8. RIEMANN'S PLAN FOR PROVING THE PRIME NUMBER THEOREM

8.1. A method to accurately estimate the number of primes. Up to the middle of the nineteenth century, every approach to estimating $\pi(x) = \#\{\text{primes} \leq x\}$ was relatively direct, based upon elementary number theory and combinatorial principles, or the theory of quadratic forms. In 1859, however, the great geometer Riemann took up the challenge of counting the primes in a very different way. He wrote just one paper that could be called "number theory," but that one short memoir had an impact that has lasted nearly a century and a half, and its ideas have defined the subject we now call *analytic number theory*.

Riemann's memoir described a surprising approach to the problem, an approach using the theory of complex analysis, which was at that time still very much a developing subject.[1] This new approach of Riemann seemed to stray far away from the realm in which the original problem lived. However, it has two key features:

- it is a potentially practical way to settle the question once and for all;
- it makes predictions that are similar, though not identical, to the Gauss prediction. Indeed it even suggests a secondary term to compensate for the overcount we saw in the data in the table in section 2.10.

Riemann's method is the basis of our main proof of the prime number theorem, and we shall spend this chapter giving a leisurely introduction to the key ideas in it. Let us begin by extracting the key prediction from Riemann's memoir and restating it in entirely elementary language:

$$\text{lcm}[1, 2, 3, \cdots, x] \quad \text{is about} \quad e^x.$$

Using data to test its accuracy we obtain:

x	Nearest integer to $\ln(\text{lcm}[1, 2, 3, \cdots, x])$	Difference
100	94	-6
1000	997	-3
10000	10013	13
100000	100052	57
1000000	999587	-413

Riemann's prediction can be expressed precisely and explicitly as

$$|\log(\text{lcm}[1, 2, \cdots, x]) - x| \leq 2\sqrt{x}(\log x)^2 \quad \text{for all } x \geq 100. \tag{8.1.1}$$

[1]Indeed, Riemann's memoir on this number-theoretic problem was a significant factor in the development of the theory of analytic functions, notably their global aspects.

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Since the power of prime p which divides $\text{lcm}[1, 2, 3, \cdots, x]$ is precisely the largest power of p not exceeding x, we have that

$$\left(\prod_{p\le x} p\right) \times \left(\prod_{p^2\le x} p\right) \times \left(\prod_{p^3\le x} p\right) \times \cdots = \text{lcm}[1, 2, 3, \cdots, x].$$

Combining this with Riemann's prediction and taking logarithms, we deduce that

$$\left(\sum_{p\le x} \log p\right) + \left(\sum_{p^2\le x} \log p\right) + \left(\sum_{p^3\le x} \log p\right) + \cdots \quad \text{is about} \quad x.$$

The primes in the first sum here are precisely the primes counted by $\pi(x)$, the primes in the second sum those counted by $\pi(x^{1/2})$, and so on. By partial summation we deduce that

$$\pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + \tfrac{1}{3}\pi(x^{1/3}) + \cdots \approx \int_2^x \frac{dt}{\ln t} = \text{Li}(x).$$

If we solve for $\pi(x)$ in an appropriate way, we find the equivalent form

$$\pi(x) \approx \text{Li}(x) - \tfrac{1}{2}\text{Li}(x^{1/2}) + \cdots .$$

Hence Riemann's method yields more-or-less the same prediction as Gauss, but with something extra, a secondary term that will hopefully compensate for the overcount that we witnessed in the Gauss prediction. Reviewing the data (where "Riemann's overcount" refers to $\text{Li}(x) - \frac{1}{2}\text{Li}(\sqrt{x}) - \pi(x)$, while "Gauss's overcount" refers to $\text{Li}(x) - \pi(x)$ as before) we have:

x	$\#\{\text{primes} \le x\}$	Gauss's overcount	Riemann's overcount
10^8	5761455	753	131
10^9	50847534	1700	-15
10^{10}	455052511	3103	-1711
10^{11}	4118054813	11587	-2097
10^{12}	37607912018	38262	-1050
10^{13}	346065536839	108970	-4944
10^{14}	3204941750802	314889	-17569
10^{15}	29844570422669	1052618	76456
10^{16}	279238341033925	3214631	333527
10^{17}	2623557157654233	7956588	-585236
10^{18}	24739954287740860	21949554	-3475062
10^{19}	234057667276344607	99877774	23937697
10^{20}	2220819602560918840	222744643	-4783163
10^{21}	21127269486018731928	597394253	-86210244
10^{22}	201467286689315906290	1932355207	-126677992

Table 6. Primes up to various x, and Gauss's and Riemann's predictions.

Riemann's prediction does seem to fare better than that of Gauss, though not a lot better. However, the fact that the error in Riemann's prediction takes both positive and negative values suggests that this might be about the best that can be done.

8.2. Linking number theory and complex analysis. Riemann showed that the number of primes up to x can be given in terms of the complex zeros of the function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

studied by Euler, which we now call the *Riemann zeta-function*. In this definition s is a complex number, which we write as $s = \sigma + it$ when we want to refer to its real and imaginary parts σ and t separately. If s were a real number, we would know from first-year calculus that the series in the definition of $\zeta(s)$ converges if and only if $s > 1$; that is, we can sum up the infinite series and obtain a finite, unique value. In a similar way, it can be shown that the series only converges for complex numbers s such that $\sigma > 1$. But what about when $\sigma \leq 1$? How do we get around the fact that the series does not sum up (that is, converge)? As we discussed in section 7.7, one can "analytically continue" $\zeta(s)$ so that it is well-defined on the whole complex plane. More than that, $\zeta(s) - \frac{1}{s-1}$ is analytic, so that ζ is meromorphic, indeed analytic other than its only pole at $s = 1$, which is a simple pole with residue 1.

Riemann showed that confirming Gauss's conjecture for the number of primes up to x is *equivalent* to gaining a good understanding of the zeros of the function $\zeta(s)$, so we now begin to sketch the key steps in the argument that link these seemingly unconnected topics. The starting point is to take the derivative of the logarithm of Euler's identity (2.2.1)

$$\zeta(s) = \sum_{\substack{n \geq 1 \\ n \text{ a positive integer}}} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{8.2.1}$$

to obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p \text{ prime}} \frac{\log p}{p^s - 1} = \sum_{p \text{ prime}} \sum_{m \geq 1} \frac{\log p}{p^{ms}}.$$

Perron's formula (7.6.2) allows one to describe a "step-function" in terms of a continuous function so that if x is not a prime power then we obtain

$$\Psi(x) := \sum_{\substack{p^m \leq x \\ p \text{ prime}, m \geq 1}} \log p = \frac{1}{2\pi i} \sum_{p \text{ prime}, m \geq 1} \log p \int_{s:\mathrm{Re}(s)=c} \left(\frac{x}{p^m}\right)^s \frac{\mathrm{d}s}{s}$$

$$= -\frac{1}{2\pi i} \int_{s:\mathrm{Re}(s)=c} \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \,\mathrm{d}s. \tag{8.2.2}$$

Here we can justify swapping the order of the sum and the integral if c is taken large enough since then everything converges absolutely. Note that we are not counting the number of primes up to x but rather the "weighted" version, $\Psi(x)$.

The next step is perhaps the most difficult. The idea is to replace the line $\mathrm{Re}(s) = c$ along which the integral has been taken, by a line far to the left, on which we can show that the integral is small, in fact smaller the further we go to the left. The difference between the values along these two integrals is given by a sum of residues, as described in sections 7.5 and 7.6. Now for any meromorphic function f, the poles of $f'(s)/f(s)$ are given by the zeros and poles of f, all of order 1, and the residue is simply the order of that zero, or minus the order of that pole. In this way we can obtain the *explicit formula*

$$\Psi(x) = \sum_{\substack{p \text{ prime}, m\geq 1 \\ p^m \leq x}} \log p = x - \sum_{\rho:\ \zeta(\rho)=0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}, \tag{8.2.3}$$

where, if ρ is a zero of $\zeta(s)$ of order k, then there are k terms for ρ in the sum. One might ask how we add up the (possibly) infinite sum over zeros ρ of $\zeta(s)$? Simple, add up by order of ascending $|\rho|$ values and it will work out. It is hard to believe that there can be such a formula, an exact expression for the number of primes up to x in terms of the zeros of a complicated function. You can see why Riemann's work stretched people's imagination and had such an amazing impact.

8.3. The functional equation. We saw in section 7.7 how to analytically continue $\zeta(s)$ to all s for which $\text{Re}(s) > 0$. Riemann made an amazing observation which allows us to easily determine the values of $\zeta(s)$ on the left side of the complex plane (where the function is not naturally defined) in terms of the right side. The idea is to multiply $\zeta(s)$ through by some simple function so that this new product $\xi(s)$ satisfies the *functional equation*

$$\xi(s) = \xi(1-s) \quad \text{for all complex numbers } s. \tag{8.3.1}$$

Riemann determined that this can be done by taking $\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$. Here $\Gamma(s)$ is a function which equals the factorial function at positive integers (that is, $\Gamma(n) = (n-1)!$); and is well-defined and continuous for all other s.

8.4. The zeros of the Riemann zeta-function. An analysis of the right side of (8.2.1) reveals that there are no zeros of $\zeta(s)$ with $\text{Re}(s) > 1$. Therefore, using (8.3.1) and (7.9.4), we deduce that the only zeros of $\zeta(s)$ with $\text{Re}(s) < 0$ are those at the negative even integers $-2, -4, \ldots$, the so-called *trivial zeros*. Thus to be able to use (8.2.3) we need to determine the zeros of $\zeta(s)$ inside the *critical strip* $0 \leq \text{Re}(s) \leq 1$. After some calculation, Riemann made yet another extraordinary observation which, if true, would allow us tremendous insight into virtually every aspect of the distribution of primes:[2]

$$\boxed{\text{THE RIEMANN HYPOTHESIS: If } \zeta(s) = 0 \text{ with } 0 \leq \text{Re}(s) \leq 1 \text{ then } \text{Re}(s) = \frac{1}{2}.}$$

Clever people have computed literally billions of zeros of $\zeta(s)$,[3] and every single zero inside the critical strip that has been computed does indeed have real part equal to 1/2. For example, the nontrivial zeros closest to the real axis are $s = 1/2 + i\gamma_1$ and $s = 1/2 - i\gamma_1$,

[2] No reference to these calculations of Riemann appeared in the literature until Siegel discovered them in Riemann's personal, unpublished notes long after Riemann's death.

[3] At least the ten billion zeros of lowest height; that is, with $|\text{Im}(s)|$ smallest.

where $\gamma_1 \approx 14.1347\ldots$. Note that if the Riemann Hypothesis is true, then we can write all the non-trivial zeros of the Riemann zeta-function in the form $\rho = \frac{1}{2} + i\gamma$ (together with their conjugates $\frac{1}{2} - i\gamma$, since $\zeta(1/2 + i\gamma) = 0$ if and only if $\zeta(1/2 - i\gamma) = 0$), where γ is a positive real number. We believe that the positive numbers γ occurring in the nontrivial zeros look more or less like random real numbers, in the sense that none of them are related to others by simple linear equations with integer coefficients (or even by more complicated polynomial equations with algebraic numbers as coefficients). However, nothing along these lines has ever been proved, indeed all we know how to do is to approximate these nontrivial zeros numerically to a given accuracy,

We will show that there are infinitely many zeros $\beta + i\gamma$ of $\zeta(s)$ in the critical strip, indeed about $\frac{T}{2\pi} \log\left(\frac{T}{2e}\right)$ with $0 \leq \gamma \leq T$. It is not difficult to find all of the zeros up to a given height T. The Riemann Hypothesis can be shown to hold for at least forty percent of all zeros, and it fits nicely with many different heuristic assertions about the distribution of primes and other sequences, yet remains an unproved hypothesis, perhaps the most famous and tantalizing in all of mathematics.

8.5. Counting primes. At first sight it seems to make sense to use partial summation on (8.2.3) to get an exact expression for $\pi(x)$, such as

$$\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \ldots = \mathrm{Li}(x) - \sum_{\rho:\ \zeta(\rho)=0} \mathrm{Li}(x^\rho) + \mathrm{Small}(x) - \log 2,$$

where $\mathrm{Small}(x) = \int_x^\infty \frac{dt}{(t^3 - t)\log t}$.[4] However this is a lot more complicated than (8.2.3), so it will be easier to do partial summation at the end of our calculations rather than before.

Since $\zeta(s)$ has infinitely many zeros in the critical strip, (8.2.3) is a difficult formula to use in practice. Indeed we would not expect to use infinitely many sine waves from the formula (7.2.1) to approximate $\{x\} - \frac{1}{2}$ in practice, but instead we would use a finite number of sine waves, as in our discussion there, presumably those with the largest amplitudes. Similarly we modify (8.2.3) to only include only a finite number of zeros, in particular those up to a certain height, T, that is in the box

$$\mathcal{B}(T) := \{\rho :\ \zeta(\rho) = 0,\ 0 \leq \mathrm{Re}(\rho) \leq 1,\ -T \leq \mathrm{Im}(\rho) \leq T\}.$$

This though is an approximation, not an exact formula, and so comes at the cost of an error term, which depends on the height T: For $1 \leq T \leq x$ we have[5]

$$\Psi(x) = x - \sum_{\substack{\rho:\ \zeta(\rho)=0 \\ 0<\mathrm{Re}(\rho)<1 \\ |\mathrm{Im}(\rho)|<T}} \frac{x^\rho}{\rho} + O\left(\frac{x \log x \log T}{T}\right). \tag{8.5.1}$$

Our goal is to show that $\Psi(x) \sim x$, so we select $T \geq (\log x)^2$, and hence we only need to bound the sum over zeros of $\zeta(s)$. Each term in this sum is a complex number and

[4] This expression appeared in Riemann's paper. The simpler expression (8.2.3) is due to Von Mangoldt.

[5] The trivial zeros lie at $-2, -4, -6, \ldots$ and so contribute $\sum_{m\leq 1} 1/(2mx^{2m}) = -\frac{1}{2}\log(1 - \frac{1}{x^2})$ in total to (8.2.3), which is $O(1)$ for $x \geq 1$.

so consists of a magnitude and direction and we might guess that there is considerable cancellation amongst these terms, resulting from the different directions in which they point. However we are unable to prove anything along these lines so, rather disappointingly, we simply bound each term in absolute value:

$$\left|\sum_{\rho\in\mathcal{B}(T)} \frac{x^\rho}{\rho}\right| \le \sum_{\rho\in\mathcal{B}(T)} \left|\frac{x^\rho}{\rho}\right| \le \max_{\rho\in\mathcal{B}(T)} x^{\mathrm{Re}(\rho)} \sum_{\rho\in\mathcal{B}(T)} \frac{1}{|\rho|} \ll x^{\beta(T)}(\log T)^2,$$

using the fact that there are about $\frac{T}{2\pi}\log\left(\frac{T}{2e}\right)$ zeros in $\mathcal{B}(T)$ for all T, where $\beta(T)$ is the largest real part of any zero in $\mathcal{B}(T)$.

The final step in proving the prime number theorem is therefore to produce *zero-free regions* for $\zeta(s)$: that is regions of the complex plane, close to the line $\mathrm{Re}(s) = 1$, without zeros of $\zeta(s)$. For instance in section 9.6 we show that we may take $\beta(T) = 1 - c/\log T$ for some constant $c > 0$. Therefore choosing T so that $\log T = (\log x)^{1/2}$ we deduce that

$$\Psi(x) = x + O\left(x/e^{c'(\log x)^{1/2}}\right)$$

for some constant $c' > 0$, which implies the prime number theorem,

$$\pi(x) = \mathrm{Li}(x) + O\left(x/e^{c'(\log x)^{1/2}}\right).$$

One can see that any improvements in the zero-free region for $\zeta(s)$ will immediately imply improvements in the error term of the prime number theorem. For example if the Riemann Hypothesis is true, so that $\beta(T) = \frac{1}{2}$ for all T, then by taking $T = \sqrt{x}$ in (8.5.1) we obtain that $\Psi(x) = x + O(x^{1/2}(\log x)^2)$,[6], and so

$$\pi(x) = \int_2^x \frac{\mathrm{d}t}{\log t} + O(\sqrt{x}\log x). \tag{8.5.2}$$

We will show that this is not only implied by the Riemann Hypothesis, but actually implies the Riemann Hypothesis. With more care one can prove that the more precise bound

$$|\pi(x) - \mathrm{Li}(x)| \le \sqrt{x}\log x \quad \text{for all } x \ge 3, \quad \text{is equivalent to the Riemann Hypothesis.}$$

8.6. Riemann's revolutionary formula. Riemann's formula (8.2.3) is a little hard to appreciate at first glance. If we assume that the Riemann Hypothesis is true then each non-trivial zeros can be written as $1/2 + i\gamma$ and therefore contributes $x^{1/2+i\gamma}/(\frac{1}{2} + i\gamma)$. Now as we vary over zeros of $\zeta(s)$ the value of γ grows and $\frac{1}{2} + i\gamma$ will be dominated by the $i\gamma$. Therefore $x^{1/2+i\gamma}/(\frac{1}{2} + i\gamma)$ is roughly $x^{1/2+i\gamma}/(i\gamma)$. Summing this together with

[6] Which is a weak form of (8.1.1) since $\Psi(x) = \log(\mathrm{lcm}[1, 2, \cdots, x])$.

the term for $\frac{1}{2} - i\gamma$, we get, roughly, $x^{1/2+i\gamma}/(i\gamma) - x^{1/2-i\gamma}/(i\gamma) = 2x^{1/2}\sin(\gamma\log x)/\gamma$. Combining this information, (8.2.3) becomes

$$\Psi(x) \text{ is roughly } x - 2x^{1/2} \sum_{\gamma>0:\ \zeta(\frac{1}{2}+i\gamma)=0} \frac{\sin(\gamma\log x)}{\gamma} + O(1).$$

We want to convert this into information about the number of primes up to x. If we proceed by partial summation then $\Psi(x)$ should be replaced by $\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \dots$, as in section 8.1, and $x^{1/2}$ by $x^{1/2}/\log x$. Therefore, after some re-arrangement,

$$\boxed{\frac{\int_2^x \frac{\mathrm{d}t}{\ln t} - \#\{\text{primes} \le x\}}{\sqrt{x}/\ln x} \approx 1 + 2 \sum_{\substack{\text{all real numbers } \gamma>0\\ \text{such that } \frac{1}{2}+i\gamma\\ \text{is a zero of } \zeta(s)}} \frac{\sin(\gamma\log x)}{\gamma}.} \tag{8.6.1}$$

The numerator of the left side of this formula is the overcount term when comparing Gauss's prediction $\mathrm{Li}(x)$ with the actual count $\pi(x)$ for the number of primes up to x. The denominator, being roughly of size $\sqrt{x}$, corresponds to the magnitude of the overcount as we observed earlier in our data. The right side of the formula bears much in common with our formula for $\{x\} - 1/2$. It is a sum of sine functions, with the numbers γ employed in two different ways in place of $2\pi n$: each γ is used inside the sine (as the "frequency"), and the reciprocal of each γ forms the coefficient of the sine (as the "amplitude"). We even get the same factor of 2 in each formula. However, the numbers γ here are much more subtle than the straightforward numbers $2\pi n$ in the corresponding formula for $x - 1/2$.

This formula can perhaps be best paraphrased as

The primes should be counted as a sum of waves.

It should be noted that this formula is valid if and only if the Riemann Hypothesis is true – and is thus widely believed to be correct. There is a similar formula if the Riemann Hypothesis is false, but it is rather complicated and technically far less pleasant. The main difficulty arises because the coefficients, $1/\gamma$, are replaced by functions of x. So we want the Riemann Hypothesis to hold because it leads to the simpler formula (8.6.1), and that formula is a delight. Indeed this formula is similar enough to the formulas for sound-waves for some experts to muse that (8.6.1) asserts that "*the primes have music in them*."