a & b \\ e & f \end{pmatrix} x + \begin{pmatrix} c & d \\ g & h \end{pmatrix} y,$$

where we have, in each added the dummy variables, x, y. The determinant, $-Q_j(x, y)$, of each $M_j(x, y)$ gives rise to a quadratic form in x and y.

$$M_1(x,y) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} x + \begin{pmatrix} e & f \\ g & h \end{pmatrix} y,$$

$$M_2(x,y) := \begin{pmatrix} a & c \\ e & g \end{pmatrix} x + \begin{pmatrix} b & d \\ f & h \end{pmatrix} y,$$

$$M_3(x,y) := \begin{pmatrix} a & b \\ e & f \end{pmatrix} x + \begin{pmatrix} c & d \\ g & h \end{pmatrix} y,$$

$Q_j(x,y) = -\det M_j(x,y)$, a bqf

Now apply an $\mathrm{SL}(2,\mathbb{Z})$ transformation in one direction. That is, if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ then we replace the face

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \beta$$

and

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \delta.$$

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$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \delta.$$

Then $M_1(x,y)$ gets mapped to

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \beta\right\} x + \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \delta\right\} y,$$

that is $M_1(\alpha x + \gamma y, \beta x + \delta y)$. Therefore

$Q_1(x,y) = -\det M_1(x,y)$ gets mapped to $Q_1(\alpha x + \gamma y, \beta x + \delta y)$. which is equivalent to $Q_1(x,y)$.

Now $M_2(x,y)$ gets mapped to

$$\begin{pmatrix} a\alpha + e\beta & c\alpha + g\beta \\ a\gamma + e\delta & c\gamma + g\delta \end{pmatrix} x + \begin{pmatrix} b\alpha + f\beta & d\alpha + h\beta \\ b\gamma + f\delta & d\gamma + h\delta \end{pmatrix} y$$

$$= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} M_2(x,y);$$

hence the determinant, $Q_2(x,y)$, is unchanged. An analogous calculation reveals that $M_3(x,y)$ gets mapped to $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} M_3(x,y)$ and its det, $Q_3(x,y)$ also unchanged.

Therefore we can act on our cube by such $\mathrm{SL}(2,\mathbb{Z})$-transformations, in each direction, and each of the three quadratic forms remains in the same equivalence class.

Another prehomogenous vector space

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Bhargava's cubes can be identified as

$$a\ e_1 \times e_1 \times e_1 + b\ e_1 \times e_2 \times e_1 + c\ e_2 \times e_1 \times e_1$$
$$+d\ \ e_2 \times e_2 \times e_1 +\ e\ e_1 \times e_1 \times e_2 + f\ e_1 \times e_2 \times e_2$$
$$+g\ e_2 \times e_1 \times e_2 + h\ e_2 \times e_2 \times e_2$$

with

the representation $\mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2$
of the group
$\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$.

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This pair is also a prehomogenous vector space

Reducing a Bhargava cube

Simplify entries using the following reduction algorithm:

- We select the corner that is to be a so that $a \neq 0$.
- Transform cube to ensure a divides b, c and e.

If not, say a does not divide e, n select integers α, β so that $a\alpha + e\beta = (a, e)$. Let $\gamma = -e/(a, e)$, $\delta = a/(a, e)$. In transformed matrix

$$a' = (a, e),\ e' = 0 \text{ and } 1 \leq a' \leq a - 1.$$

If a' does not divide b' or c', repeat the process.
Each time we reduce a, so a finite process.

- Transform cube to ensure $b = c = e = 0$. Select $\alpha = 1$, $\beta = 0, \gamma = -e/a$, $\delta = 1$, so that $e' = 0, b' = b, c' = c$. We repeat this in each of the three directions to ensure that $b = c = e = 0$.

Reducing a Bhargava cube, II

Replacing a by $-a$, we have that the three matrices are:

$$M_1 = \begin{pmatrix} -a & 0 \\ 0 & d \end{pmatrix} x + \begin{pmatrix} 0 & f \\ g & h \end{pmatrix} y, \text{ so } Q_1 = adx^2 + ahxy + fgy^2;$$

$$M_2 = \begin{pmatrix} -a & 0 \\ 0 & g \end{pmatrix} x + \begin{pmatrix} 0 & d \\ f & h \end{pmatrix} y, \text{ so } Q_2 = agx^2 + ahxy + dfy^2;$$

$$M_3 = \begin{pmatrix} -a & 0 \\ 0 & f \end{pmatrix} x + \begin{pmatrix} 0 & d \\ g & h \end{pmatrix} y, \text{ so } Q_3 = afx^2 + ahxy + dgy^2.$$

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All discrim $(Q_j) = (ah)^2 - 4adfg$, and

$$Q_1(fy_2x_3+gx_2y_3+hy_2y_3, ax_2x_3-dy_2y_3) = Q_2(x_2, y_2)Q_3(x_3, y_3)$$

$x_1 = fy_2x_3 + gx_2y_3 + hy_2y_3$ and $y_1 = ax_2x_3 - dy_2y_3$.

Reducing a Bhargava cube, II

Replacing a by $-a$, we have that the three matrices are:

$$M_1 = \begin{pmatrix} -a & 0 \\ 0 & d \end{pmatrix} x + \begin{pmatrix} 0 & f \\ g & h \end{pmatrix} y, \text{ so } Q_1 = adx^2 + ahxy + fgy^2;$$

$$M_2 = \begin{pmatrix} -a & 0 \\ 0 & g \end{pmatrix} x + \begin{pmatrix} 0 & d \\ f & h \end{pmatrix} y, \quad \text{so } Q_2 = agx^2 + ahxy + dfy^2;$$

$$M_3 = \begin{pmatrix} -a & 0 \\ 0 & f \end{pmatrix} x + \begin{pmatrix} 0 & d \\ g & h \end{pmatrix} y, \quad \text{so } Q_3 = afx^2 + ahxy + dgy^2.$$

All discrim $(Q_j) = (ah)^2 - 4adfg$, and

$$Q_1(fy_2x_3+gx_2y_3+hy_2y_3, ax_2x_3-dy_2y_3) = Q_2(x_2,y_2)Q_3(x_3,y_3)$$

$x_1 = fy_2x_3 + gx_2y_3 + hy_2y_3$ and $y_1 = ax_2x_3 - dy_2y_3$.

Dirichlet: $a = 1$. So

Includes *every* pair of bqfs of same discriminant.

$\mathrm{SL}(2,\mathbb{Z})$-transformations. Forms-Ideals-Transformations

Generators of SL$(2,\mathbb{Z})$ $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

Let $z_f := \frac{-b+\sqrt{d}}{2a}$ in the upper half plane.
z, z' equivalent if $\exists M \in \mathrm{SL}(2,\mathbb{Z})$ such that $z' = u/v$ where $\begin{pmatrix}u\\ v\end{pmatrix} = M\begin{pmatrix}z\\ 1\end{pmatrix}$. Hence $z \sim z+1$ and $z \sim -1/z$.
Now $\langle 2a, -b+\sqrt{d}\rangle = 2a(1, z_f)$ so equivalent.

Generators of SL$(2,\mathbb{Z})$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. correspond to two basic ops in Gauss's reduction algorithm
The first is $x \to x+y, y \to y$, so that

$$f(x,y) \sim g(x,y) := f(x+y,y) = ax^2+(b+2a)xy+(a+b+c)y^2.$$

Note that $I_g = (2a, -(b+2a)+\sqrt{d}) = I_f$,
and $z_g = \frac{-b-2a+\sqrt{d}}{2a} = z_f - 1$.

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The second is $x \to y,\ y \to -x$ so that

$$f(x,y) \sim h(x,y) := f(y,-x) = cx^2 - bxy + ay^2.$$

Note that $I_h = (2c, b+\sqrt{d})$, and $z_h = \frac{b+\sqrt{d}}{2c}$.

$$z_f \cdot z_h = \frac{-b+\sqrt{d}}{2a} \cdot \frac{b+\sqrt{d}}{2c} = \frac{d-b^2}{4ac} = -1$$

that is $z_h = -1/z_f$. Then

$$I_h \sim (1, z_h) = (1, -1/z_f) \sim (1, -z_f) = (1, z_f) \sim I_f.$$

Since any $\mathrm{SL}(2,\mathbb{Z})$-transformation can be constructed out of the basic two transformation we deduce

Theorem $f \sim f'$ if and only if $I_f \sim I_{f'}$ if and only if $z_f \sim z_{f'}$.

The ring of integers of a quadratic field, revisited

$$\text{Integer solutions } x, y \text{ to } x^2 + 19 = y^3 \quad ?$$

If so, y is odd else $x^2 \equiv 5 \pmod 8$ #. Also $19 \nmid y$ else $19|x \implies 19 \equiv x^2 + 19 = y^3 \equiv 0 \pmod{19^2}$.

$$\text{Hence } (y, 38) = 1.$$

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Hence the ideals $(x + \sqrt{-19})$ and $(x - \sqrt{-19})$ are coprime

Their product is a cube and so they are both cubes

.

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The ring of integers of $\mathbb{Q}[\sqrt{-19}]$ has class number one. So every ideal is principal. Hence

$$x + \sqrt{-19} = u(a + b\sqrt{-19})^3 \text{ where } u \text{ is a unit.}$$

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Only units: 1 and -1. Change a, b, to ua, ub. Hence

$$\begin{aligned} x + \sqrt{-19} &= (a + b\sqrt{-19})^3 \\ &= a(a^2 - 57b^2) + b(3a^2 - 19b^2)\sqrt{-19}, \end{aligned}$$

so that $b(3a^2 - 19b^2) = 1$.

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Therefore $b = \pm 1$ and so $3a^2 = 19b^2 \pm 1 = 19 \pm 1$ which is impossible. We deduce:

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However what about $\quad 18^2 + 19 = 7^3$

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However what about $18^2 + 19 = 7^3$

The mistake: The ring of integers of $\mathbb{Q}[\sqrt{-19}]$ is **not** the set of numbers of the form $a + b\sqrt{-19}$ with $a, b \in \mathbb{Z}$. *It is $(a + b\sqrt{-19})/2$ with $a, b \in \mathbb{Z}$ and $a \equiv b \pmod 2$.*

Integer solutions x, y to $x^2 + 19 = y^3$?

The ring of integers of $\mathbb{Q}[\sqrt{-19}]$ has class number one. So every ideal is principal. Hence

$$x + \sqrt{-19} = (\tfrac{a+b\sqrt{-19}}{2})^3$$

$$\begin{aligned} 8x + 8\sqrt{-19} &= (a + b\sqrt{-19})^3 \\ &= a(a^2 - 57b^2) + b(3a^2 - 19b^2)\sqrt{-19}, \end{aligned}$$

so that $b(3a^2 - 19b^2) = 8$. Therefore
$b = \pm 1, \pm 2, \pm 4$ or ± 8 and so
$3a^2 = 19 \pm 8, 19 \cdot 4 \pm 4, 19 \cdot 16 \pm 2$ or $19 \cdot 64 \pm 1$.
The only solution is $b = 1, a = \pm 3$ leading to

$$x = \mp 18, \ y = 7, \text{ the only solutions.}$$