1. *(1 point.)* Each relation clearly holds for $n = 1$. Assume each relation holds for some $n \geq 1$. Then

$$\sum_{k=1}^{n+1} k(k+1) = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}$$

$$\sum_{k=1}^{n+1} k^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2}{4}\left(n^2 + 4(n+1)\right) = \frac{(n+1)^2(n+2)^2}{4}$$

$$\sum_{k=1}^{n+1} k(k!) = (n+1)! - 1 + (n+1)(n+1)! = (n+1)!(1+n+1) - 1 = (n+2)! - 1.$$

2. *(1 point.)* (a) *Alternative 1.*

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} 1^k 1^{n-k} \binom{n}{k} = (1+1)^n = 2^n.$$

Alternative 2. Let $A = \{1, 2, \ldots, n\}$ and let $P(A)$ be the set of subsets of A. For any given subset S of A, k is either in S or it is not ($k = 1, 2, \ldots, n$). Therefore $\#P(A) = 2^n$. On the other hand, there are clearly $\binom{n}{k}$ ways of forming a k-element subset S of A, therefore

$$2^n = \#P(A) = \sum_{k=0}^{n} \binom{n}{k}.$$

Alternative 3. The relation clearly holds for $n = 1$. Assume it holds for some $n \geq 1$. Then

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1}{k} &= \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n+1}{k} + \binom{n+1}{n+1} \\
&= 1 + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) + 1 \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k} + \sum_{k=1}^{n} \binom{n}{k-1} + \binom{n}{n} \\
&= 2^n + \sum_{k=0}^{n-1} \binom{n}{k} + \binom{n}{n} \\
&= 2^n + 2^n \\
&= 2^{n+1}.
\end{aligned}$$

(b) *Alternative 1.*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} 1^{n-k}(-1)k\binom{n}{k} = (1-1)^n = 0.$$

Alternative 2. If n is odd, then

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k \binom{n}{k} &= \frac{1}{2} \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} + \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{n-k} \right) \\
&= \frac{1}{2} \sum_{k=0}^{n} \left(\binom{n}{k} - \binom{n}{k} \right) \\
&= 0.
\end{aligned}$$

If n is even, then

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k} &= 1 + \sum_{k=1}^{n-1}(-1)^k\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)+\binom{n-1}{n-1}\\ &= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}+\sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k-1}+\binom{n-1}{n-1}\\ &= 0-\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\\ &= 0.\end{aligned}$$

Alternative 3. The relation clearly holds for $n = 1$. Assume it holds for some $n \geq 1$. Then

$$\begin{aligned}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k} &= 1 + \sum_{k=1}^{n}(-1)^k\left(\binom{n}{k}+\binom{n}{k-1}\right)+(-1)^{n+1}\\ &= (-1)^0\binom{n}{0}+\sum_{k=1}^{n}(-1)^k\binom{n}{k}+\sum_{k=1}^{n}(-1)^k\binom{n}{k-1}+(-1)^{n+1}\binom{n}{n}\\ &= 0-\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}-(-1)^n\binom{n}{n}\\ &= -\sum_{k=0}^{n}(-1)^k\binom{n}{k}\\ &= 0.\end{aligned}$$

5. Observe that

$$n^3 = n^2\cdot n = \left(\frac{n^2-n}{2}+n+\frac{n^2-n}{2}\right)\left(\frac{n^2-n}{2}+n-\frac{n^2-n}{2}\right) = \left(\frac{n^2+n}{2}\right)^2-\left(\frac{n^2-n}{2}\right)^2,$$

and $(n^2 \pm n)/2 = n(n \pm 1)/2$ are integers. How did we know to "observe" that? We want to write n^3 as a difference of two squares, say $c^2 - a^2$. Now we can write $c = a + b$ for some b, so we want integers a, b such that

$$n^3 = (a+b)^2 - a^2 = 2ab + b^2.$$

If we put $b = n$, we can divide through by n (if $n = 0$, all is trivial):

$$n^2 = 2a + n,$$

that is $a = (n^2 - n)/2$, which is indeed an integer.

7. *(6 points.)* (a) The product is $n(n + 1)(n + 2)(n + 3)$ for some integer n. One of the integers is divisible by 4, and another must be 2 modulo 4, that is divisible by 2. At least on of the integers must be divisible by 3. Hence the product is divisible by $2 \cdot 3 \cdot 4 = 24$.
(b) We must have $n = 4k + 2$ and $m = 4\ell + 2$ for some integers k, ℓ. Then $m + n = 4(k + \ell) + 4$.
(c) One can observe that $(n + 1)^3 - (n + 1) = n^3 - n + 3(n^2 + n)$, and the result follows by induction. But the intended idea is as follows. We have

$$n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1).$$

One of $n-1, n, n+1$ must be divisible by 3, and at least one of them must be even. Hence n^3-n is divisible by $2\cdot 3 = 6$.
(d) Again, one can observe that $(n+1)^5 - (n+1) = n^5 - n + 5(n^4 + 2n^3 + 2n^2 + n)$, and the result follows by induction. But the idea is as follows. We have

$$n^5 - n = (n^3 - n)(n^2 + 1),$$

so 6 divides $n^5 - n$. If 5 divides $n^3 - n = n(n-1)(n+1)$, then $5 \cdot 6 = 30$ divides $n^5 - n$. If not, then 5 must divide $n+2$ or $n+3$. Then

$$n^2 + 1 = (n + 2 - 2)^2 + 1 = (n+2)^2 + 4(n+2) + 5$$

or

$$n^2 + 1 = (n + 3 - 3)^2 + 1 = (n+3)^2 + 6(n+3) + 10.$$

In either case, 5 divides $n^2 + 1$. Hence $5 \cdot 6 = 30$ divides $n^5 - n$.
(e) An odd integer $n = 4k + r$ where $r = 1$ or 3. Then

$$n^2 = 16k^2 + 8rk + r^2 = 8m + 1,$$

where $m = 2k^2 + rk$ if $r = 1$, and $m = 2k^2 + rk + 1$ if $r = 3$.
(f) If m, n are odd then for some integers k, ℓ, we have

$$m^2 + n^2 = (8k + 1) + (8\ell + 1) = 4(2k + 2\ell) + 2.$$

9. For some integers a, b, c we have

$$c = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}.$$

Then

$$a = abc - b = b(ac - 1) \quad \text{and} \quad b = abc - a = a(bc - 1),$$

that is, $b \mid a$ and $a \mid b$. Hence $a = \pm b$, and $a = b$ if a, b are positive. If $0 < a = b \geq 3$ then $0 < 1/a + 1/b \leq 2/3$ is not an integer, so in fact $a = b = 1$ or 2.

11. *(3 points.)* Let $d = \prod_p p^{a_p}$ be the canonical factorization of d. If $p \mid d$, then $p \mid mn$, and so p divides m or n, but not both (for otherwise $p \mid (m, n)$). Let

$$r = \prod_{\substack{p \mid d \\ p \mid m}} p^{a_p} \quad \text{and} \quad s = \prod_{\substack{p \mid d \\ p \mid n}} p^{a_p}.$$

Then clearly $rs = d$, $r \mid m$, $s \mid n$ and $(r, s) = 1$.

In other words, let $r = (d, m) \mid m$ and let $s = (d, n) \mid n$. Then, since $(m, n) = 1$, $rs = (d, mn) = d$. It is clear that $(r, s) = 1$, for if p divides (r, s) then p divides m and n, so $p \mid (m, n)$. (Or: if $m = rr'$, $n = ss'$, then we have integers x, y such that $1 = mx + ny = r(r'x) + s(s'y)$.)

12. *(2 points.)* $(629, 357) = 17$, for

$$\begin{aligned} 629 &= 357 + 272 \\ 357 &= 272 + 85 \\ 272 &= 85 \cdot 3 + 17 \\ 85 &= 17 \cdot 5 + 0. \end{aligned}$$

Now

$$\begin{aligned} 17 &= 272 - 85 \cdot 3 \\ &= 272 - (357 - 272) \cdot 3 \\ &= 272 \cdot 4 - 357 \cdot 3 \\ &= (629 - 357) \cdot 4 - 357 \cdot 3 \\ &= 629 \cdot 4 - 357 \cdot 7, \end{aligned}$$

therefore the integers (x, y) such that $629x - 357y = 17$ are precisely the integers of the form $(4 + 357t/17, 7 + 629t/17)$, $t \in \mathbb{Z}$.

14. $(x_0, y_0) = 1$ because $a/d, b/d$ are integers and $(a/d)x_0 + (b/d)y_0 = 1$, and x_0, y_0 are not unique because $a(x_0 + bt/d) + b(y_0 - at/d) = d$ for every integer t.

17. *(4 points.)* Write $a = \prod_p p^{a_p}$, $b = \prod_p p^{b_p}$. Then $(a, b) = \prod_p p^{c_p}$, where $c_p = \max\{a_p, b_p\}$. Now $a^2 = \prod_p p^{2a_p}$, $b^2 = \prod_p p^{2b_p}$ and $(a^2, b^2) = \prod_p p^{d_p}$, where $d_p = \max\{2a_p, 2b_p\} = 2\max\{a_p, b_p\} = 2c_p$. Hence $(a^2, b^2) = c^2$.

Or, note that $(a/c, b/c) = 1$, and so $(a^2/c^2, b^2/c^2) = 1$, for if $p \mid a^2/c^2, b^2/c^2$ then $p \mid a/c, b/c$, so $p = 1$. Hence $(a^2, b^2) = c^2$.

Or: clearly c^2 divides a^2 and b^2, hence (a^2, b^2). We will show that (a^2, b^2) divides c^2. Let x, y be integers such that $ax + by = c$, and note that

$$ab = \frac{ab}{c} \cdot c = a^2 x \frac{b}{c} + b^2 y \frac{a}{c},$$

where $a/c, b/c$ are integers. Now

$$c^2 = a^2x^2 + 2abxy + b^2y^2 = a^2x^2 + 2a^2x^2y\frac{b}{c} + 2b^2xy^2\frac{a}{c} + b^2y^2 = a^2X + b^2Y,$$

where $X = x^2(1 + 2yb/c$ and $Y = y^2(1 + 2xa/c)$ are integers. Hence c^2 is a multiple of (a^2, b^2).

17 points, plus 3 points for a reasonable attempt at all questions = 20 points.