1. *(1/2 point.)* $(x, y) = (1 - 4t, -2 - 3t), t \in \mathbb{Z}$.

2. *(1/2 point.)* $ax + by = b + c$ iff $ax + b(y - 1) = c$.

3. *(2 points.)* Obviously $(a, b, c) \mid (a, b)$ so it suffices to show that $ax + by = c$ iff $(a, b) \mid (a, b, c)$ iff $(a, b) \mid c$. Let $(a, b) = d$ and suppose $a = da', b = db'$. If $ax + by = c$ then $da'x + db'y = c$, hence $d \mid c$. Now suppose $d \mid c$, say $c = dc'$. Since a', b' are coprime, there exist integers x, y such that $a'x + b'y = 1$. Then $a(xc') + b(yc') = da'xc' + db'yc' = dc' = c$.

5. *(3 points.)* Suppose $a^2 + (a + k)^2 = (a + 2k)^2$, a, k positive integers. Then $(a + k)(a - 3k) = 0$, so $a = 3k$, for $a = -k$ is not allowed. Therefore all pythagorean triples forming an arithmetic progression are given by $3k, 4k, 5k$, k a positive integer.

Alternatively, we know that all primitive pythagorean triples x, y, z are given by $x = r^2 - s^2, y = 2rs, z = r^2 + s^2$, where $(r, s) = 1$, $r > s$ and exactly one of r, s is even. If x, y, z is an arithmetic progression, then $z - y = y - x$, that is $2r^2 = 4rs$, or $r = 2s$. Now $s = (2s, s) = (r, s) = 1$, so there is just one primitive pythagorean triple that forms an arithmetic progression, namely $x = 3, y = 4, z = 5$. Therefore all pythagorean triples that form an arithmetic progression are given by: $x = 3k, y = 4k = x + k, z = 5k = y + k$, k a positive integer.

10. *(1 point.)* $n^2 + (n + 1)^2 = 2(n^2 + n) + 1$ is odd, but $2m^2$ is even.

11. *(1 point.)* $4z + 7 \equiv 3 \bmod 4$, but $x^2 + y^2 \equiv 0, 1$ or $2 \bmod 4$.

16. *(4 points.)* Suppose there exists a nontrivial solution. Then there exists a nontrivial solution x, y, z with $(x, y, z) = 1$, for if $a^3 + 3b^3 = 9c^3$ and $a = gx, b = gy, c = gz$, then $g^3x^3 + 3g^3y^3 = 9g^3z^3$, that is $x^3 + 3y^3 = z^3$, and $(x, y, z) = 1$ if $g = (a, b, c)$. Let x, y, z be a solution with $(x, y, z) = 1$. Now $x^3 = 3(y^3 - 3z^3)$, so 3 divides x^3 and hence x: say $x = 3a$. Then $3y^3 = 9(z^3 - 3a^3)$, so 3 divides y^3 and hence y: say $y = 3b$. Then $9z^3 = 27(a^3 + 3b^3)$, so 3 divides z^3 and hence z. Then 3 divides $(x, y, z) = 1$, a contradiction.

22. *(2 points.)* Suppose $n^x + n^y = n^z$, n a positive integer, x, y, z nonnegative integers. Without loss of generality, $x \leq y$, and obviously $y < z$. Then $n^{y-x}(n^{z-y} - 1) = n^{z-x} - n^{y-x} = 1$, that is $n^{y-x} = n^{z-y} - 1 = 1$, hence $n = 2$, $y = x$ and $z = y + 1$.

1. *(1 point.)* $(3/13) \equiv 3^{(13-1)/2} \equiv (3^3)^2 \equiv 1^2 \equiv 1 \bmod 13$, so 3 is a square modulo 13 (indeed, $4^2 \equiv 3 \bmod 13$). $(7/3) \equiv 3^{(7-1)/2} \equiv -1 \bmod 7$, so 3 is not a square mod 7.

Supplementary problem. *(5 points.)* For any integer n, there exist integers a, b such that $pa + qb = n$, and all solutions to $px + qy = n$ are given by $x = a + tq, y = b - tp$, t an integer. Let x, y be a solution with $0 \leq y \leq p - 1$. If $n \geq pq - p - q + 1 = (p - 1)(q - 1)$, then $px = n - qy \geq (p - 1)(q - 1) - q(p - 1) = -(p - 1)$, so $x \geq -1 + 1/p$, that is $x \geq 0$.

If $a, b \geq 0$ and $ap + bq = pq - p - q$ then $(a + 1)p + (b + 1)q = pq$, so p divides $b + 1 \geq 1$ and q divides $a + 1 \geq 1$, therefore $a + 1 \geq q$ and $b + 1 \geq p$. Then $pq = (a + 1)p + (b + 1)q \geq pq + pq$, that is $pq \leq 0$, a contradiction because p, q are positive. Therefore $pq - p - q$ is the greatest integer not of the form $ap + bq$ with $a, b \geq 0$.

For any integer n, there exist integers a, b such that $ap - bq = n$. Whatever a and b are, if t is a sufficiently large positive integer then $a + tq$ and $b + tp$ are nonnegative. Then $(a + tq)p - (b + tp)q = ap - bq = n$. Hence $\{ap - bq : a, b \geq 0\} = \mathbb{Z}$.

Total: 20 points.