10. THE EXPLICIT FORMULA AND THE PROOF OF THE PRIME NUMBER THEOREM.

10.1. THE EXPLICIT FORMULA. We saw in section 2.11 that it is okay to work with

$$\Psi(x) := \sum_{\substack{p \text{ prime, } m\geq 1 \\ p^m \leq x}} \log p.$$

The reason that we prefer this to $\pi(x)$ is that when we use Perron's formula (7.6.2), we obtain (provided x is not a prime power), the rather elegant formula

$$\begin{aligned} \Psi(x) &= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \sum_{p \text{ prime, } m\geq 1} \frac{\log p}{p^{ms}} \, \frac{x^s}{s} \, ds \\ &= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, ds \end{aligned} \tag{10.1.1}$$

provided $\mathrm{Re}(s) > 1$ so that we can justify swapping the order of summation and integration. We now move the contour away to the left, so to $\mathrm{Re}(s) = -N$ for some large odd integer N. What are the poles of the integrand? For each zero and pole of $\zeta(s)$ with $-N < \mathrm{Re}(s) < c$ we have a pole of order 1, as well as a pole at $s = 0$. So the contribution of the pole at $s = 0$ is $-\zeta'(0)/\zeta(0)$ and of the pole of $\zeta(s)$ at $s = 1$ is x. If ρ is a zero of $\zeta(s)$ of order m then its contribution is $-mx^\rho/\rho$. Hence we obtain

$$\Psi(x) = x - \sum_{\substack{\rho:\ \zeta(\rho)=0 \\ \mathrm{Re}(\rho) > -N}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2i\pi} \int_{-N-i\infty}^{-N+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, ds$$

where we count a zero ρ of $\zeta(s)$ of multiplicity m, m times in the sum. Now, the integrand can be proved to go to 0 as $N \to \infty$. Moreover the zero $\rho = -2m$ contributes $1/(2mx^{2m})$, which gives a total of $-\frac{1}{2}\log(1 - \frac{1}{x^2})$ when we sum up from 1 to ∞. Hence we have the *explicit formula*

$$\boxed{\Psi(x) = x - \sum_{\substack{\rho:\ \zeta(\rho)=0 \\ 0 < \mathrm{Re}(\rho) < 1}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log\left(1 - \frac{1}{x^2}\right).} \tag{10.1.2}$$

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This amazing exact formula appears to give the number of primes up to x, a discontinuous function, as a sum of continuous functions; this is possible, as in our discussion of Fourier analysis in sections 7.2 and 7.3, when the sum is infinite. (note: Discuss this in section 8). However, as beautiful as the formula is, it has the drawback that it involves infinitely many terms and it is far from clear how to manipulate such a sum. It is more convenient for us to truncate the sum at a height T, that is to consider only those zeros $\beta + i\gamma$ inside $C = \{s : \ 0 \leq \mathrm{Re}(s) \leq 1, \ -T \leq \mathrm{Im}(s) \leq T\}$, and we have a spectacularly accurate estimate for the number of zeros inside this box thanks to (9.8.2). To do so we must use Proposition 7.6 rather than (7.6.2). Proceeding as above we obtain that

$$\Psi(x) - \frac{1}{2i\pi}\int_{c-iT}^{c+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\, ds \ll \sum_{p \text{ prime},\ m\geq 1} \frac{\log p}{1+T|\log(x/p^m)|}\left(\frac{x}{p^m}\right)^c .$$

If $|\log(x/p^m)| \leq 1$ then let d be the closest integer to $x - p^m$, so that $|\log(x/p^m)| = |\log(1+(p^m-x)/x)| \gg |d|/x$. Therefore the right side here is, taking $c = 1 + 1/\log x$,

$$\begin{aligned} &\ll \sum_{p \text{ prime},\ m\geq 1} \frac{\log p}{1+T}\left(\frac{x}{p^m}\right)^c + \sum_{0\leq |d| \ll x} \frac{\log x}{1+T|d|/x} \\ &\ll \frac{x^c}{T}\left(-\frac{\zeta'(c)}{\zeta(c)}\right) + \frac{x\log x\log T}{T} + \log x \ll \frac{x\log x\log T}{T} + \log x. \end{aligned}$$

Next we wish to evaluate the integral, and idea will be to make a closed contour, namely the rectangle with vertices at $c-iT, c+iT, -N+iT, -N-iT$, and the integral around this contour is the sum of the contributions of the residues inside the contour, namely

$$x - \sum_{\substack{\rho:\ \zeta(\rho)=0 \\ 0<\mathrm{Re}(\rho)<1 \\ |\mathrm{Im}(\rho)|<T}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \sum_{\substack{1\leq m\leq N \\ m \text{ even}}} \frac{1}{mx^m}.$$

We shall show that the integral around the three "new" sides of the rectangle is small: By using (9.7.5) we obtain, since $|x^s| = x^{-N}$,

$$\frac{1}{2i\pi}\int_{-N-iT}^{-N+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\, ds \ll \int_{-T}^{T} \log(N+|t|)\frac{x^{-N}}{N+|t|}\, dt \ll \frac{\log^2 T + \log N}{x^N}.$$

By exercise 9.8b there exists a value $t \in [T, T+1]$ which is at a distance $\gg 1/\log T$ from the nearest zero of $\zeta(s)$; we change the value of T to this value t, so that we can apply the estimate (9.7.4) for the integrals from $c \pm iT$ to $U \pm iT$, to obtain

$$\frac{1}{2i\pi}\left\{\int_{c+iT}^{-N+iT} + \int_{-N-iT}^{c-iT}\right\}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\, ds \ll \frac{(\log T)^2}{T}\int_{-N}^{c} x^\sigma\, d\sigma \ll \frac{x(\log T)^2}{T\log x}.$$

Combining these estimates and letting $N \to \infty$ we obtain

$$\Psi(x) = x - \sum_{\substack{\rho:\ \zeta(\rho)=0 \\ 0<\mathrm{Re}(\rho)<1 \\ |\mathrm{Im}(\rho)|<T}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) + O\left(\frac{x\log x \log T}{T} + \log x\right) \tag{10.1.3}$$

In our proof we replaced T by $t \in [T, T+1]$; now we try to change it back: When we do so the key difference comes in the sum. In section 9.7 we saw that there are $\ll \log T$ zeros with $T \le |\gamma| < t$ and each of these contributes $\ll |x^\rho/\rho| \le x/T$ to the sum. Thus the change in (10.1.3) when we revert back from t to T is $\ll (x\log T)/T$ which is smaller than the given error term. Therefore (10.1.3) holds for all $T \ge 2$.

We proved (10.1.3) under the assumption that x is not a prime power. In this case Perron's formula provides a weight of $\frac{1}{2}$ to $\log x$ which is smaller than the given error term. Therefore

$$\Psi(x) = x - \sum_{\substack{\rho:\ \zeta(\rho)=0 \\ 0<\mathrm{Re}(\rho)<1 \\ |\mathrm{Im}(\rho)|<T}} \frac{x^\rho}{\rho} + O\left(\frac{x\log x \log T}{T} + \log x\right) \tag{10.1.4}$$

holds for all $x, T \ge 2$.

10.2. Proving the prime number theorem. Our objective, as we saw in section 2, is to prove that $\psi(x) \sim x$. Therefore we want to pick T to be somewhat bigger than $(\log x)^2$ so that the error term in (10.1.4) is $o(x)$. This leaves us with sum over the zeros of the Riemann zeta-function and this is

$$\sum_{\substack{\rho:\ \zeta(\rho)=0 \\ 0<\mathrm{Re}(\rho)<1 \\ |\mathrm{Im}(\rho)|<T}} \frac{x^\rho}{\rho} \le \sum_{\substack{\rho:\ \zeta(\rho)=0 \\ 0<\mathrm{Re}(\rho)<1 \\ |\mathrm{Im}(\rho)|<T}} \frac{x^{\mathrm{Re}(\rho)}}{|\rho|} \le x^{1-1/(71\log T)}\left(\frac{(\log T)^2}{2\pi} + O(1)\right)$$

by (9.6.6) for T sufficiently large, and (9.8.5). Now, selecting $T = \exp(\frac{1}{9}(\log x)^{1/2})$ and inserting the last estimate into (10.1.4) we obtain

$$\Psi(x) = x + O\left(xe^{-\frac{1}{10}(\log x)^{1/2}}\right). \tag{10.2.1}$$

By partial summation one can deduce that

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-\frac{1}{10}(\log x)^{1/2}}\right). \tag{10.2.2}$$

10.3. Assuming the Riemann Hypothesis. If we now assume that $\mathrm{Re}(\rho) \le \lambda$ for some fixed $\lambda \in [\frac{1}{2}, 1)$ then the sum in (10.1.4) is $\ll x^\lambda(\log T)^2$. Taking $T = x$ we deduce that

$$\Psi(x) = x + O\left(x^\lambda(\log x)^2\right) \text{ and } \pi(x) = \mathrm{Li}(x) + O\left(x^\lambda \log x\right). \tag{10.3.1}$$

We can take λ if the Riemann Hypothesis is true.

We can prove a converse result. If $\mathrm{Re}(s) > 1$ then

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{\substack{p \text{ prime, } m\geq 1 \\ p^m \leq x}} \frac{\log p}{p^{ms}} = \frac{s}{s-1} + s\int_1^\infty \frac{(\Psi(x) - x)}{x^{s+1}}\, dx. \tag{10.3.2}$$

If (10.3.1) holds then the right side of (10.3.2) converges for $\mathrm{Re}(s) > \lambda$. This implies that $\zeta(s)$ has no zeros with $\mathrm{Re}(s) > \lambda$ else the left side would diverge at that point. Therefore $\Psi(x) = x + O(x^{1/2}(\log x)^2)$ if and only if the Riemann Hypothesis is true.