9. THE FUNDAMENTAL PROPERTIES OF $\zeta(s)$

9.1. Representations of $\zeta(s)$. Let us begin this section by noting that for $\mathrm{Re}(s) > 1$ we have

$$\left(1-\frac{2}{2^s}\right)\zeta(s) = \left(1-\frac{2}{2^s}\right)\sum_{n\geq 1}\frac{1}{n^s} = \sum_{n\geq 1}\frac{1}{n^s} - 2\sum_{m\geq 1}\frac{1}{(2m)^s}$$
$$= \sum_{m\geq 1}\left(\frac{1}{(2m-1)^s} - \frac{1}{(2m)^s}\right).$$

Just as in (3.3.5) we find that with the terms grouped like this the right side converges for $\mathrm{Re}(s) > 0$. This defines an analytic continuation for $\zeta(s)$ except perhaps where $s-1$ is an integer multiple of 2π. In fact the analogy to (3.3.6) yields that the right side is $\ll |s|/\mathrm{Re}(s)$.

Another approach is given by noting that if $\mathrm{Re}(s) > 1$ then

(9.1.1) $$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx$$

which gives an analytic continuation of $\zeta(s)$ to $\mathrm{Re}(s) > 0$, and implies that $|\zeta(s)| \ll |s|$ provided $\mathrm{Re}(s), |s-1| > c > 0$.

Exercises

9.1a.a) Combine (9.1.1) with exercise 2.2a.d to prove that

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma$$

b) Deduce that

$$\lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) = \gamma$$

9.1b.a) Use (9.1.1) to deduce that $\zeta(\overline{s}) = \overline{\zeta(s)}$.

b) Deduce that if $\zeta(\sigma+it) = 0$ then $\zeta(\sigma-it) = 0$.

9.2. A functional equation.

Lemma 9.2. *For any $a \in \mathbb{R}$ and $x > 0$ we have*

(9.2.1) $$\sum_{n\in\mathbb{Z}} e^{-\pi(n+a)^2/x} = \sqrt{x}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x - 2i\pi n a}.$$

Proof. By (7.3.2) we have, taking $t = xu - a$,

$$\sum_{n\in\mathbb{Z}} e^{-\pi(n+a)^2/x} = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi(t+a)^2/x+2i\pi mt} dt$$
$$= x \sum_{m\in\mathbb{Z}} e^{-\pi(xm^2+2ima)} \int_{-\infty}^{\infty} e^{-\pi x(u-im)^2} du.$$

If we change variables $v = u - im$ in the final integral we are integrating the function $e^{-\pi xv^2}$ from $-\infty$ to ∞ along a path shifted a little bit up or down. The value of the integral does not change since there are no singularities of this function, so its value is $\int_{-\infty}^{\infty} e^{-\pi xv^2} dv = C/\sqrt{x}$, letting $w = \sqrt{x}v$, where $C := \int_{-\infty}^{\infty} e^{-\pi w^2} dw$. This gives (9.2.1) with the right side multiplied through by C; taking $a = 0, x = 1$ we deduce that $C = 1$ and hence our result.

If we differentiate (9.2.1) with respect to a we obtain

$$\sum_{n\in\mathbb{Z}} (n+a)e^{-\pi(n+a)^2/x} = ix^{3/2} \sum_{n\in\mathbb{Z}} ne^{-\pi n^2x-2i\pi na}. \tag{9.2.2}$$

9.3. A functional equation for the Riemann zeta function. Suppose that $\text{Re}(s) > 1$. Writing $\omega(x) := \sum_{n\geq 1} e^{-\pi n^2 x}$, we obtain from (9.2.1) with $a = 0$ that $2\omega(1/x) + 1 = \sqrt{x}(2\omega(x) + 1)$. Therefore

$$\int_0^1 x^{\frac{s}{2}-1}\omega(x)dx = \int_1^{\infty} x^{-\frac{s}{2}-1}\omega(1/x)dx = \int_1^{\infty} x^{-\frac{s}{2}-1} \left(\frac{\sqrt{x}-1}{2} + \sqrt{x}\omega(x)\right) dx$$
$$= \frac{1}{s-1} - \frac{1}{s} + \int_1^{\infty} x^{-\frac{s+1}{2}}\omega(x)dx = \frac{1}{s(s-1)} + \int_1^{\infty} x^{\frac{1-s}{2}-1}\omega(x)dx.$$

Hence by (7.9.3) we obtain

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \sum_{n\geq 1} \int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx = \int_0^{\infty} x^{\frac{s}{2}-1}\omega(x)dx$$
$$= -\frac{1}{s(1-s)} + \int_1^{\infty} (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})\omega(x)\frac{dx}{x} \tag{9.3.1}$$

This equation is important for two reasons. Firstly since $\omega(x)$ gets small very rapidly as x gets larger, we see that the integral on the right of (9.3.1) converges for all s, not just those with $\text{Re}(s) > 1$. Thus this formula provides an analytic continuation of $\Gamma(\frac{s}{2})\zeta(s)$ except at the points $s = 0, 1$ where we get poles of order 1. Moreover, one can see that (9.3.1) remains unchanged if we replace s by $1-s$. A convenient way to write this information is to define $\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ so that $\xi(s)$ is analytic and satisfies the functional equation

$$\xi(s) = \xi(1-s). \tag{9.3.2}$$

Now $\frac{s}{2}\Gamma(\frac{s}{2})$ has no zeros (see section 7.9), and so the poles of $(s-1)\zeta(s)$ are the same as those of $\xi(s)$ (of which there are none). Therefore the only pole of $\zeta(s)$ lies at $s=1$; and if $\mathrm{Re}(s)<0$ then $\zeta(s)$ has *trivial* zeros at $s=-2,-4,-6,\dots$

Exercises

9.3a. Use (7.9.5) to rewrite (9.3.2) as $\zeta(1-s)=2^{1-s}\pi^{-s}(\cos\frac{\pi}{2}s)\Gamma(s)\zeta(s)$.

9.4. A FUNCTIONAL EQUATION FOR MODULAR FUNCTIONS. For $f(z)=\sum_{n\geq 1}c_n e^{2i\pi nz}$, define the *Mellin transform* as

$$\Lambda(s,f):=\int_0^\infty f(iz)z^{s-1}dz=\sum_{n\geq 1}c_n\int_0^\infty e^{-2\pi nz}z^{s-1}dz$$
$$=\sum_{n\geq 1}\frac{c_n}{(2\pi n)^s}\Gamma(s):=(2\pi)^{-s}\Gamma(s)L(s,f),$$

changing variable $t=2\pi nz$, where $L(s,f):=\sum_{n\geq 1}c_n/n^s$. Now suppose that f satisfies $f(-1/t)=\pm t^k f(t)$ for some even integer k. Taking $t=iz$ we obtain $f(i/z)=\pm(iz)^k f(iz)$, so that

$$\Lambda(s,f)=\pm i^{-k}\int_0^1 f(i/z)z^{s-1-k}dz+\int_1^\infty f(iz)z^{s-1}dz$$
$$=\pm i^{-k}\int_1^\infty f(iy)y^{k-1-s}dy+\int_1^\infty f(iz)z^{s-1}dz=\int_1^\infty\left(\pm i^{-k}z^{k-s}+z^s\right)f(iz)\frac{dz}{z}.$$

Therefore $\Lambda(k-s,f)=\pm(-1)^{k/2}\Lambda(s,f)$. We would like this integral to converge absolutely for all s, which can be proved in certain interesting circumstances.

More generally one has a functional equation like $g(-1/(Nt))=\pm N^{k/2}t^k g(t)$. Writing $u=\sqrt{N}t$ and $f(z)=g(z/\sqrt{N})$ one has $f(-1/u)=\pm u^k f(u)$, which takes us back to the situation above.

9.5. PROPERTIES OF $\xi(s)$. Using section 9.1 and Stirling's formula we see that for $\xi(s)(=\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s))$ we have $\log|\xi(s)|\sim|s|\log|s|$ for $\mathrm{Re}(s)\geq 1/2$, as $|s|\to\infty$. We also get this inequality in $\mathrm{Re}(s)\leq 1/2$ using the functional equation (9.3.2). Therefore $\xi(s)$ is an analytic function of order 1 and so we can write

$$\xi(s)=e^{As+B}\prod_{\rho:\ \xi(\rho)=0}(1-s/\rho)e^{s/\rho} \tag{9.5.1}$$

by (7.4.3). The zeros of $\xi(s)$ are precisely the non-trivial zeros of $\zeta(s)$; that is, the zeros in the *critical strip* $0\leq\mathrm{Re}(s)\leq 1$, the others having been cancelled by the zeros of $\Gamma(\frac{s}{2})$. From section 7.4 we know that $\sum_{\rho:\ \xi(\rho)=0}1/|\rho|^{1+\epsilon}$ converges for every $\epsilon>0$. However $\sum_{\rho:\ \xi(\rho)=0}1/|\rho|$ must diverge, else, as noted at the end of section 7.4, we would have the bound $\log|\zeta(s)|\ll|s|$. (Note that this implies that $\zeta(s)$ has infinitely many zeros in the critical strip.) Taking the logarithmic derivative of (9.5.1) gives

$$\frac{\xi'(s)}{\xi(s)}=A+\sum_{\rho:\ \xi(\rho)=0}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right), \tag{9.5.2}$$

so that, by (7.9.4) we obtain, noting that the zeros of $\zeta(s)$ are precisely those of $\xi(s)$ together with the trivial zeros $-2, -4, -6, \ldots$,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + A' + \sum_{\rho:\ \zeta(\rho)=0} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right), \tag{9.5.3}$$

where $A' = A + \frac{\gamma}{2} + \frac{1}{2}\log \pi$. By (9.3.2), we have $\frac{\xi'(s)}{\xi(s)} + \frac{\xi'(1-s)}{\xi(1-s)} = 0$, and that if $\xi(\rho) = 0$ then $\xi(1-\rho) = 0$; hence, by (9.5.2),

$$0 = 2A + \sum_{\rho:\ \xi(\rho)=0} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + \sum_{\rho':\ \xi(\rho')=0} \left(\frac{1}{1-s-\rho'} + \frac{1}{\rho'} \right) = 2A + 2 \sum_{\rho:\ \xi(\rho)=0} \frac{1}{\rho},$$

adding the terms $1/(s-\rho)$ and $1/(1-s-\rho')$ where $\rho' = 1-\rho$. Therefore

$$A = -\sum_{\rho:\ \xi(\rho)=0} \frac{1}{\rho}. \tag{9.5.4}$$

We have seen that this sum does not converge absolutely, but if we pair up the ρ and $1-\rho$ terms, or the ρ and $\overline{\rho}$ terms, then it does. Note that if $\rho = \beta + i\gamma$ then $\mathrm{Re}(1/\rho) = \beta/(\beta^2+\gamma^2)$, so every term in the sum in (9.5.4) is negative, and therefore $A < 0$.

Exercises

9.5a. In this exercise we evaluate A and B in (9.5.1).

a) Use (7.9.5) to show that $\Gamma(1/2) = \pi^{1/2}$, and deduce, using the definition of ξ, that $e^B = \xi(0) = \xi(1) = 1/2$.

b) Use (9.5.2), the functional equation, and exercises 7.9a and 9.1a and to show that $A = \xi'(0)/\xi(0) = -\xi'(1)/\xi(1) = \frac{1}{2}\log 4\pi - 1 - \frac{\gamma}{2} = -.0230957084\ldots$.

c) Deduce, using (9.5.4), that if $\xi(\rho) = 0$ with $\mathrm{Re}(\rho) \geq 1/2$ then $|\rho| \geq 6.580128218\ldots$.

9.6. A zero-free region for $\zeta(s)$. We begin by proving that $\zeta(1+it) \neq 0$ for all real t. This was the final step in the proof of the prime number theorem in 1896, and the proof is quite beautiful. Starting from the Euler product we have

$$\log \zeta(\sigma+it) = -\sum_{p \text{ prime}} \log \left(1 - \frac{1}{p^{\sigma+it}} \right) = \sum_{p \text{ prime}} \sum_{m \geq 1} \frac{1}{mp^{m(\sigma+it)}}$$

for $\sigma > 1$, so that

$$\log |\zeta(\sigma+it)| = \mathrm{Re}\left(\log \zeta(\sigma+it)\right) = \sum_{p \text{ prime}} \sum_{m\geq 1} \frac{\cos(mt\log p)}{mp^{m\sigma}}. \tag{9.6.1}$$

Now if $\zeta(1+it) = 0$ then (9.6.1) yields that the $\cos(mt\log p)$ have a bias as we vary over prime powers p^m, pointing significantly more often in the negative than positive direction. But this implies that $\cos(2mt\log p)$ should point significantly more often in the positive than negative direction, so that $\zeta(1+2it)$ is unbounded, which we know is impossible. The proof (of Mertens) that we now give formalizes this heuristic. The first thing to notice is that for any θ,

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geq 0,$$

so that $3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \geq 0$ by (9.6.1), and hence

$$\zeta(\sigma)^3 \cdot |\zeta(\sigma+it)|^4 \cdot |\zeta(\sigma+2it)| \geq 1. \tag{9.6.2}$$

Now assume that $\zeta(1+it) = 0$ so that $\zeta(\sigma+it) \sim C(\sigma-1)^r$ for some integer $r \geq 1$ and constant $C \neq 0$, as $\sigma \to 1^+$. We also know that $\zeta(\sigma) \sim 1/(\sigma-1)$ as $\sigma \to 1^+$. But then (9.6.2) implies that there exists $\epsilon > 0$ such that if $|\sigma - 1| < \epsilon$ then $|\zeta(\sigma+2it)| \geq 1/(2C^4(\sigma-1)^{4r-3}) \geq 1/(2C^4(\sigma-1))$. This implies that $1+2it$ is a pole of $\zeta(s)$, giving a contradiction.

We can extend this proof to obtain a *zero-free region* for $\zeta(s)$, that is a region of the complex plane without zeros of $\zeta(s)$. Now, by (9.5.3) and exercise 7.9c, we have

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} - \log|s| + O(1) + \sum_{\substack{\rho:\ \zeta(\rho)=0 \\ 0\leq \mathrm{Re}(\rho)\leq 1}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{9.6.3}$$

Now $\mathrm{Re}(1/\rho) \geq 0$ as $0 \leq \mathrm{Re}(\rho) \leq 1$, and if $\mathrm{Re}(s) \geq 1 > \mathrm{Re}(\rho)$ then $\mathrm{Re}(1/(s-\rho)) \geq 0$. Suppose that $s = \sigma + it$ where $\sigma > 1$ so that $\mathrm{Re}(\zeta'(s)/\zeta(s)) \geq \mathrm{Re}(1/(1-s)) - \log|s| + O(1)$; and if $\zeta(\beta+it) = 0$ for some $\beta < 1$ then we can add a $1/(\sigma-\beta)$ to the lower bound.

Next we again use the cosine inequality, this time with the series

$$-\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \sum_{p \text{ prime}} \log p \sum_{m\geq 1} \frac{\cos(mt\log p)}{p^{m\sigma}},$$

so that

$$0 \leq -3\ \mathrm{Re}\left(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right) - 4\ \mathrm{Re}\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right) -\ \mathrm{Re}\left(\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right). \tag{9.6.4}$$

Assuming that $\zeta(\beta+it) = 0$ and σ is close to 1, this is

$$\leq \frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + 5\log(|t|+2) + O(1) \tag{9.6.5}$$

since $\mathrm{Re}(1/(\sigma+it-1)) \leq 1/|t-1| \ll 1$. Selecting $\sigma = 1 + 1/(10\log(|t|+2))$, we deduce that

$$\beta \leq 1 - \frac{1}{70\log(|t|+2)} + O\left(\frac{1}{(\log(|t|+2))^2}\right). \tag{9.6.6}$$

Since there are no zeros near to $\sigma = 1$ (by exercise 9.5a.c), one can prove by these methods, the more convenient

$$\beta \leq 1 - \frac{1}{71\log(|t|+2)}.$$

In 1922, Littlewood enlarged the width of the zero-free region to $\gg \log\log|t|/\log|t|$, and in 1958 by Korobov and Vinogradov to $\gg 1/(\log|t|)^{2/3+\epsilon}$, a central result that has not been improved in fifty years.

9.7. Approximations to $\zeta'(s)/\zeta(s)$. Following (9.5.4) and (9.6.3) we have

$$\sum_{\substack{\rho:\ \zeta(\rho)=0\\ 0\le \mathrm{Re}(\rho)\le 1}} \frac{1}{s-\rho} = \frac{\zeta'(s)}{\zeta(s)} + \log|s| + \frac{1}{s-1} + O(1). \tag{9.7.1}$$

The right side equals $\log T + O(1)$ when $s = 2+iT$ for large T. For $0\le\beta\le 1$, the real part of $1/(2+iT-(\beta+i\gamma))$ is $(2-\beta)/((2-\beta)^2+(T-\gamma)^2) \ge 1/(4+(T-\gamma)^2)$, and so

$$\sum_{\substack{\rho:\ \zeta(\beta+i\gamma)=0\\ 0\le\beta\le 1}} \frac{1}{4+(T-\gamma)^2} \le \log T + O(1). \tag{9.7.2}$$

We deduce that there are $\le 8\log T + O(1)$ zeros $\beta+i\gamma$ for which $|T-\gamma|\le 2$.

Now take (9.7.1) with $s=\sigma+iT$ (which is not a zero of $\zeta(s)$) and σ bounded, and subtract (9.7.1) with $s = 2+iT$. The terms corresponding to a zero ρ give

$$\left|\frac{1}{s-\rho} - \frac{1}{2+iT-\rho}\right| = \frac{2-\sigma}{|s-\rho|\cdot|2+iT-\rho|} \le \frac{2-\sigma}{|T-\gamma|^2}.$$

We will use this bound when $|T-\gamma|\ge 2$; and note that $1/|2+iT-\rho|\le 1$ for all such ρ by considering the real part. Therefore for $s=\sigma+iT$ we deduce from (9.7.2) that

$$\begin{aligned}\left|\frac{\zeta'(s)}{\zeta(s)} - \sum_{\substack{\rho:\ \zeta(\rho)=0\\ |T-\gamma|\le 2}}\frac{1}{s-\rho}\right| &\le \sum_{\substack{\rho:\ \xi(\rho)=0\\ |T-\gamma|\le 2}}\frac{1}{|2+iT-\rho|} + \sum_{\substack{\rho:\ \xi(\rho)=0\\ |T-\gamma|\ge 2}}\frac{2(2-\sigma)}{4+|T-\gamma|^2} + O(1)\\ &\le (12-2\sigma)\log T + O(1)\end{aligned} \tag{9.7.3}$$

Suppose that we select s which is not too close to any zero of $\zeta(s)$, that is $|s-\rho|\gg 1/\log T$ for every ρ such that $\zeta(\rho)=0$. Then the contribution of the sum on the left side of (9.7.3) is $\ll (\log T)^2$, as the sum contains $\ll \log T$ terms, and so we can deduce that for $|\sigma|\le 2$

$$\left|\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right| \ll (\log(|t|+2))^2. \tag{9.7.4}$$

If $\sigma\le -1$ we can do better by using the functional equation as presented in exercise 9.3a. Thus if $s = 1-(\sigma+it)$ then

$$\begin{aligned}\frac{\zeta'(1-s)}{\zeta(1-s)} &= -\log(2\pi) - \frac{\pi}{2}\tan\left(\frac{\pi}{2}s\right) + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}\\ &= \frac{1}{s-2m-1} + \log|s| + O(1)\end{aligned} \tag{9.7.5}$$

using exercise 7.9c, where $2m$ is the even integer nearest to $-\sigma$.

9.8. On the number of zeros of $\zeta(s)$. We know that $\zeta(s)$ has the same zeros in the critical strip as $\xi(s)$; and $\xi(s)$ has the advantage that it is analytic. Therefore the number of zeros, $N(T)$, of $\zeta(s)$ inside $C := \{s : \ 0 \leq \text{Re}(s) \leq 1, \ 0 \leq \text{Im}(s) \leq T\}$ is given by

$$N(T) = \frac{1}{2i\pi} \oint_C \frac{\xi'(s)}{\xi(s)} \, ds = \frac{1}{2\pi} \, \triangle_C(\arg(\xi(s))) \tag{9.8.1}$$

by the argument principle as discussed in section 7.5, so long as there are no zeros on C: We showed in section 9.6 that $\xi(s)$ has no zeros with $\text{Re}(s) = 1$, which implies via the functional equation (9.3.2) that $\zeta(s)$ has no zeros with $\text{Re}(s) = 0$. By exercise 9.5a.c there are no zeros in this region with $\text{Re}(s) = 0$ (or even small). We need only to make sure that there is no zero with $\text{Im}(s) = T$. Now, from (9.3.2) we know that $\xi(s) = \xi(1-s) = \overline{\xi(1-\overline{s})}$; in particular $\xi(\sigma + it) = \overline{\xi(1 - \sigma + it)}$; and so the change of argument as we proceed along the path P which goes from $1/2$ to 1, then 1 to $1 + iT$, and then $1 + iT$ to $1/2 + iT$, is the same as when we proceed around the rest of C. Moreover $\xi(s)$ is real-valued (by definition) and positive for $-1 \leq s \leq 2$ since it has no zeros close to 0, hence there is no change in $\arg(\xi(s))$ as we go along this line. We have therefore proved that $N(T)$ equals $\frac{1}{\pi}$ times the change in argument of $\xi(s)$ along the path L which goes from 1 to $1+iT$, and then from $1 + iT$ to $1/2 + iT$.

For the next part of the calculation it is easiest if we widen C, to allow $-1 \leq \text{Re}(s) \leq 2$: this does not change the value of (9.8.1) since there are no further zeros of $\xi(s)$ in this region, nor any of the arguments above. By definition $\arg(\xi(s)) = \arg(s) + \arg(s - 1) - \frac{\log \pi}{2} \, \text{Im}(s) + \arg(\Gamma(\frac{s}{2})) + \arg(\zeta(s))$. Now the arguments of both s and $s-1$ change from 0 to $\frac{\pi}{2} + O(1/T)$. Stirling's formula (see exercise 7.9a below) tells us that $\arg(\Gamma(\frac{s}{2}))$ changes from 0 to $\frac{T}{2} \log\left(\frac{T}{2e}\right) - \frac{\pi}{8} + O\left(\frac{1}{T}\right)$. Therefore

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2e}\right) + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right), \tag{9.8.2}$$

where $S(T) := \frac{1}{\pi} \ \arg\zeta(\frac{1}{2} + iT)$ (since $\arg\zeta(2) = 0$).

By exercise 9.8b we see that $\arg\zeta(2 + iT)$ is bounded, and so, using (9.7.3),

$$\begin{aligned}
\pi S(T) &= \left(\arg\zeta(\frac{1}{2} + iT) - \arg\zeta(2 + iT)\right) + O(1) = -\int_{\frac{1}{2}+iT}^{2+iT} \text{Im}\left(\frac{\zeta'(s)}{\zeta(s)}\right) ds + O(1) \\
&= -\sum_{\substack{\rho:\ \zeta(\rho)=0 \\ |T-\gamma|\leq 2}} \int_{\frac{1}{2}+iT}^{2+iT} \text{Im}\left(\frac{1}{s-\rho}\right) ds + O(\log T) \\
&= -\sum_{\substack{\rho:\ \zeta(\rho)=0 \\ |T-\gamma|\leq 2}} \left(\arg(\frac{1}{2} + iT - \rho) - \arg(2 + iT - \rho)\right) + O(\log T).
\end{aligned}$$

Evidently each such change in argument contributes at most π, and we have seen that the sum has $\ll \log T$ terms, and so we deduce that

$$S(T) \ll \log T, \tag{9.8.3}$$

which implies that

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2e}\right) + O\left(\log T\right). \tag{9.8.4}$$

We deduce that

$$\sum_{\substack{\rho:\ \zeta(\rho)=0\\ 0<\mathrm{Re}(\rho)<1\\ |\mathrm{Im}(\rho)|<T}} \frac{1}{|\rho|} = \frac{(\log T)^2}{2\pi} + O(1). \tag{9.8.5}$$

Exercises

9.8a. Use (7.9.6) and the Taylor series for $\log(1+z)$ to show that

$$-4\log\left(\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right)\right) = \pi T + \log\left(\frac{T}{2e}\right) + 1 - 2\log(2\pi) + i\left(\frac{\pi}{2} - 2T\log\left(\frac{T}{2e}\right)\right) + O\left(\frac{1}{T}\right).$$

9.8b. Use (9.7.2) to show that $N(T+1) - N(T) \ll \log T$. Use this together with (9.8.2) to show that $N(T+1) - N(T) \gg \log T$ for at least a positive proportion of integers T. Deduce also that that there exists $t \in [T, T+1]$ which is at a distance $\gg 1/\log T$ from the nearest zero.

9.8c. Show that there exists a constant Δ_0 such that if $\Delta \geq \Delta_0$ then $N(T+\Delta) - N(T) \asymp \Delta \log T$ for all sufficiently large T.

9.8d. Prove that the argument of $\zeta(2+it)$ is bounded.