27. *(3 points.)* We have

$$\frac{n}{\varphi(n)} = \prod_{p^{\nu_p}\|n} \frac{p^{\nu_p}}{\varphi(p^{\nu_p})} = \prod_{p^{\nu_p}\|n} \frac{p^{\nu_p}}{p^{\nu_p} - p^{\nu_p-1}} = \prod_{p^{\nu_p}\|n} \frac{p}{p-1} = \prod_{p|n}\left(1 + \frac{1}{p-1}\right),$$

and

$$\sum_{d|n} \frac{\mu^2(d)}{\varphi(d)} = \prod_{p^{\nu_p}\|n} \sum_{d|p^{\nu_p}} \frac{\mu^2(d)}{\varphi(d)} = \prod_{p^{\nu_p}\|n}\left(1 + \frac{1}{\varphi(p)}\right) = \prod_{p|n}\left(1 + \frac{1}{p-1}\right).$$

28. *(2 points.)*For any prime power p^ν, we have $\sum_{d|p^\nu} \mu(d)g(p^\nu/d) = g(p^\nu) - g(p^{\nu-1})$. Hence

$$\sum_{d|n} \mu(d)g(p^\nu/d) = \prod_{p^{\nu_p}\|n} \sum_{d|p^{\nu_p}} \mu(d)g(p^\nu/d) = \prod_{p^{\nu_p}\|n} (g(p^{\nu_p}) - g(p^{\nu_p-1})).$$

36. *(2 points.)* We have

$$\sum_{d|n} \mu(d)\varphi(d) = \prod_{p^{\nu_p}\|n} \sum_{d|p^{\nu_p}} \mu(d)\varphi(d) = \prod_{p^{\nu_p}\|n} (1 - \varphi(p)) = \prod_{p^{\nu_p}\|n} (1 - (p-1)) = \prod_{p|n} (2 - p),$$

and this is 0 if 2 divides n.

37. *(4 points.)* For any prime power p^ν,

$$\sum_{d|p^\nu} \sigma(d) = \sigma(1) + \sigma(p) + \cdots + \sigma(p^\nu) = 1 + (1+p) + \cdots + (1 + p + \cdots + p^\nu) = (\nu+1) + p\nu + \cdots + p^\nu,$$

and

$$p^\nu \sum_{d|p^\nu} \frac{\tau(d)}{d} = p^\nu\left(\frac{\tau(1)}{1} + \frac{\tau(p)}{p} + \cdots + \frac{\tau(p^\nu)}{p^\nu}\right) = p^\nu\left(1 + \frac{2}{p} + \cdots + \frac{\nu+1}{p^\nu}\right) = p^\nu + 2p^{\nu-1} + \cdots + (\nu+1) = \sum_{d|p^\nu} \sigma(d).$$

Hence

$$\sum_{d|n} \sigma(d) = \prod_{p\|p^{\nu_p}} \sum_{d|p^{\nu_p}} \sigma(d) = \prod_{p\|p^{\nu_p}} p^{\nu_p} \sum_{d|p^{\nu_p}} \frac{\tau(d)}{d} = n \prod_{p\|p^{\nu_p}} \sum_{d|p^{\nu_p}} \frac{\tau(d)}{d} = n \sum_{d|n} \frac{\tau(d)}{d}.$$

43. *(4 points.)* We have $p = \sum_{d|p} f(d) = 1 + f(p)$, that is $f(p) = p - 1$, for every prime p. Suppose $f(p^k) = p^k - p^{k-1}$ for every k up to some ν. Then $p^{\nu+1} = \sum_{d|p^{\nu+1}} f(p) = 1 + p - 1 + p^2 - p + \cdots + p^\nu - p^{\nu-1} + f(p^{\nu+1}) = p^\nu + f(p^{\nu+1})$, thus $f(p^{\nu+1}) = p^{\nu+1} - p^\nu$. That $f(p^\nu) = p^\nu - p^{\nu-1} = \varphi(p^\nu)$ for every prime power p^ν follows by induction. Since f and φ are multiplicative, $f(n) = \prod_{p^{\nu_p}\|n} f(p^{\nu_p}) = \prod_{p^{\nu_p}\|n} \varphi(p^{\nu_p}) = \varphi(n)$ for every integer n.

Alternatively, we have $\varphi(1) = f(1) = 1$, and if $\varphi(k) = f(k)$ for $k = 1, \ldots, n-1$, then

$$0 = n - n = \sum_{\substack{d|n \\ d<n}} \varphi(d) + \varphi(n) - \sum_{\substack{d|n \\ d<n}} f(d) - f(n) = \varphi(n) - f(n),$$

so the result follows by induction.

As a third alternative, we could use Möbius inversion: $f(n) = \sum_{d|n} \mu(d)n/d = n \prod_p (1 - 1/p) = \varphi(n)$.

2. *(5 points.)* Let $\pi(x,y)$ be the number of $n \le x$ that are not divisible by any prime $p \le z$, equivalently the number of $n \le x$ such that $(n, P_z) = 1$, where $P_z = \prod_{p \le z} p$. Using the fact that $\sum_{d|n} \mu(d)$ is equal to 1 if $n = 1$ and to 0 otherwise, we have

$$\pi(x,z) = \sum_{n \le x} \sum_{d|(n,P_z)} \mu(d) = \sum_{n \le x} \sum_{\substack{d|n \\ d|P_z}} \mu(d) = \sum_{d|P_z} \mu(d) \sum_{\substack{n \le x \\ d|n}} = \sum_{d|P_z} \mu(d)[x/d].$$

The primes $p \in (\sqrt{x}, x]$ and 1 are precisely the integers $n \le x$ that are not divisible by any prime $p \le \sqrt{x}$. Thus $\pi(x, \sqrt{x}) = \pi(x) - \pi(\sqrt{x}) + 1$.

Alternatively, let $p_1, \ldots, p_r$ be the primes up to $\sqrt{x}$. The primes in $(\sqrt{x}, x]$, and 1, are precisely the integers $n \le x$ that are divisible by none of the p_i. There are $[x/p_i]$ integers $n \le x$ that are divisible by p_i, $[x/p_i p_j]$ that are divisible by $p_i p_j$, and so on. By inclusion-exclusion, $\pi(x) - \pi(\sqrt{x}) + 1$ equals

$$[x] - \sum_{p_i} [x/p_i] + \sum_{p_i < p_j} [x/p_i p_j] - \sum_{p_i < p_j < p_k} [x/p_i p_j p_k] + \cdots + (-1)^r [x/p_1 \cdots p_r] = \sum_{d|p_1 \cdots p_r} \mu(d)[x/d].$$

Total: 20 points.