Question 1

Give an example of a sequence of random variables:

(a) \((X_n, n \geq 1)\) that converges almost surely but does not converge in \(L^1\);
(b) \((Y_n, n \geq 1)\) that converges in \(L^1\) but does not converge almost surely.

Question 2

Let \((X_i, i \geq 1)\) be a sequence of independent random variables that are uniformly distributed on \([-1, 1]\). One defines \(Y_i = X_i^2\).

(a) Show that \(\frac{Y_1 + Y_2 + \cdots + Y_n}{n} \to \frac{1}{3}\) in probability when \(n \to \infty\).

(b) For \(0 < \varepsilon < 1\), let \(A_{n,\varepsilon}\) be the subset of \(\mathbb{R}^n\) defined by

\[ A_{n,\varepsilon} = \{ x \in \mathbb{R}^n : \sqrt{(1-\varepsilon)\frac{n}{3}} < \|x\| < \sqrt{(1+\varepsilon)\frac{n}{3}} \} , \]

where \(\|x\|\) is the Euclidean norm of \(x\). Deduce that

\[ \frac{|A_{n,\varepsilon} \cap [-1,1]^n|}{|[-1,1]^n|} \to 1 , \]

where \(|B|\) is the volume of the subset \(B\) in \(\mathbb{R}^n\).

*Suggestion: Interpret the left-hand side in probabilistic terms.*

Question 3

Let \((X_n, n \geq 1)\) be a sequence of independent real random variables that follow a standard normal distribution \(\mathcal{N}(0, 1)\) et \((\alpha_n, n \geq 1)\) be a sequence of real numbers. Define \(F_n = \sigma(X_1, \ldots, X_n)\) and

\[ M_n = \exp \left( \sum_{k=1}^{n} \alpha_k X_k - \frac{1}{2} \sum_{k=1}^{n} \alpha_k^2 \right) , \ n \geq 1 . \]
(a) For $\alpha \in \mathbb{R}$, show that
$$E[e^{\alpha X_1}] = e^{\alpha^2/2}.$$  

(b) Show that $(M_n, n \geq 1)$ is a martingale with respect to the filtration $(\mathcal{F}_n, n \geq 1)$ that converges almost surely.

(c) Suppose $\sum_{k=1}^{+\infty} \alpha_k^2 = +\infty$. Show that $\lim_{n \to \infty} M_n = 0$ almost surely.

(Suggestion: Compute $E[M_n^{1/3}]$ and use Fatou.)

(d) Is the convergence to 0 in (c) in $L^1$ too?

Question 4

Let $(X_n, n \geq 1)$ be a random walk on the vertices $V$ of a connected graph such that

$$P(u, v) = \mathbb{P}(X_{n+1} = v | X_n = u) = \frac{1}{d(u)},$$

for every vertex $v \in V(u)$, where $V(u)$ is the set of vertices adjacent to $u$ and $d(u)$ is the number of elements in $V(u)$.

(a) Show that

$$\pi(u) = \frac{d(u)}{\sum_{v \in V} d(v)},$$

for every $u \in V$, is a stationary distribution for the random walk.

(b) What is the expected number of transitions to return to $u$ starting from $u$ for every $u \in V$?

Question 5

Let $(X_n, n \geq 1)$ be a sequence of independent real random variables with $E[X_n] = \mu$ and $Var(X_n) = \sigma^2 > 0$, for every $n \geq 1$. Show that

$$\frac{S_n - n\mu}{\sqrt{\sum_{i=1}^{n} (X_i - \overline{X_n})^2}} \to Z \text{ in distribution when } n \to \infty,$$

where $S_n = X_1 + \cdots + X_n$ and $\overline{X_n} = S_n/n$, with $Z$ being a $\mathcal{N}(0, 1)$ random variable.

(Suggestion: Show first that if $Z_n \to Z$ in distribution and $Y_n \to 1$ in probability, then $Z_n/Y_n \to Z$ in distribution.)