11. THE PRIME NUMBER THEOREM FOR ARITHMETIC PROGRESSIONS

11.1. Representations of $L(s,\chi)$. Let χ be a character mod q. We have

$$\sum_{n\geq 1}\chi(n)x^n = \frac{1}{1-x^q}\ \sum_{n=1}^{q-1}\chi(n)x^n.$$

Using (7.9.2) we obtain

$$\begin{aligned}\Gamma(s)L(s,\chi) &= \sum_{n\geq 1}\chi(n)\Gamma(s)n^{-s} = \sum_{n\geq 1}\chi(n)\int_0^1 x^{n-1}(\log x^{-1})^{s-1}dx\\ &= \int_0^1 \frac{1}{1-x^q}\ \sum_{n=1}^{q-1}\chi(n)x^{n-1}(\log x^{-1})^{s-1}dx.\end{aligned}$$

This proves that $L(s,\chi)$ is analytic on the whole of $\mathbb{C}$.

Exercises

11.1a. If χ is a primitive, real, non-principal character mod q then the qth *Fekete polynomial* is defined as $f_q(t) := \sum_{n=0}^{q-1}\chi(n)t^n$. By noting that $(\log x^{-1})^{s-1}/x(1-x^q) > 0$ if $0 < x < 1$ and $s\in\mathbb{R}$, prove that if $f_q(t)$ has no zeros with $t\in[0,1]$ then neither does $L(s,\chi)$.

11.2. A functional equation for Dirichlet L-functions with $\chi(-1)=1$. Define $\omega(x,\chi) := \sum_{n\in\mathbb{Z}}\chi(n)e^{-\pi n^2x/q}$. By (3.5.1) and (9.2.1), and then taking $m = qn-a$,

$$\begin{aligned}g(\overline{\chi})\omega(x,\chi) &= \sum_{a=0}^{q-1}\overline{\chi}(a)\sum_{n\in\mathbb{Z}}e^{-\pi n^2x/q+2i\pi an/q} = \frac{1}{\sqrt{x/q}}\ \sum_{a=0}^{q-1}\overline{\chi}(a)\sum_{n\in\mathbb{Z}}e^{-\pi(qn-a)^2/qx}\\ &= \frac{1}{\sqrt{x/q}}\sum_{m\in\mathbb{Z}}\overline{\chi}(-m)e^{-\pi m^2/qx} = \sqrt{q/x}\ \omega(1/x,\overline{\chi}),\end{aligned}$$

as $\chi(-1)=1$. Therefore, by changing x to x/q in (7.9.3), and proceeding as in section 9.2,

$$\xi(s,\chi) := \left(\frac{q}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi) = \int_1^\infty \left\{x^{\frac{s}{2}}\omega(x,\chi) + \frac{\sqrt{q}}{g(\overline{\chi})}\ x^{\frac{1-s}{2}}\ \omega(x,\overline{\chi})\right\}\frac{dx}{x} \tag{11.2.1}$$

Thus $\xi(s,\chi)$ is analytic, and hence this provides the analytic continuation of $L(s,\chi)$ to the whole complex plane. Therefore if $\epsilon_\chi := \sqrt{q}/g(\overline{\chi})$ (so that $|\epsilon_\chi| = 1$ by section 3.5) we deduce that

$$\xi(1-s,\overline{\chi}) = \epsilon_\chi\xi(s,\chi). \tag{11.2.2}$$

For $\mathrm{Re}(s) < 0$ we know that $L(1-s,\overline{\chi}) \neq 0$ and so $\xi(1-s,\overline{\chi}) \neq 0$. Therefore, by (11.2.2), $L(s,\chi)$ has zeros in this region only at the (simple) poles of $\Gamma\left(\frac{s}{2}\right)$, that is at $s = 0, -2, -4, -6, \dots$.

11.3. A functional equation for Dirichlet L-functions with $\chi(-1) = -1$. Define $\omega(x,\chi) := \sum_{n\in\mathbb{Z}} \chi(n) n e^{-\pi n^2 x/q}$. By (3.5.1) and (9.2.2), taking $m = qn - a$ as in section 11.2, and using that $\chi(-1) = -1$ we obtain

$$g(\overline{\chi})\omega(x,\chi) = i\ \frac{q^{1/2}}{x^{3/2}}\ \omega(1/x,\overline{\chi}).$$

Therefore, by changing x to x/q and s to $s+1$ in (7.9.3), and proceeding as above,
(11.3.1)

$$\xi(s,\chi) := \left(\frac{q}{\pi}\right)^{\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) = \int_1^\infty \left\{ x^{\frac{s}{2}} \omega(x,\chi) + \epsilon_\chi x^{\frac{1-s}{2}}\ \omega(x,\overline{\chi}) \right\} \frac{dx}{x^{1/2}}$$

where $\epsilon_\chi := i\sqrt{q}/g(\overline{\chi})$. Therefore (11.2.2) holds and the only zeros of $L(s,\overline{\chi})$ in the region $\mathrm{Re}(s) < 0$ are simple zeros at $s = -1, -3, -5, \dots$.

11.4. Properties of $\xi(s,\chi)$. One can develop a theory for $\xi(s,\chi)$ that is very similar to that developed for $\xi(s)$ in section 9.5. Indeed we obtain a product exactly as in (9.5.1) for $\xi(s,\chi)$, and a representation in (9.5.2). The analogy to (9.5.3) is

$$\frac{L'(s,\chi)}{L(s,\chi)} = A'_\chi + \sum_{\rho:\ L(\rho,\chi)=0} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right), \tag{11.4.1}$$

where if $\chi(-1) = 1$ we have $L(0,\chi) = 0$ so we denote this term in the sum as $1/s$, and $A'_\chi = A_\chi + \frac{\gamma}{2} - \frac{1}{2}\log(\frac{q}{\pi})$, plus $\log 2$ if $\chi(-1) = -1$. When trying to imitate the proof of (9.5.4) we find that we have, from (11.2.2), if $\xi(\rho,\chi) = 0$ then $\xi(1-\rho,\overline{\chi}) = 0$ and so $\xi(1-\overline{\rho},\chi) = 0$. One can then deduce that

$$\mathrm{Re}(A_\chi) = -\sum_{\xi(\rho,\chi)=0} \mathrm{Re}\left(\frac{1}{\rho}\right),$$

and again note that this gives that $\mathrm{Re}(A_\chi) < 0$. We will see in exercise 11.4a that the value of A_χ depends heavily on the value of $L'(1,\overline{\chi})/L(1,\overline{\chi})$ which may be large if there is a zero of $L(s,\chi)$ with s close to 1. Inserting this last equation into (11.4.1), and proceeding as we did in (9.6.3), yields

$$-\mathrm{Re}\left(\frac{L'(s,\chi)}{L(s,\chi)}\right) = \log|s| + \frac{1}{2}\log q + O(1) - \sum_{\substack{\rho:\ L(\rho,\chi)=0 \\ 0\le \mathrm{Re}(\rho)\le 1}} \mathrm{Re}\left(\frac{1}{s-\rho}\right). \tag{11.4.2}$$

As in section 9.6, if we take $s = \sigma + it$ with $\sigma > 1$ then each $\mathrm{Re}\left(\frac{1}{s-\rho}\right) > 0$ and so

$$-\mathrm{Re}\left(\frac{L'(s,\chi)}{L(s,\chi)}\right) < \log(|t|+2) + \frac{1}{2}\log q + O(1). \tag{11.4.3}$$

This is true for primitive χ and can be extended to imprimitive but non-principal χ at a cost of an extra $O(\log\log q)$ on the right side of (11.4.3)– see exercise 11.4b.

Exercises

11.4a.a) Show that $e^{B_\chi} = \epsilon_{\overline{\chi}} q^{1/2} L(1,\overline{\chi})$ if $\chi(-1)=1$, and $= \epsilon_{\overline{\chi}}(q/\pi)L(1,\overline{\chi})$ if $\chi(-1)=-1$.

b) So $A_\chi = \xi'(0,\chi)/\xi(0,\chi) = -\xi'(1,\overline{\chi})/\xi(1,\overline{\chi}) = -\frac{1}{2}\log(q/\pi) + \frac{\gamma}{2} - L'(1,\overline{\chi})/L(1,\overline{\chi})$, plus $\log 2$ if $\chi(-1) = -1$.

11.4b. Show that if $\chi \pmod q$ is induced by $\psi \neq 1$ and $\mathrm{Re}(s)\geq 1$ then $\left|\frac{L'(s,\chi)}{L(s,\chi)} - \frac{L'(s,\psi)}{L(s,\psi)}\right| \leq \sum_{p|q}\frac{\log p}{p-1} \ll \log\log q$.

11.5. Zero-free regions for $L(s,\chi)$. The arguments of sections 9.6 to 9.8 are easily generalized to $L(s,\chi)$ for a fixed character χ. However in many applications one wants results in which the zero-free regions depend on the modulus q of χ:

Theorem 11.5. *There exists a constant $c>0$ such that if χ is a non-principal character mod q then there is at most one zero $\beta+i\gamma$ of $L(s,\chi)$ inside the region*

$$\sigma \geq 1 - \frac{c}{\log(q(|\gamma|+2))}.$$

If such a zero exists that it is simple and real (that is, $\gamma=0$), and χ is a real character of order 2.

Proof. As χ is non-principal character so $q\geq 3$. The analogy to (9.6.4) is

$$0 \leq -3\ \mathrm{Re}\left(\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)}\right) - 4\ \mathrm{Re}\left(\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)}\right) - \ \mathrm{Re}\left(\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)}\right).$$

Assume that $L(\beta+i\gamma,\chi)=0$ and take $t=\gamma$. Let $\mathcal{L} := 5\log(|t|+2)+3\log q$ and $\sigma = 1+1/2\mathcal{L}$. If $\chi^2\neq\chi_0$ then by using (9.6.3) and proceeding as in section 9.6, though now using (11.4.2) and (11.4.3), we obtain

$$\frac{4}{\sigma-\beta} \leq \frac{3}{\sigma-1} + \mathcal{L} + O(1).$$

Therefore $\beta \leq 1 - 1/(14\mathcal{L}) + O(1/\mathcal{L}^2)$. If $\chi^2=\chi_0$ we can make use the estimates of section 9.6 for the third term, introducing an additional $\mathrm{Re}(\frac{1}{\sigma-1+2i\gamma}) = \frac{\sigma-1}{(\sigma-1)^2+4\gamma^2}$. If $|\gamma|\geq\sigma-1$ then this additional term is $\leq 1/5(\sigma-1)$, so the above becomes

$$\frac{4}{\sigma-\beta} \leq \frac{16}{5(\sigma-1)} + \mathcal{L} + O(1),$$

and we obtain $\beta \leq 1 - 3/(74\mathcal{L}) + O(1/\mathcal{L}^2))$.

Now suppose that χ is real and $0<|\gamma|<1/2\mathcal{L}$, so that we also have the zero $\beta - i\gamma$. Let $\sigma = 1+1/4\mathcal{L}$ so that $|\gamma| < |\sigma-\beta|/2$. Then (11.4.2) yields for $s=\sigma$

$$\frac{8}{5(\sigma-\beta)} < \frac{2(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2} = \mathrm{Re}\left(\frac{1}{\sigma-\beta-i\gamma} + \frac{1}{\sigma-\beta+i\gamma}\right)$$
$$< \left|\frac{L'(\sigma,\chi)}{L(\sigma,\chi)}\right| + \frac{1}{2}\log q + O(1) \leq \frac{\zeta'(\sigma,\chi)}{\zeta(\sigma,\chi)} + \frac{\mathcal{L}}{6} + O(1) \leq \frac{1}{\sigma-1} + \frac{\mathcal{L}}{6} + O(1),$$

and therefore $\beta \leq 1 - 67/(500\mathcal{L}) + O(1/\mathcal{L}^2)$.

Finally suppose that χ is real and $\gamma = 0$, and we have two zeros $\beta < \beta'$. We can immediately replace $8/5(\sigma - \beta)$ by $2/(\sigma - \beta)$ in the inequality obtained in the previous case, and so at least as good a result follows.

Exercises

11.5a. Show that the proof of Theorem 11.5 yields constant $c = 1/124$ provided $q + |\gamma|$ is sufficiently large.

11.6. Approximations to $L'(s,\chi)/L(s,\chi)$. Following the method of section 9.7 we deduce from (11.4.2) that

$$\sum_{\substack{\rho:\ L(\beta+i\gamma,\chi)=0\\ 0\leq\beta\leq 1}} \frac{1}{4+(t-\gamma)^2} \leq \log(|t|+2) + \frac{1}{2}\log q + O(1). \tag{11.6.1}$$

We deduce that there are $\leq 8(|t|+2) + 4\log q + O(1)$ zeros $\beta + i\gamma$ for which $|t - \gamma| \leq 2$. Therefore for $s = \sigma + it$ with σ bounded, we have

$$\left| \frac{L'(s,\chi)}{L(s,\chi)} - \sum_{\substack{\rho:\ L(\rho,\chi)=0\\ |t-\gamma|\leq 2}} \frac{1}{s-\rho} \right| \leq (6-\sigma)(2\log(|t|+2) + \log q) + O(1) \tag{11.6.2}$$

11.7. On the number of zeros of $L(s,\chi)$. Suppose that χ is a primitive character mod q, and let $N(T,\chi)$ denote that number of zeros of $L(s,\chi)$ inside $C := \{s :\ 0 \leq \mathrm{Re}(s) \leq 1,\ -T \leq \mathrm{Im}(s) \leq T\}$.[1] Our approach will be more-or-less that of section 9.8, with a few minor differences: We use the contour with corners at $\frac{5}{2} - iT, \frac{5}{2} + iT, -\frac{3}{2} + iT, -\frac{3}{2} - iT$ so as to avoid possible zeros at -1 or -2; we now have one trivial zero inside the contour (at 0 or -1). The contribution of the left half of the contour is the same as the right half since $\arg(\xi(1-\overline{s},\chi)) = -\arg(\epsilon_\chi) - \arg(\xi(s,\chi))$ by (11.2.2), and thus the change in each half, as one goes round the contour, is the same. Next, applying Stirling's formula, etc., as in section 9.8, we obtain for any $T > 0$

$$\frac{1}{2}\ N(T,\chi) = \frac{T}{2\pi}\log\left(\frac{qT}{2\pi e}\right) - \frac{\chi(-1)}{8} + S(T,\chi) + O\left(\frac{1}{T+1}\right), \tag{11.7.1}$$

where $S(T,\chi) := \frac{1}{2\pi}(\arg L(\frac{1}{2}+iT,\chi) - \arg L(\frac{1}{2}-iT,\chi))$. Now, using (11.6.2) we see that $S(T,\chi) \ll \log qT$ (as in the proof of (9.8.3)), and therefore

$$\frac{1}{2}\ N(T,\chi) = \frac{T}{2\pi}\log\left(\frac{qT}{2\pi e}\right) + O(\log qT). \tag{11.7.2}$$

[1] We cannot just restrict to $0 \leq \mathrm{Im}(s) \leq T$ since there is no reason now for zeros to be symmetric in the real axis.

11.8. The explicit formula. We will obtain an explicit formula for

$$\Psi(x,\chi) := \sum_{\substack{p \text{ prime, } m\geq 1\\ p^m\leq x}} \chi(p^m)\log p, \tag{11.8.1}$$

for each Dirichlet character $\chi \pmod q$, since then we have

$$\Psi(x;q,a) := \sum_{\substack{p \text{ prime, } m\geq 1\\ p^m\leq x\\ p^m\equiv a \pmod q}} \log p = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a)\Psi(x,\chi). \tag{11.8.2}$$

We proceed as in section 10.1 but now with L'/L in place of ζ'/ζ. In analogy to (10.1.2) we have the explicit formula when χ is primitive and $q\geq 2$, if x is not a prime power and $x\geq 2$ then

$$\Psi(x,\chi) = -\sum_{\substack{\rho:\ L(\rho,\chi)=0\\ 0<\mathrm{Re}(\rho)<1}} \frac{x^\rho}{\rho} - c_\chi - \frac{1}{2}\log(x-1) - \frac{\chi(-1)}{2}\log(x+1), \tag{11.8.3}$$

for an appropriate constant c_χ. The key differences in the calculations are that the x term disappears because there is no pole at $s=0$; if $\chi(-1)=-1$ then the trivial zeros appear at the negative odd integers; and if $\chi(-1)=1$ there is now a double zero at $s=0$, leaving a residue of $\log x$.

We again seek to replace this by a sum over zeros up to a given height T, using Proposition 7.6. The analogous proof works since $|L'(\sigma+iT,\chi)/L(\sigma+iT,\chi)| \ll (\log qT)^2$ if $-1\leq\sigma\leq 2$ by (11.6.1) and (11.6.2), and is $\ll 1/|s+n|+\log q|s|$ if $\sigma\leq -1$, and $-n$ is the nearest trivial zero of $L(\rho,\chi)$ to s, using the functional equation (11.2.2). We deduce that

$$\Psi(x,\chi) = -\sum_{\substack{\rho:\ L(\rho,\chi)=0\\ 0<\mathrm{Re}(\rho)<1\\ -T<\mathrm{Im}(\rho)<T}} \frac{x^\rho}{\rho} - c_\chi + O\left(\frac{x(\log qT)(\log x)}{T} + \log x\right). \tag{11.8.3}$$

This formula is surprisingly useless, as it stands, since we need to determine a bound on c_χ before we can know how large x must be before this gives a good estimate. Working through the earliest steps of the argument we find that

$$c_\chi = O(1) - \sum_\rho \left(\frac{1}{\rho} + \frac{1}{2-\rho}\right).$$

If $\rho=\sigma+it$ then this term in the sum is $\ll 1/t^2$, so that the sum over ρ with $|t|\geq 1$ is $\ll \log q$ by (11.7.2). Since each $|2-\rho|\geq 1$ the sum of these terms with $|t|\leq 1$ is $\ll \log q$ by the sentence after (11.6.1); similarly we get a bound $\ll (\log q)^2$ for the terms with $1\geq|\rho|\gg 1/\log q$. By Theorem 11.5 and the symmetry of zeros because of the functional equation, we know that there is at most one zero that has not been included in such considerations, and if such a zero exists then χ is real, and the zero is real and simple. We will denote it by $1-\beta$ so that β is also a zero with $\beta \geq 1 - c/\log q$. Therefore our formula becomes for $x \geq q$ and $x \geq T$,

$$\Psi(x,\chi) = -\sum_{\rho \in \mathcal{B}'(T)} \frac{x^\rho}{\rho} - \frac{x^\beta}{\beta} - \frac{x^{1-\beta}-1}{1-\beta} + O\left(\frac{x(\log qT)(\log x)}{T}\right). \tag{11.8.4}$$

where the terms involving β are there only if there exists some $\beta \in \mathcal{B}(T) \setminus \mathcal{B}'(T)$, where

$$\mathcal{B}'(T) := \left\{\rho = \sigma + i\gamma :\ \zeta(\rho) = 0,\ \frac{c}{\log qT} \leq \sigma \leq 1 - \frac{c}{\log qT},\ -T \leq \gamma \leq T\right\},$$

and c is as in Theorem 11.5. We show in exercise 11.8a that the $\frac{x^{1-\beta}-1}{1-\beta}$ term may be omitted in (11.8.4) when $x \geq q, e^{2c}T$ to obtain

$$\Psi(x,\chi) = -\sum_{\rho \in \mathcal{B}'(T)} \frac{x^\rho}{\rho} - \frac{x^\beta}{\beta} + O\left(\frac{x(\log qT)(\log x)}{T}\right). \tag{11.8.5}$$

Exercises

11.8a. Show that if $x \geq e^{2c}T$ then $Tx^{c/\log T} \leq x$ and deduce that the $\frac{x^{1-\beta}-1}{1-\beta}$ term in (11.8.4) may be subsumed into the error term.

11.8b. Deduce that (11.8.5) holds for imprimitive characters as well as primitive characters.

11.9. The prime number theorem for arithmetic progressions. As in the proof of the prime number theorem, we simply bound the sum over zeros in (11.8.4). Each $|x^\rho| \leq x^{1-c/\log qT}$, and for the sum $\sum_{\rho \in \mathcal{B}'(T)} 1/|\rho|$, we have $\ll (\log q)^2$ for the ρ with $|\rho| \leq 1$, as in the previous section, and $\ll (\log qT)(\log T)$ for the ρ with $|\rho| > 1$ since there are $\ll t \log qt$ zeros up to height $t \geq 1$. Inserting this all into (11.8.5) we obtain

$$\left|\Psi(x,\chi) + \frac{x^\beta}{\beta}\right| \ll x^{1-c/\log qT}(\log qT)^2 + \frac{x(\log qT)(\log x)}{T}.$$

Selecting $q \leq T := e^{\sqrt{(c/2)\log x}}$ and inserting this last estimate into (11.8.2), we obtain that whenever $(a,q) = 1$ we have

$$\Psi(x;q,a) = \frac{x}{\phi(q)} - \frac{\overline{\chi}(a)}{\phi(q)}\,\frac{x^\beta}{\beta} + O(xe^{-\sqrt{(c/3)\log x}}), \tag{11.9.1}$$

where $L(\beta,\chi) = 0$ as in Theorem 11.5. The main difficulty in the theory of the distribution of primes comes from the possible existence of the zero β. Of course we believe that β never exists but we are unable to rule out that possibility. We shall explore such putative β in detail in the next chapter.

Let us suppose that we do have such a zero β with $\beta > 1 - c/\log q$. By (slightly modifying) exercise 4.7b we deduce that

$$L(1,\chi) = L(1,\chi) - L(\beta,\chi) = \int_\beta^1 L'(\sigma,\chi)d\sigma \le C(1-\beta)(\log q)^2$$

for some constant $C > 0$, so that

$$\beta \le 1 - \frac{L(1,\chi)}{C(\log q)^2} \le 1 - \frac{c}{\sqrt{q}(\log q)^2} \tag{11.9.2}$$

by (4.7.1). Inserting this bound into (11.9.1) we deduce that if $q \le (\log x)^2/(\log\log x)^7$ then

$$\Psi(x;q,a) = \frac{x}{\phi(q)}\left(1 + O\left(\frac{1}{(\log x)^A}\right)\right), \tag{11.9.2}$$

for any fixed $A > 0$. From the proof one can give values for all of the implicit constants.

The *Generalized Riemann Hypothesis* states that all zeros ρ of $L(\rho,\chi) = 0$ in the critical strip satisfy $\text{Re}(\rho) = \frac{1}{2}$. This implies that, in (11.8.5), we have

$$|\Psi(x,\chi)| \ll x^{1/2}(\log qT)^2 + \frac{x(\log qT)(\log x)}{T} \ll x^{1/2}(\log qx)^2$$

selecting $T = x^{1/2}$, so that

$$\Psi(x;q,a) = \frac{x}{\phi(q)} + O(x^{1/2}(\log qx)^2), \tag{11.9.3}$$

Therefore the Generalized Riemann Hypothesis implies that

$$\pi(x;q,a) \sim \pi(x)/\phi(q), \tag{11.9.4}$$

holds only once x is a little bigger than $q^2 \log^5 q$, and with a more precise argument we can improve this but not to better than "q^2". However calculations reveal that (11.9.4) seems to hold when x is just a little bigger than q (rather than q^2) and so even the Riemann Hypothesis, and its generalizations are inadequate for giving us a precise behaviour of the distribution of primes. This difference, between q and q^2 is enormous, and one finds that this same question, with a constant "2", appears all over the subject.

11.10. A FINAL REMARK. If the exceptional zero β, of the previous section, exists, suppose that it is a zero of $L(s,\chi)$ for a primitive character χ (mod m). If m does not divide q then the x^β/β term does not appear in (11.9.1). Therefore we can deduce

Theorem 11.10. *Select c as in Theorem 11.5. There exist constants $c_1, c_2 > 0$ such that for any given x we have*

$$\Psi(x;q,a) = \frac{x}{\phi(q)} + O(xe^{-c_1\sqrt{\log x}}),$$

for all $(a,q) = 1$ and all $q \le e^{c_2\sqrt{\log x}}$, except perhaps those q that are multiples of an exceptional modulus m. This modulus m exists only if there is a primitive character χ (mod m) and a real, simple zero β of $L(s,\chi) = 0$, for which $\beta > 1 - c/\sqrt{\log x}$.