3. *(1 point.)*

$$\begin{aligned} x/2 &= [x/2] + \{x/2\}, \quad 0 \le \{x/2\} < 1 \\ x = 2(x/2) &= 2[x/2] + 2\{x/2\}, \quad 0 \le 2\{x/2\} < 2 \\ 2[x/2] \le [x] &\le 2[x/2] + 1 \\ 0 \le [x] - 2[x/2] &\le 1 \end{aligned}$$

4.(a) *(1 point.)*

$$\begin{aligned} x &= [x] + \{x\} \\ x + 1/2 &= [x] + \{x\} + 1/2 \\ [x + 1/2] &= \begin{cases} [x] & \text{if } 0 \le \{x\} < 1/2 \\ [x] + 1 & \text{if } 1/2 \le \{x\} < 1 \end{cases} \\ 2x &= 2[x] + 2\{x\} \\ [2x] &= \begin{cases} 2[x] & \text{if } 0 \le \{x\} < 1/2 \\ 2[x] + 1 & \text{if } 1/2 \le \{x\} < 1 \end{cases} \\ &= [x] + [x + 1/2] \end{aligned}$$

6. *(2 points.)* Let $\nu_p(a)$ denote the highest power of the prime p that divides a nonzero integer a. Clearly $\nu_p(ab) = \nu_p(a) + \nu_p(b)$ for any nonzero integers a, b, thus $\nu_p(n!) = \nu_p(n) + \nu_p(n-1) + \cdots + \nu_p(1)$. Let $s(j) = \#\{k \in \{1, 2, \ldots, n\} : \nu_p(k) = j\}$. Then

$$\begin{aligned} \nu_p(n!) &= \nu_p(n) + \nu_p(n-1) + \cdots + \nu_p(1) \\ &= s(1) + 2s(2) + 3s(3) + \cdots \\ &= (s(1) + s(2) + s(3) + \cdots) + (s(2) + s(3) + s(4) + \cdots) + (s(3) + s(4) + s(5) + \cdots) + \cdots \\ &= t(1) + t(2) + t(3) + \cdots, \end{aligned}$$

where $t(i) = s(i) + s(i+1) + \cdots = \#\{k \in \{1, 2, \ldots, n\} : p^i \mid k\} = [n/p^i]$. Thus

$$\nu_p(n!) = \sum_{i \ge 1} [n/p^i] = \sum_{i=1}^{[\frac{\log n}{\log p}]} [n/p^i].$$

(a) $\nu_2(435!) = [435/2] + \cdots + [435/2^8] = 429$.
(b) $\nu_3(435!) = [435/3] + \cdots + [435/3^5] = 215$.
(c) $435! = 2^{429} \cdot 3^{215} P = 6^{215} \cdot 2^{429-215} P$, where $(P, 6) = 1$, therefore $6^{215} \| 435!$.
(d) $435! = 2 \cdot (2^2)^{214} \cdot 3^{214} \cdot 3P = 12^{214} \cdot 2 \cdot 3P$, where $(P, 6) = 1$, therefore $12^{214} \| 435!$.
(e) $\nu_7(435!) = [435/7] + [435/7^2] + [435/7^3] = 71$, so $435! = 2^{429} \cdot 3^{215} \cdot 7^{71} P = 42^{71} \cdot 2^{429-71} \cdot 3^{215-71} P$, where $(P, 42) = 1$, therefore $42^{71} \| 435!$.

7. *(4 points.)* In **3** we showed that $[2x] \ge 2[x]$ for every real number x. Now if p is any prime, then

$$\nu_p((2n)!) = \sum_{i \ge 1} [2n/p^i] \ge \sum_{i \ge 1} 2[n/p^i] = 2 \sum_{i \ge 1} [n/p^i] = 2\nu_p(n!) = \nu_p((n!)^2).$$

That is, if $p^\nu \| (n!)^2$ then $p^\nu \mid (2n)!$, hence $(2n)!/(n!)^2$ is an integer. Now if $2^a \le n < 2^{a+1}$, then $2^{a+1} \le 2n < 2^{a+2}$, hence

$$\nu_2((2n)!) = \sum_{i=1}^{a+1} [2n/2^i] = 1 + \sum_{i=1}^{a} [2n/2^i] \ge 1 + 2 \sum_{i=1}^{a} [n/2^i] = 1 + 2\nu_2(n!) = 1 + \nu_2((n!)^2).$$

That is, if $2^\nu \| (n!)^2$, then $2^{\nu+1} \mid (2n)!$, hence $(2n)!/(n!)^2$ is even. Alternatively, we could observe that

$$\binom{(2n)!}{(n!)^2} = \frac{2n(2n-1)!}{n(n-1)!n!} = 2\binom{2n-1}{n},$$

but we would still have to prove that $\binom{2n-1}{n}$ is an integer[1].

10. *(1 point.)* If f and g are multiplicative, then $fg(1) = f(1)g(1) = 1 \cdot 1 = 1$, and for any coprime m, n, we have $fg(mn) = f(mn)g(mn) = f(m)f(n)g(m)g(n) = f(m)g(m)f(n)g(n) = fg(m)fg(n)$. Hence fg is multiplicative. If k is an integer and kf is multiplicative, then $1 = kf(1) = k$, hence kf is multiplicative if and only if $k = 1$.

16. *(2 points.)* By multiplicativity,

$$\sum_{d|n} \tau(d) = \prod_{p^{\nu_p}\|n} (\tau(1) + \tau(p) + \cdots + \tau(p^{\nu_p})) = \prod_{p^{\nu_p}\|n} (1 + 2 + \cdots + (\nu_p + 1)) = \prod_{p^{\nu_p}\|n} \frac{1}{2}(\nu_p + 1)(\nu_p + 2).$$

17. *(1 point.)* $\left(\prod_{d|n} d\right)^2 = \prod_{d|n} d(n/d) = \prod_{d|n} n = n^{\tau(n)}$, hence $\prod_{d|n} d = n^{\tau(n)/2}$.

19. *(3 points.)* We have $p^{a(\nu_p+1)} - 1 = (p^a - 1)(p^{a\nu_p} + p^{a(\nu_p-1)} + \cdots + 1)$, hence for any prime p and $a > 0$,

$$\sigma_a(p^{\nu_p}) = 1^a + p^a + (p^2)^a + \cdots + (p^{\nu_p})^a = \frac{p^{a(\nu_p+1)} - 1}{(p^a - 1)}.$$

Also, $\sigma_0(p^{\nu_p}) = \tau(p^{\nu_p}) = \nu_p + 1$. Now σ_a is multiplicative, so

$$\sigma_a(n) = \begin{cases} \displaystyle\prod_{p^{\nu_p}\|n} (\nu_p + 1) & \text{if } a = 0 \\ \displaystyle\prod_{p^{\nu_p}\|n} \frac{p^{a(\nu_p+1)} - 1}{(p^a - 1)} & \text{if } a > 0. \end{cases}$$

20. *(2 points.)* We have $\sigma(2^{\nu_0} p_1^{\nu_1} \cdots p_r^{\nu_r}) = (2^{\nu_0+1} - 1) \prod_{i=1}^r \frac{p_i^{\nu_i+1}-1}{p_i-1}$, and this is odd if and only if $(p_i^{\nu_i+1} - 1)/(p_i - 1) = 1 + p_i + \cdots + p_i^{\nu_i}$ is odd for every i. Each of the $\nu_i + 1$ terms in this sum is odd, so $\nu_i + 1$ must be odd in order for the sum to be odd. Hence $\sigma(2^{\nu_0} p_1^{\nu_1} \cdots p_r^{\nu_r})$ is odd if and only if each ν_i is even, in other words $2^{\nu_0} p_1^{\nu_1} \cdots p_r^{\nu_r}$ is a square (ν_0 even) or twice a square (ν_0 odd).

21. *(1 point.)* If n is perfect then $2 = \sigma(n)/n = (1/n)\sum_{d|n} d = (1/n)\sum_{d|n} n/d = \sum_{d|n} 1/d$.

26. *(2 points.)* By multiplicativity, and since $\mu(p^i) = 0$ if $i \geq 2$, $\sum_{d|n} \mu(d)f(d) = \prod_{p^{\nu_p}\|n} (\mu(1)f(1) + \mu(p)f(p)) = \prod_{p|n}(1 - f(p))$. Hence

$$\begin{aligned}
\sum_{d|n} |\mu(d)| &= \prod_{p|n}(1 - \mu(p)) = \prod_{p|n} 2 = 2^{\omega(n)} \qquad (|\mu(d)| = \mu(d)\mu(d)) \\
\sum_{d|n} \mu(d)\tau(d) &= \prod_{p|n}(1 - \tau(p)) = \prod_{p|n}(1 - 2) = (-1)^{\omega(n)} \\
\sum_{d|n} \mu(d)\sigma(d) &= \prod_{p|n}(1 - \sigma(p)) = \prod_{p|n}(1 - (p + 1)) = (-1)^{\omega(n)} \prod_{p|n} p \\
\sum_{d|n} \mu(d)\varphi(d) &= \prod_{p|n}(1 - \varphi(p)) = \prod_{p|n}(1 - (p - 1)) = (-1)^{\omega(n)} \prod_{p|n}(p - 2).
\end{aligned}$$

Total: 20 points.

[1] If you want to prove $\binom{2n-1}{n}$ or $\binom{2n}{n}$ is an integer by saying that $\binom{n}{k}$ is the number of ways of choosing k objects from n, then prove this in detail.