7. PRIMER ON ANALYSIS.

To study analytic number theory it helps to know some of the basics of "hard analysis". We made a start in section 2 with $o(.), O(.), \ll, \gg, \asymp, \sim$, etc., but now it is time to discuss, possibly review, the techniques and ideas that can be first encountered in courses in real analysis, complex analysis, Fourier analysis, applied analysis etc.

7.1. INEQUALITIES. The *Cauchy-Schwarz inequality* is remarkably useful. It states that if $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k \in \mathbb{C}$ then

$$\left|\sum_{i=1}^{k} a_i b_i\right|^2 \leq \left(\sum_{i=1}^{k} |a_i|^2\right)\left(\sum_{j=1}^{k} |b_j|^2\right). \tag{7.1.1}$$

There are many proofs in the literature. The main use comes from the separating of variables, the a_i from the b_j, which might have arisen for quite different reasons. The Cauchy-Schwarz inequality is a special case of *Holder's inequality* which states that if p and q are positive with $1/p + 1/q = 1$ then

$$\left|\sum_{i=1}^{k} a_i b_i\right| \leq \left(\sum_{i=1}^{k} |a_i|^p\right)^{1/p}\left(\sum_{j=1}^{k} |b_j|^q\right)^{1/q}. \tag{7.1.2}$$

There is a continuous version of this, namely, if $f(t), g(t) \in \mathbb{C}(t)$ then

$$\left|\int_u^v f(t)g(t)\right| dt \leq \left(\int_u^v |f(t)|^p \, dt\right)^{1/p}\left(\int_u^v |g(t)|^q \, dt\right)^{1/q}. \tag{7.1.3}$$

Given $h(t)$ define $e^{I(m)} := \int_u^v |h(t)|^m \, dt$. If $\ell < m < n$ then we can take $\frac{1}{p} = \frac{n-m}{n-\ell}$ so that $\frac{1}{q} = \frac{m-\ell}{n-\ell}$, and then $f(t) = h(t)^{\ell/p}$ and $g(t) = h(t)^{n/q}$, in (7.1.3) to obtain

$$(n-\ell)I(m) \leq (n-m)I(\ell) + (m-\ell)I(n); \tag{7.1.4}$$

that is, $I(m)$ can be bounded by the appropriate convex linear combination of $I(\ell)$ and $I(n)$, a *convexity bound*.

Exercises

7.1a. You interview a set of one parent families, first asking each parent the number of children in their family and determining the average of the responses, and then asking each child the number of children

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX in their family and determining the average of the responses. Which average is larger, or are they always the same?

7.1b. Use the method of Lagrange multipliers to prove Holder's inequality. Show that one gets equality if and only if there exists a non-zero constant c such that $|a_i|^p = c|b_i|^q$ for all i. (Note that $\frac{1}{p} + \frac{1}{q} = 1$ if and only if $(p-1)(q-1) = 1$.)

7.2. Fourier series. You have probably wondered what "radio waves" and "sound waves" have to do with waves. Sounds don't usually seem to be very "wavy," but rather are fractured, broken up, stopping and starting and changing abruptly. So what's the connection? The idea is that all sounds can be converted into a sum of waves. For example, let's imagine that our "sound" is represented by the gradually ascending line $y = x - \frac{1}{2}$, considered on the interval $0 \leq x \leq 1$, which is shown in the first graph of Figure 1.

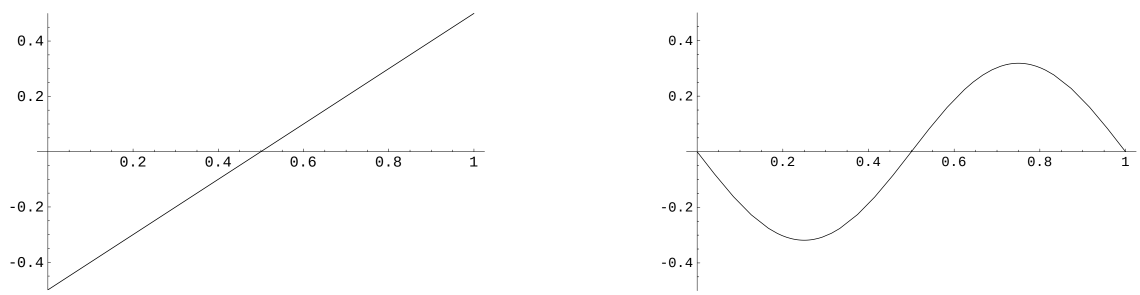

Figure 1. The line $y = x - \frac{1}{2}$ and the wave $y = -\frac{1}{\pi}\sin 2\pi x$.

If we try to approximate this with a "wave," we can come pretty close in the middle of the line using the function $y = -\frac{1}{\pi}\sin(2\pi x)$. However, as we see in the second graph of Figure 1, the approximation is rather poor when $x < 1/4$ or $x > 3/4$.

How can we improve this approximation? The idea is to "add" a second wave to the first, this second wave going through two complete cycles over the interval $[0, 1]$ rather than only one cycle. This corresponds to hearing the sound of the two waves at the same time, superimposed; mathematically, we literally add the two functions together. As it turns out, adding the function $y = -\frac{1}{2\pi}\sin(4\pi x)$ makes the approximation better for a range of x-values that is quite a bit larger than the range of good approximation we obtained with one wave, as we see in Figure 2.

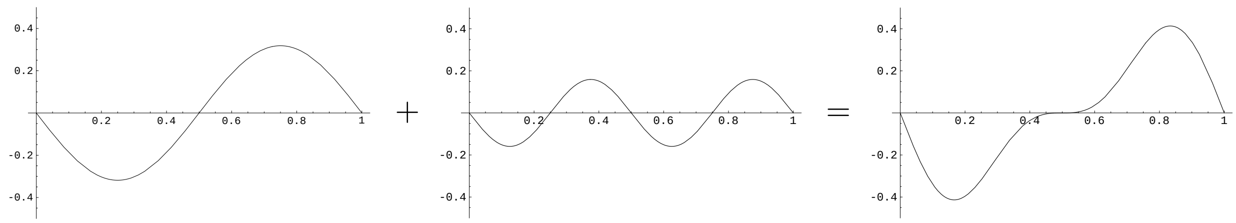

Figure 2. Adding the wave $y = -\frac{1}{2\pi}\sin 4\pi x$ to the wave $y = -\frac{1}{\pi}\sin 2\pi x$

We can continue in this way, adding more and more waves that go through three, four, or five complete cycles in the interval, and so on, to get increasingly better approximations to the original straight line. The approximation we get by using one hundred superimposed waves is really quite good, except near the endpoints 0 and 1.

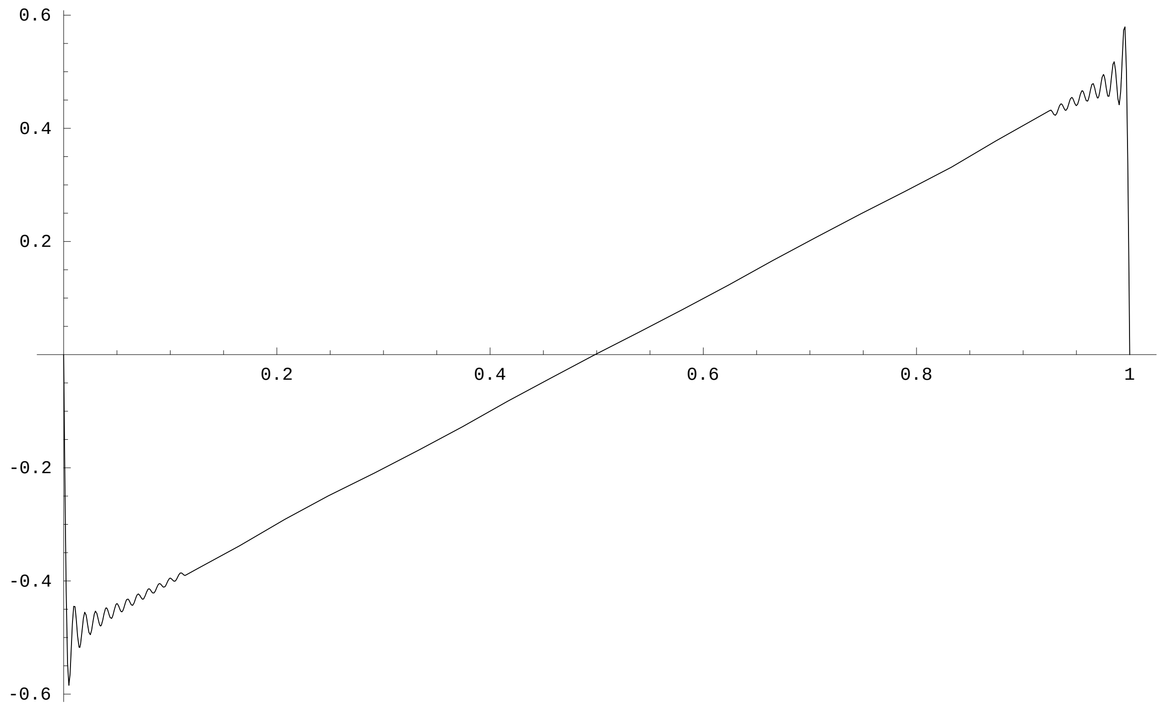

FIGURE 3. The sum of one hundred carefully chosen waves

If we were to watch these one hundred approximations being built up one additional wave at a time, we would quickly be willing to wager that the more waves we allowed ourselves, the better the resulting approximation would be, perhaps becoming as accurate as could ever be hoped for. As long as we allow a tiny bit of error in the approximation (and shut our eyes to what happens very near the endpoints[1]), we can in fact construct a sufficiently close approximation if we use enough waves. However, to get a "perfect" copy of the original straight-line, we would need to use infinitely many sine waves—more precisely, the ones on the right-hand side of the formula

$$x - \frac{1}{2} = -2 \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{2\pi n}, \tag{7.2.1}$$

which can be shown to hold whenever $0 < x < 1$. (We can read off, in the $n = 1$ and $n = 2$ terms of this sum, the two waves that we chose for Figures 1 and 2.) This formula is not

[1]The inability of these finite sums of waves to approximate an original function well near a very "unwavelike" feature, like an endpoint or a discontinuity, is a persistent problem in Fourier analysis known as the *Gibbs phenomenon*.

of much practical use, since we can't really transmit infinitely many waves at once—but it's a gorgeous formula nonetheless!

In general, for any function $f(x)$ defined on the interval $[0,1]$ that is not "too irregular," we can find numbers a_n and b_n such that $f(x)$ can be written as a sum of trigonometric functions, namely,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\big). \tag{7.2.2}$$

This formula, together with a way to calculate the coefficients a_n and b_n, is one of the key identities from "Fourier analysis," and it and its many generalizations are the subject of the field of mathematics known as harmonic analysis. In terms of waves, the numbers $2\pi n$ are the *frequencies* of the various component waves (controlling how fast they go through their cycles), while the coefficients a_n and b_n are their *amplitudes* (controlling how far up and down they go).

In fact the right side of (7.2.2) defines a function on all the reals that is evidently periodic of period 1 (being a sum of periodic functions of period 1), that is $f(x+n) = f(x)$ for all $x \in \mathbb{R},\ n \in \mathbb{Z}$. We may write

$$a_0 + \sum_{m\geq 1} (a_m \cos(2\pi m x) + b_m \sin(2\pi m x)) = \sum_{m\in\mathbb{Z}} c_m e^{2i\pi m x}, \tag{7.2.3}$$

where $a_0 = c_0$ and $a_m = c_m + c_{-m},\ b_m = i(c_m - c_{-m})$ with

$$c_m = \int_0^1 f(t) e^{-2i\pi m t} dt, \tag{7.2.2}$$

for each $m \in \mathbb{Z}$. Notice that if we integrate the series in (7.2.3) multiplied by $e^{-2i\pi j t}$, over $[0,1)$, we obtain c_j provided that we can swap the order of summation and integration without anything surprising happening. Dirichlet proved that this is true at any point of continuity of f, and that the value of (7.2.3) is $\frac{1}{2}(\lim_{t\to x^-} f(t) + \lim_{t\to x^+} f(t))$ at any point of discontinuity of f. An amazingly useful result.

Note that we can rewrite (7.2.1) as

$$\{x\} - \frac{1}{2} = -\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{e^{2i\pi k x}}{2i\pi k}.$$

7.3. Poisson summation. Suppose that we have a function $g(x)$ that is real valued and piecewise continuous. If $f(t) = g(x+t)$ on $t \in [0,1)$ and is periodic, of period 1, then it is also real valued and piecewise continuous, so we can use the Fourier series (7.2.1) and (7.2.2) to prove that

$$\frac{1}{2}(g(x) + g(x+1)) = \sum_{m\in\mathbb{Z}} \int_0^1 g(x+u) e^{-2i\pi m u} du = \sum_{m\in\mathbb{Z}} e^{2i\pi m x} \int_x^{x+1} g(t) e^{-2i\pi m t} dt.$$

Repeating this process by shifting $x \to x+1$, etc, we obtain, since $e^{2i\pi mx} = e^{2i\pi m(x+1)}$, the *Poisson summation formula*, for $y - x \in \mathbb{Z}$,

$$\frac{1}{2}g(x)+g(x+1)+g(x+2)+\cdots+g(y-1)+\frac{1}{2}g(y) = \sum_{m\in\mathbb{Z}} e^{2i\pi mx}\int_x^y g(t)e^{-2i\pi mt}dt. \tag{7.3.1}$$

This implies that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m) \text{ where } \hat{f}(u) := \int_{-\infty}^{\infty} f(t)e^{-2i\pi ut}dt. \tag{7.3.2}$$

One can easily deduce that if $h = f * g$, that is h is the *convolution* of f and g defined by

$$h(u) := \int_{-\infty}^{\infty} f(t)g(u-t)dt \text{ then } \hat{h}(t) = \hat{f}(t)\hat{g}(t).$$

One can "invert" the Fourier transform to obtain

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t)e^{2i\pi tx}dt.$$

One can deduce *Parseval's identity*

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} \hat{f}(t)\overline{\hat{g}(t)}dt,$$

which itself implies

$$\int_{-\infty}^{\infty} |f(x)|^2dx = \int_{-\infty}^{\infty} |\hat{f}(t)|^2dt.$$

If f has period q then (7.3.1) implies

$$\sum_{n=0}^{q-1} f(n) = \sum_{m\in\mathbb{Z}} \int_0^q f(t)e^{-2i\pi mt}dt.$$

For the example $f(t) = e^{2i\pi t^2/q}$ we obtain, swapping m and $-m$,

$$\begin{aligned}
\sum_{n=0}^{q-1} e^{2i\pi n^2/q} &= \sum_{m\in\mathbb{Z}} \int_0^q e^{2i\pi(t^2/q+mt)}dt = q\sum_{m\in\mathbb{Z}} \int_0^1 e^{2i\pi q(u^2+mu)}du \\
&= q\sum_{m\in\mathbb{Z}} e^{-i\pi qm^2/2} \int_0^1 e^{2i\pi q(u+m/2)^2}du \\
&= q\sum_{\substack{m\in\mathbb{Z}\\ m \text{ even}}} \int_{m/2}^{m/2+1} e^{2i\pi qv^2}dv + q(-i)^q \sum_{\substack{m\in\mathbb{Z}\\ m \text{ odd}}} \int_{m/2}^{m/2+1} e^{2i\pi qv^2}dv \\
&= q(1+(-i)^q)\int_{-\infty}^{\infty} e^{2i\pi qv^2}dv = \sqrt{q}(1+(-i)^q)\int_{-\infty}^{\infty} e^{2i\pi w^2}dw,
\end{aligned}$$

taking $t = uq$ and completing the square, then putting $v = u + m/2$ and combining the integrals, and finally letting $w = \sqrt{q}v$. Taking $q = 1$ we see that the integral here equals $1/(1-i)$, and therefore if q is prime and χ is the character of order 2 mod q then

$$g(\chi) := \sum_{a \pmod q} (1+\chi(a)) \exp\left(2i\pi \frac{a}{q}\right) = \sum_{n \pmod q} \exp\left(2i\pi \frac{n^2}{q}\right)$$

$$= \sqrt{q}\frac{(1+(-i)^q)}{1-i} = \sqrt{q} \cdot \begin{cases} 1 & \text{if } q \equiv 1 \pmod 4 \\ i & \text{if } q \equiv 3 \pmod 4 \end{cases} \tag{7.3.3}$$

Thus we have evaluated the sign of Gauss sum we discussed in section 3.5. Therefore if $q \equiv 3 \pmod 4$ then by (3.6.3) and (4.5.2) we obtain

$$\sum_{1 \le a < q/2} \left(\frac{a}{q}\right) = \frac{4 - 2\chi(2)}{w} h(-q) > 0; \tag{7.3.4}$$

which shows that there are more quadratic residues than non-residues $\pmod q$ between 0 and $\frac{q}{2}$, when q is a prime $\equiv 3 \pmod 4$. Moreover if $q \equiv 1 \pmod 4$ then by section 3.6 and (4.6.5) we obtain

$$\epsilon_q^{h(q)} = \prod_{a \pmod q} (1-\alpha^a)^{-\left(\frac{q}{a}\right)} \quad \text{where } \alpha := \exp\left(\frac{2i\pi}{q}\right) \tag{7.3.5}$$

We define a *finite Fourier transform* by

$$\hat{f}(k) = \frac{1}{q}\sum_{n=1}^{q} f(n)e^{-kn/q},$$

so that

$$f(n) = \sum_{k=1}^{q} \hat{f}(k)e^{kn/q}.$$

The analogy to Parseval's identity is

$$\sum_{n=1}^{q} |f(n)|^2 = q\sum_{k=1}^{q} |\hat{f}(k)|^2.$$

7.4. The order of a function. An analytic function $f(z)$ has finite order d if $\log|f(z)| \ll |z|^{d+o(1)}$, and d is the smallest such number. If f has no zeros we can write it in the form $e^{g(z)}$; where $g(z) = \sum_{n\ge 0} c_n z^n$. Hence

$$|c_n|R^n = \left|\int_0^1 g(Re^{2i\pi t})e^{-2i\pi nt}dt\right| \ll R^{d+o(1)},$$

so that $c_n = 0$ if $n > k$, and therefore g must be a polynomial of degree d, which must be an integer. Note that this argument works if we simply have an infinite sequence of values of R, getting arbitrarily large.

Mahler's measure gives a useful meaure of a polynomial: If $f(x) = a_0 \prod_{1\leq i\leq k}(x-\alpha_i)$ then $M(f) := |a_0| \prod_{1\leq i\leq k} \max\{1, |\alpha_i|\}$. *Jensen's formula* links this definition that depends on the algebraic part of f (that is, its roots), with the analytic part of f (its size):

$$\log M(f) = \int_0^1 \log |f(e^{2i\pi t})| \, dt. \tag{7.4.1}$$

This is easily proved by factoring f and proving this for each linear polynomial f. Rather more generally, if g is an analytic function of finite order with $g(0) \neq 0$ then

$$\exp\left(\int_0^1 \log |g(Re^{2i\pi t})| \, dt\right) = |g(0)| \prod_{\substack{z:\ |z|\leq R \\ g(z)=0}} \frac{R}{|z|}$$

where we count multiple zeros with their multiplicity. This can be proved by writing $g(z) = f(z)h(z)$ where f is a polynomial the same zeros as g, and h is analytic without zeros so we can write it as $e^{P(z)}$ where P is a polynomial, as above.

If one orders the zeros $z_1, z_2, \ldots$ of g to have absolute value $0 < |z_1| = r_1 \leq |z_2| = r_2 \leq \ldots$ and there are k zeros with $|z| \leq R$ then

$$\log\left(\prod_{\substack{z:\ |z|\leq R \\ g(z)=0}} \frac{R}{|z|}\right) = \sum_{i=1}^{k} \log\left(\frac{R}{r_i}\right) = \sum_{i=1}^{k-1} i \log\left(\frac{r_{i+1}}{r_i}\right) + k\log\left(\frac{R}{r_k}\right) = \int_0^R n(r)\frac{dr}{r}$$

where $n(r) := \#\{i :\ r_i \leq r\}$. Therefore

$$n(R) \leq n(R)\int_R^{eR} \frac{dr}{r} \leq \int_R^{eR} n(r)\frac{dr}{r} \leq \int_0^1 \log |g(Re^{2i\pi t})/g(0)| \, dt \ll R^{d+o(1)},$$

and hence

$$\sum_{\substack{z:\ |z|\leq R \\ g(z)=0}} \frac{1}{|z|^\alpha} = \alpha \int_0^\infty \frac{n(r)}{r^{1+\alpha}} dr \ll 1 \tag{7.4.2}$$

if $\alpha > d$. Let us now focus on the case $d = 1$, and consider the function

$$G(z) := \prod_{n\geq 1} (1 - z/z_n) e^{z/z_n}.$$

Taking logarithms we see that this is an analytic function that converges everywhere by (7.4.2). Since $n(R) \ll R^{1+o(1)}$ we can certainly find arbitrarily large values of R such that if $|z| = R$ then $|z_n - z| \geq 1/R^{o(1)}$ for all n. Hence $|1 - z/z_n| \geq 1/R^{1+o(1)}$ if $|z_n| \leq 2R$, implying that

$$-\log|G(z)| \ll \sum_{n:\ |z_n|\leq 2R} (\log R + R/|z_n|) + \sum_{n:\ |z_n|>2R} |R/z_n|^2$$
$$\ll n(2R)\log R + R\int_0^{2R} \frac{n(t)}{t^2}dt + R^2\int_{2R}^{\infty} \frac{n(t)}{t^3}dt \ll R^{1+o(1)}.$$

Therefore $g(z)/G(z)$ is an analytic function of order 1 with no zeros and so must be of the form e^{Az+B} for some constant A, B. Hence we may rewrite $g(z)$ as

$$g(z) = e^{Az+B}\prod_{n\geq 1}(1 - z/z_n)e^{z/z_n} \tag{7.4.3}$$

which is everywhere convergent. If $\sum_n 1/|z_n|$ converges then one can modify the above to show that $\log|g(z)| \ll |z|$.

7.5. Complex analysis. If one wants to make sense of calculus when one is working with functions of a complex variable then one needs to determine how to evaluate $\int_a^b f(z)dz$. If a and b are real numbers and z a real variable, then the Riemann-Stieltje's interpretation of the integral involves allowing the z variable to travel along a path from a to b, there being only one choice along the x-axis, though one can, if needs be, re-interpret the path as going from a to c, and then from c to b, whether or not c lies in-between a and b. In all cases this should give the same answer. When we work with functions of a complex variable, and $a, b \in \mathbb{C}$, we want this to again be true; that is that the value of $\int_a^b f(z)dz$ does not depend on the path taken to get from a to b; and, in the complex plane, there are an infinite number of possible choices of path. In fact if we wish to compare the value of the integral along two paths from a to b, we could append the paths into a circuit C by going from a to b along one path, and then going backwards on the other from b to a. We would then want that $\int_C f(z)dz = 0$. This is true for any closed circuit C when $f(z)$ can be written at each point in terms of a Taylor series (that is, f is *analytic*); and in fact it is true for C when f is analytic inside and on C. If f is analytic except for a finite number of points, say $z_1, \ldots, z_k$, called *poles*, for each of which there exists an integer $q_1, \ldots, q_k > 0$ such that $(z - z_j)^{q_j} f(z)$ can be written as a Taylor series at $z = z_k$, then we say that f is *meromorphic*. From what we have just written, to be able to interpret all integrals involving f we simply need to determine the value of the integral in a closed circuit around each z_j, that circuit not enclosing z_i for any $i \neq j$.

Suppose that $f(z)$ can be written as $\sum_{J\geq r} a_j(z - z_0)^j$ with $a_J \neq 0$, for some J (which equals $= -q_\ell$ when $z_0 = z_\ell$) in a circle C of radius r around c_0. Then, taking $z = z_0 + re^{2i\pi t}$,

$$\frac{1}{2i\pi}\oint_C f(z)\ dz = \sum_{j\geq J} a_j \cdot \frac{1}{2i\pi}\oint_C (z - z_0)^j dz = a_{-1} \tag{7.5.1}$$

by (3.1.1). Hence only the coefficient of $(z - z_0)^{-1}$ contributes to the integral around a closed circuit like this. If $a_{-1} \neq 0$ then we say that the pole at $z = z_0$ of $f(z)$ *contributes a* *residue of* a_{-1} *to the integral.* We assumed in (3.1.1) that t grows in the positive direction in the integral, that is we went anti-clockwise around C; had we gone clockwise the answer here would be $-a_{-1}$.

If we wish to determine $\frac{1}{2i\pi}\oint_C f(z)\, dz$ for a function f which is meromorphic inside C then we can break the region C up into a finite number of pieces, each containing at most one pole of f. If $C_1, C_2, \dots, C_r$ denote the boundaries of these pieces then $\frac{1}{2i\pi}\oint_C f(z)\, dz = \sum_{i=1}^r \frac{1}{2i\pi}\oint_{C_i} f(z)\, dz$, where all of the C_i are traversed in an anti-clockwise direction, so that the contribution of any internal boundary cancels, having been traversed once in each direction.

The birth of analytic number theory came in recognizing that one could use (7.5.1) to sum numbers a_{-1}, by summing up functions f, and then using the tools of calculus to determine the answer.

One example of such a tool is the *argument principle*. Suppose that $f(z)$ is a meromorphic function and we wish to count the number of zeros of f inside a closed circuit C. To simplify matters we carefully select our circuit to never run though a zero or pole of f. If $f = \sum_{j\geq J} a_j(z-z_0)^j$ with $a_J \neq 0$, for some J then $f'(z)/f(z) - J/(z-z_0)$ is analytic at z_0, the extra term $J/(z-z_0)$ only contributing if $J \neq 0$, that is if z_0 is either a pole of zero of f. Hence, by (7.5.1)

$$\frac{1}{2i\pi}\oint_C \frac{f'(z)}{f(z)}\, dz = \#\{\text{zeros of } f \text{ inside } C\} - \#\{\text{poles of } f \text{ inside } C\}, \tag{7.5.2}$$

where we count a zero of order $J > 0$, J times, and a pole of order $J < 0$, $|J|$ times. Now if $f(z) = e^{r(z)+2i\pi a(z)}$ on the boundary then $f'(z)/f(z) = r'(z) + 2i\pi a'(z)$. Comparing the imaginary parts of both sides of (7.5.2), we see that $\oint_C r'(z)dz = 0$, and so the left side of (7.5.2) equals $\oint_C a'(z)\, dz$, which is the change in $a(z)$ as we go round the circuit C. Therefore the number of zeros of f minus the number of poles inside C equals the change in argument of $f(z)$, divided by 2π, as we go around the circuit C.

7.6. Perron's formula and its variants. We begin with a way of identifying whether a real number t is positive or negative, using the discontinuous integral

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{su}\frac{ds}{s} = \begin{cases} 0 & \text{if } u<0 \\ 1/2 & \text{if } u=0 \\ 1 & \text{if } u>0 \end{cases} \tag{7.6.1}$$

which holds for any $c > 0$. (This can be viewed as the continuous version of (7.2.3).) It is proved using the ideas of complex analysis: The key idea is that the value of the integral in a closed contour C, is given by the contribution of the poles of e^{su}/s (as discussed in the previous section). Since the only pole lies at $s = 0$, the value of the integral equals 0 if $s = 0$ does not lie within C, and equals 1 if it does. Here we take the line from $c - i\infty$ to $c + i\infty$ to be part of the boundary, and the rest of the boundary going from $N + i\infty$ to $N - i\infty$ for some $N \neq c$.[2] For $u < 0$ we take N to be a very large positive real number

[2]If one is uncomfortable with the idea that all vertical lines end in the same place, then one can truncate the lines at $c \pm iT, N \pm iT$, as we will explain.

and we will bound the size of the integral: We employ the same ides that has served us well when studying L-functions, which is that the size of the integrand changes little while the argument goes through a full revolution. Thus writing $s = N + it$ we find that $e^{su}/s = e^{Nu}e^{itu}/(N + it)$. Allowing t to vary from $t_0 > 0$ to $t_0 + 2\pi/|u|$ we obtain:

$$\begin{aligned}\frac{1}{2i\pi}\int_{N+it_0}^{N+i(t_0+2\pi/|u|)} e^{su}\frac{ds}{s} &= \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi/|u|}\frac{e^{Nu}e^{itu}}{N+it}dt\\ &= \frac{e^{Nu}}{2\pi}\int_{t_0}^{t_0+2\pi/|u|}\left(\frac{e^{itu}}{N+it}-\frac{e^{itu}}{N+it_0}\right)dt,\end{aligned}$$

which is $\leq \pi e^{Nu}/(u^2(N^2 + t_0^2))$ in absolute value. Thus taking $t_0 = 2\pi m/|u|$ for $m = 0, 1, 2, \dots$ (and the conjugate argument for negative t), we obtain

$$\left|\frac{1}{2i\pi}\int_{N-i\infty}^{N+i\infty} e^{su}\frac{ds}{s}\right| \leq 2\sum_{m\leq N|u|/2\pi}\frac{e^{Nu}}{(uN)^2} + 2\sum_{m\geq N|u|/2\pi}\frac{e^{Nu}}{(2\pi m)^2} \leq \frac{2e^{Nu}}{\pi N|u|};$$

and this $\to 0$ as we allow N to get larger and larger. Thus we have proved the case with $u < 0$. The result with $u > 0$ is similar though now we move our contour to the left, that is letting $N \to -\infty$, and therefore the pole at $s = 0$ now contributes to the value of the integral. The case with $u = 0$ is easiest approached by adding the contributions of $c - it$ and $c + it$, and changing variable $t = cv$ to obtain $\frac{1}{\pi}\int_0^\infty \frac{1}{1+v^2}dv = \frac{1}{2}$ via the substitution $v = \tan\theta$.

Typically we use Perron's formula to recognize integers $n \leq x$ so we apply it with $u = \log(x/n)$ to obtain

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{n}\right)^s\frac{ds}{s} = \begin{cases} 0 & \text{if } n > x \\ 1/2 & \text{if } n = x \\ 1 & \text{if } n < x\end{cases} \tag{7.6.2}$$

Therefore, to estimate $\sum_{n\leq x}\phi(n)/n^2$ from section 2.14, we have, assuming x is not an integer so we can avoid the $n = x$ case of (7.6.2),

$$\sum_{n\leq x}\frac{\phi(n)}{n^2} = \sum_{n\geq 1}\frac{\phi(n)}{n^2}\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{n}\right)^s\frac{ds}{s} = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\sum_{n\geq 1}\frac{\phi(n)}{n^{2+s}}\,x^s\frac{ds}{s}$$

where we did not justify swapping the order of the summation and the integration though, as usual, it can easily be justified if everything is convergent enough, which in practice involves taking c large enough. Now the Dirichlet series $\sum_{n\geq 1}\phi(n)/n^{2+s}$ involves the sum of a multiplicative function, so is the product of the same sum taken over the powers of each prime p, which is $1 + (1 - \frac{1}{p})(\frac{1}{p^{1+s}} + \frac{1}{(p^2)^{1+s}} + \dots) = (1 - \frac{1}{p^{2+s}})/(1 - \frac{1}{p^{1+s}})$. Therefore our integral becomes

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(1+s)}{\zeta(2+s)}\,x^s\frac{ds}{s}.$$

To evaluate this we again shift the contour in the same way. The key thing to note is that $|x^s| = x^c$ so we shrink the size of the integrand by moving the contour to the left. When we do this the first pole that we meet lies at $s = 0$ and is in fact a pole of order two, since a pole is contributed by the $\zeta(1+s)$ term. Let's compute the residue at this point by working out the series for the integrand: $s\zeta(1+s) = 1 + \gamma s + O(s^2)$, $\zeta(2+s) = \zeta(2)+s\zeta'(2)+O(s^2)$, $x^s = 1+s\log x+O(s^2)$, so altogether we obtain that the coefficient of $1/s$ in our integrand, and thus our residue, is $(\log x+\gamma-\zeta'(2)/\zeta(2))/\zeta(2) = \frac{6}{\pi^2}\log x+O(1)$. At this point we expect that this is a good estimate for our original sum. The next pole of our integrand has $\mathrm{Re} s < -1$,[3] so if we move the contour to the vertical line at $\mathrm{Re}(s) = -1$ then we simply need to show that this integrand is small to achieve our original objective, and this can be done.

Typically it is more useful to describe Perron's formula in terms of an integral that goes to a large height T, rather than to ∞.

Proposition 7.6. *For $c, y > 0$ we have We have*

$$\left| \frac{1}{2i\pi} \int_{c-iT}^{c+iT} e^{su} \frac{ds}{s} - \begin{cases} 0 & \text{if } u < 0 \\ 1/2 & \text{if } u = 0 \\ 1 & \text{if } u > 0 \end{cases} \right| \ll \frac{e^{cu}}{1+T|u|}.$$

Proof. In the bounds we gave above for the integrand on the line $\mathrm{Re}(s) = N$, one can simply take $N = c$ and $m_0 = [|u|T/2\pi]$, so that our missing integral is no more twice than the sum of $e^{cu}/4\pi m^2$ over all $m \geq m_0$, which is $\leq e^{cu}/(T|u|)$. The missing integral when $u = 0$ equals $\frac{1}{\pi}\int_{T/c}^{\infty} \frac{1}{1+v^2} dv =\leq \frac{1}{\pi}\int_{T/c}^{\infty} \frac{1}{v^2} dv = \frac{c}{\pi T}$.

Suppose that $u < 0$. We will create a closed circuit C by going up the path from $c-iT$ to $c+iT$, and returning on most of a semi-circle, to the right of this line. Thus our integral is minus the integral on this arc. Since the arc is to the right of the line we have $|e^{su}| \leq e^{cu}$ throughout, and evidently $|s| = r$ where $r = c^2+T^2$. Therefore as the arc has length $\leq 2\pi r$ the integral is $\leq \frac{1}{2\pi} \cdot 2\pi r \cdot e^{cu}/r = e^{cu}$. An analogous proof works when $u > 0$ by going to the right.

Exercises

7.6a. Use Perron's formula to give an integral involving L-functions that gives a precise formula for the number of distinct integers that are the sum of two squares. Why is this difficult to estimate by the methods described above?

7.6b. Prove that $\sum_{n\leq x} \frac{\phi(n)}{n^2} = \frac{6}{\pi^2}\log x + O(1)$, by writing $\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$.

7.7. Analytic continuation. There is a beautiful phenomenon in the theory of functions of a complex variable called "analytic continuation." It tells us that functions that are originally defined only for certain complex numbers often have unique "sensible" definitions for other complex numbers. For example the definition of $\zeta(s)$, that

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}$$

[3]This follows from work in section 9.6 where we show that ζ is non zero on the 1-line.

only works when $\sigma = \mathrm{Re}(s) > 1$, since it is only is this domain that the sum is absolutely convergent. However "analytic continuation" will allow us to define $\zeta(s)$ for every complex number s other than $s = 1$. This description of analytic continuation looks disconcertingly magical. Fortunately, there is a quite explicit way to show how $\zeta(\sigma + it)$ can be "sensibly" defined at least for the larger region where $\sigma > 0$. We start with an expression for teh product $(1 - 2^{1-s})\zeta(s)$ and then perform some sleight-of-hand manipulations:

$$\begin{aligned}
\left(1 - 2^{1-s}\right)\zeta(s) &= \left(1 - \frac{2}{2^s}\right)\zeta(s) = \zeta(s) - \frac{2}{2^s}\zeta(s) \\
&= \sum_{n\geq 1} \frac{1}{n^s} - 2\sum_{m\geq 1}\frac{1}{(2m)^s} \\
&= \sum_{n\geq 1} \frac{1}{n^s} - 2\sum_{\substack{n\geq 1 \\ n \text{ even}}} \frac{1}{n^s} = \sum_{\substack{m\geq 1 \\ m \text{ odd}}} \frac{1}{m^s} - \sum_{\substack{n\geq 1 \\ n \text{ even}}} \frac{1}{n^s} \\
&= \left(\frac{1}{1^s} - \frac{1}{2^s}\right) + \left(\frac{1}{3^s} - \frac{1}{4^s}\right) + \left(\frac{1}{5^s} - \frac{1}{6^s}\right) + \cdots .
\end{aligned}$$

Solving for $\zeta(s)$, we find that

$$\zeta(s) = \frac{1}{(1 - 2^{1-s})}\left\{\left(\frac{1}{1^s} - \frac{1}{2^s}\right) + \left(\frac{1}{3^s} - \frac{1}{4^s}\right) + \left(\frac{1}{5^s} - \frac{1}{6^s}\right) + \cdots\right\}.$$

All of these manipulations were valid for complex numbers $s = \sigma + it$ with $\sigma > 1$. However, it turns out that the infinite series in curly brackets actually converges whenever $\sigma > 0$. Therefore, we can take this last equation as the new "sensible" definition of the Riemann zeta-function on this larger domain. Note that the special number $s = 1$ causes the otherwise innocuous factor of $1/(1 - 2^{1-s})$ to be undefined; the Riemann zeta-function intrinsically has a problem there, one that cannot be swept away with clever rearrangements of the infinite series.

7.9. The Gamma function. Euler defined

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt, \tag{7.9.1}$$

which converges whenever $\mathrm{Re}(s) > 0$. Writing $t = -n \log x$ we obtain

$$\Gamma(s) = n^s \int_0^1 x^{n-1} (\log x^{-1})^{s-1}, \tag{7.9.2}$$

for any $n > 0$; and taking $t = n\pi^2 x$ we get

$$\Gamma(\frac{s}{2}) = \pi^{\frac{s}{2}} n^s \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx. \tag{7.9.3}$$

The Gamma function is best known as a continuous extrapolation of the factorial function: First integrate (7.9.1) by parts to obtain $\Gamma(s+1) = s\Gamma(s)$ whenever $\text{Re}(s) > 0$. Noting that $\Gamma(1) = 1$ we deduce, by induction, that $\Gamma(n+1) = n!$ for all integers $n \geq 0$. By showing that $\Gamma(s)$ is an analytic function of order 1, Weierstrass used (7.4.3) to give the formula

$$\frac{1}{s\Gamma(s)} = e^{\gamma s} \prod_{n\geq 1} (1 + s/n)e^{-s/n} \tag{7.9.4}$$

which is valid for all $s \in \mathbb{C}$, and shows that $\Gamma(s)$ has no zeroes, and simple poles at $s = 0, -1, -2, \ldots$ and nowhere else. Show that $A = \gamma, B = 0$ in (7.4.3). Deduce that $\Gamma(s+1) = s\Gamma(s)$ for all $s \in \mathbb{C}$ by this formula.

Taking the logarithmic derivative we deduce from (7.9.4) that

$$-\frac{\Gamma'(s)}{\Gamma(s)} = \gamma + \frac{1}{s} + \sum_{n\geq 1} \left(\frac{1}{s+n} - \frac{1}{n} \right) \tag{7.9.5}$$

One can deduce from (7.9.4) that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \text{ and } \Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = 2^{1-2s}\pi^{1/2}\Gamma(2s). \tag{7.9.6}$$

Stirling's formula gives, provided $|\arg s| < \pi - \delta$ for fixed $\delta > 0$,

$$\Gamma(s) = \left(\frac{s}{e}\right)^s \sqrt{2\pi/s} \left(1 + O_\delta\left(\frac{1}{|s|}\right)\right). \tag{7.9.7}$$

Exercises

7.9a. Show that $-\Gamma'(1)/\Gamma(1) = \gamma$ and so deduce, from the definition of $\Gamma(s)$ that we have $\int_0^\infty e^{-t} \log t dt = \Gamma'(1) = -\gamma$.

7.9b. Using the Taylor series for $-\log(1-t)$ at $t = -1$, and (7.9.5), deduce that $-\Gamma'(1/2)/\Gamma(1/2) = \gamma + \log 4$.

7.9c. Prove that if $|\arg s| < \pi - \delta$ for fixed $\delta > 0$ then $\frac{\Gamma'(s)}{\Gamma(s)} = \log|s| + O_\delta(1)$. Why is the restricted range necessary?