Metric Diophantine Approximation
Lecture 1: Khinchin’s theorem and its limitations

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Fundamental Question

Given an irrational number x, find fractions a/q that approximate it "well".

- the error $|x - a/q|$ must be small
- q must be small (fractions of "low complexity")

Remark

In various application, we might need q to lie in a restricted set of denominators (e.g. primes, squares etc.)

Dirichlet (c.1840):

$\forall x$ irrational, we have i.o.

$$\left|x - \frac{a}{q}\right| < \frac{1}{q^2}.$$

Improving Dirichlet's theorem:

1. Can we replace $1/q^2$ by something smaller?
2. Can we restrict q to lie in some special set of denominators?

Irrationality measure:

$$\mu(x) := \sup\left\{ E \geqslant 0 \,:\, 0 < \left| x - \frac{a}{q} \right| \leqslant \frac{1}{q^E} \text{ i.o.} \right\}$$

Results:

- Roth (1955): $\mu(x) = 2$ for every algebraic irrational x.
- Zeilberger–Zudilin (2020): $\mu(\pi) \leqslant 7.10320533\ldots$

Zaharescu (1995):

Fix $\varepsilon > 0$ and $x \in \mathbb{R} \setminus \mathbb{Q}$. Then

$$\left| x - \frac{a}{q^2} \right| \leqslant \frac{1}{q^{8/3-2\varepsilon}} = \frac{1}{(q^2)^{4/3-\varepsilon}} \quad \text{i.o.}$$

Matomäki (2009):

Fix $\varepsilon > 0$ and $x \in \mathbb{R} \setminus \mathbb{Q}$. Then

$$\left| x - \frac{a}{p} \right| \leqslant \frac{1}{p^{4/3-\varepsilon}} \quad \text{i.o. with } p \text{ prime.}$$

Hard open problems:

Improve “4/3” to “3/2” in Zaharescu’s theorem and “4/3” to “2” in Matomäki’s theorem.

Diophantine approximation:

Approximate a fixed irrational number x
$\rightsquigarrow$ hard open problems

Metric Diophantine approximation:

- Prove results about almost all numbers.
- Exclusion of small pathological sets
  $\rightsquigarrow$ simple-to-state, general results

The basic set-up:

Given “permissible errors” $\Delta_1, \Delta_2, \ldots \geqslant 0$, let

$$\mathcal{A} := \left\{ x \in [0,1] : \left| x - \frac{a}{q} \right| < \Delta_q \quad \text{i.o.} \right\}$$

Khinchin (1924):

Let $\mathcal{A} = \left\{x \in [0,1] : \left|x - a/q\right| < \Delta_q \quad \text{i.o.}\right\}$.

1. If $\sum q\Delta_q < \infty$, then $\mathrm{m}(\mathcal{A}) = 0$.
2. If $\sum q\Delta_q = \infty$ and $q^2\Delta_q \searrow$, then $\mathrm{m}(\mathcal{A}) = 1$.

The Borel–Cantelli lemmas:

$E_1, E_2, \ldots$ events; E event that ∞-many E_j occur.

1. If $\sum \mathbb{P}(E_j) < \infty$, then $\mathbb{P}(E) = 0$.
2. If $\sum \mathbb{P}(E_j) = \infty$ and the E_j's are independent, then $\mathbb{P}(E) = 1$.

Proof of 1: consider the events $\mathcal{A}_q := \left\{x \in [0,1] : \left|x - a/q\right| < \Delta_q\right\}$.

Step 1: Cassels' $0-1$ law about $\mathrm{m}(\mathcal{A})$

- Let $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the multiplication by 2, i.e. $\psi(\alpha) = 2\alpha \pmod 1$.
- This is an ergodic map; in particular, if $\psi(A) \subseteq A$, then $\mathrm{m}(A) \in \{0,1\}$.
- $\mathcal{A} = \limsup_{q\to\infty} \mathcal{A}_q = \{\alpha \in [0,1] : \exists\infty - \text{many } (a,q) \text{ s.t. } |\alpha - a/q| < \Delta_q\}$.
- Corollary of Lebesgue's density theorem: for any $r > 0$, we have $\mathrm{m}(r\mathcal{A} \,\triangle\, \mathcal{A}) = 0$ with

$$r\mathcal{A} := \{\alpha \in [0,1] : \exists\infty - \text{many } (a,q) \text{ s.t. } |\alpha - a/q| < r\Delta_q\}.$$

- Consider $\tilde{\mathcal{A}} := \bigcup_{j\in\mathbb{Z}} 2^j\mathcal{A}$, for which $\mathrm{m}(\tilde{\mathcal{A}}) = \mathrm{m}(\mathcal{A})$ and $\psi(\tilde{\mathcal{A}}) \subseteq \tilde{\mathcal{A}}$.
- Conclusion: $\mathrm{m}(\mathcal{A}) \in \{0,1\}$.

Step 2: weakening the independence assumption in Borel-Cantelli

- Fix $R > Q$ and let $N(\alpha) = \#\{(a, q) : q \in [Q, R],\ |\alpha - a/q| < \Delta_q\}$.
- $\mathrm{supp}(N) = \bigcup_{q \in [Q,R]} \mathcal{A}_q$
- $N = N \cdot \mathbb{1}_{N>0} \overset{\text{Cauchy–Schwarz}}{\Longrightarrow} \mathrm{m}(\mathrm{supp}(N)) \geqslant \dfrac{(\int_0^1 N(\alpha)\mathrm{d}\alpha)^2}{\int_0^1 N(\alpha)^2 \mathrm{d}\alpha}$
- $\displaystyle\int_0^1 N(\alpha)\mathrm{d}\alpha = \sum_{q \in [Q,R]} 2q\Delta_q = \sum_{q \in [Q,R]} \mathrm{m}(\mathcal{A}_q).$
- $\displaystyle\int_0^1 N(\alpha)^2 \mathrm{d}\alpha = \sum_{q,r \in [Q,R]} \mathrm{m}\left(\mathcal{A}_q \cap \mathcal{A}_r\right) \overset{?}{\leqslant} 10^{10^{10}} \cdot \Big(\sum_{q \in [Q,R]} \mathrm{m}(\mathcal{A}_q) \Big)^2$

Step 3: controlling pairwise intersections

- $\mathcal{A}_q = [0,1] \cap \bigcup_{a=0}^{q} [\frac{a}{q} - \Delta_q, \frac{a}{q} + \Delta_q]$, $\quad \mathcal{A}_r = [0,1] \cap \bigcup_{b=0}^{r} [\frac{b}{r} - \Delta_r, \frac{b}{r} + \Delta_r]$
- $\delta = \min\{\Delta_q, \Delta_r\} \quad \Delta = \max\{\Delta_q, \Delta_r\}$
- $\mathcal{A}_q \cap \mathcal{A}_r \approx [0,1] \cap \left(\bigcup_{|\frac{a}{q} - \frac{b}{r}| < 2\Delta} \underbrace{\text{Interval}(a/q, b/r)}_{\text{length} \leqslant 2\delta} \right) = \underbrace{\text{Diagonal}}_{\frac{a}{q} = \frac{b}{r}} \bigcup \underbrace{\text{Off-diagonal}}_{\frac{a}{q} \neq \frac{b}{r}}$
- Problem: the diagonal part could have Lebesgue measure that is too large and forces 2nd moment to explode!
- Fix: consider instead $\mathcal{A}_q^* = [0,1] \cap \bigcup_{\substack{0 \leqslant a \leqslant q \\ \gcd(a,q)=1}} \left[\frac{a}{q} - \Delta_q, \frac{a}{q} + \Delta_q \right]$.

Revised step 3: controlling pairwise intersections for reduced fractions

- $\mathcal{A}_q^* = [0,1] \cap \bigcup_{\substack{0 \leqslant a \leqslant q \\ \gcd(a,q)=1}} \left[\frac{a}{q} - \Delta_q, \frac{a}{q} + \Delta_q\right]$; $\qquad \mathcal{A}^* = \limsup_{q\to\infty} \mathcal{A}_q^*$
- New goal: show that $\mathrm{m}(\mathcal{A}^*) > 0$ if $\sum_q q\Delta_q = \infty$ and $q\Delta_q \searrow$.
- Pollington–Vaughan: if $D(q,r) := \max\{\Delta_q, \Delta_r\} \cdot \mathrm{lcm}[q,r]$, then

$$\mathrm{m}(\mathcal{A}_q^* \cap \mathcal{A}_r^*) \ll \underbrace{\mathrm{m}(\mathcal{A}_q^*)\,\mathrm{m}(\mathcal{A}_r^*)}_{=\Delta_q\phi(q)\Delta_r\phi(r)} \cdot \mathbb{1}_{D(q,r)\geqslant 1} \cdot \underbrace{\prod_{\substack{p \mid qr/\gcd(q,r)^2 \\ p > D(q,r)}} \left(1 + \frac{1}{p}\right)}_{\text{loss of factor } \ll \frac{q}{\phi(q)}\cdot\frac{r}{\phi(r)}},$$

- When $q\Delta_q \searrow$, it's easy to show that $\sum_q q\Delta_q \asymp \sum_j 4^j \Delta_{2^j} \asymp \sum_q \phi(q)\Delta_q$. □

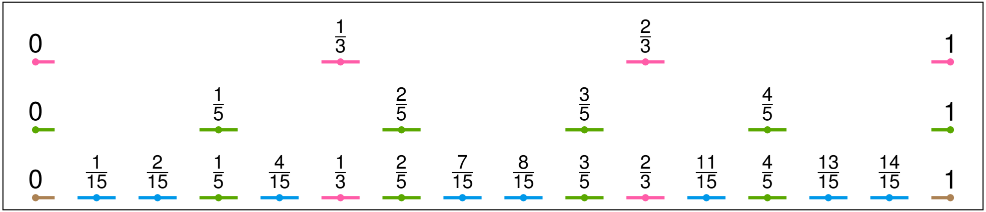
The Duffin–Schaeffer counterexample

Duffin–Schaeffer (1941):

Khinchin's theorem fails in full generality: $\exists \Delta_1, \Delta_2, \ldots \geqslant 0$ such that

$$\sum q\Delta_q = \infty \quad \text{and yet} \quad \mathrm{m}(\mathcal{A}) = 0.$$

Strategy: Construct $q_1, q_2, \ldots$ s.t. $\sum_j q_j \Delta_{q_j} < \infty$ but $\sum_j \sum_{q|q_j} q\Delta_{q_j} = \infty$.



Example with $\mathcal{A}_3, \mathcal{A}_5 \subset \mathcal{A}_{15}$

Removing repetitions:

$$\mathcal{A}^* := \left\{ x \in [0,1] : \left| x - \frac{a}{q} \right| < \Delta_q \quad \text{i.o. with } \gcd(a,q) = 1 \right\}$$

The Duffin–Schaeffer conjecture (1941):

1. If $\sum \phi(q)\Delta_q < \infty$, then $\mathrm{m}(\mathcal{A}^*) = 0$.
2. If $\sum \phi(q)\Delta_q = \infty$, then $\mathrm{m}(\mathcal{A}^*) = 1$.

History of results on the Duffin–Schaeffer conjecture

- Duffin–Schaeffer (1941): DSC is true when $\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q} \Delta_q \phi(q)}{\sum_{q\leqslant Q} \Delta_q q} > 0$.
- Gallagher (1961): There is a $0-1$ law, i.e. $\mathrm{m}(\mathcal{A}^*) \in \{0,1\}$.
- Erdős (1970) & Vaaler (1978): DSC is true when $\Delta_q = O(1/q^2)$ for all q.
- Pollington–Vaughan (1990): DSC is true in all dimensions > 1.
- Haynes–Pollington–Velani (2012), Beresnevich–Harman–Haynes–Velani (2013), Aistleitner, Harman, Haynes, Lachman, Munsch, Technau, Zafeiropoulos (2018), Aistleitner (2019): DSC is true “with extra divergence” (i.e. $\sum_q \Delta_q \phi(q)/L(q) = \infty$ with various functions $L(q) \to \infty$).
- K.–Maynard (2019): the Duffin–Schaeffer conjecture is true.

Catlin's conjecture (1976):

Let $\Delta'_q := \sup\{\Delta_q, \Delta_{2q}, \dots\}$. Then

$$\mathrm{m}(\mathcal{A}) = 1 \quad \iff \quad \sum \phi(q)\Delta'_q = \infty.$$

Proof: If $\alpha \notin \mathbb{Q}$ and $\Delta_q \to 0$, then $\alpha \in \mathcal{A}$ if-f $\alpha \in \mathcal{A}' := \mathcal{A}^*(\Delta'_1, \Delta'_2, \dots)$.

Hausdorff dimensions

Using a mass-transference principle of Beresnevich–Velani (2006), we have:

- If $\sum \phi(q)\Delta_q < \infty$ so that $\mathrm{m}(\mathcal{A}^*) = 0$, then

$$\dim(\mathcal{A}^*) = \inf\{s > 0 : \sum \phi(q)\Delta_q^s < \infty\}.$$

- If $\sum \phi(q)\Delta'_q < \infty$ so that $\mathrm{m}(\mathcal{A}) = 0$, then

$$\dim(\mathcal{A}) = \inf\{s > 0 : \sum \phi(q)(\Delta'_q)^s < \infty\}.$$

How to prove that $\mathrm{m}(\mathcal{A}^*)$ when $\sum_q \phi(q)\Delta_q = \infty$:

- Gallagher's $0-1$ law: enough to show $\mathrm{m}(\mathcal{A}^*) > 0$
- Cauchy–Schwarz: enough to show $\mathrm{m}(\mathcal{A}_q^* \cap \mathcal{A}_r^*) \ll \mathrm{m}(\mathcal{A}_q^*)\,\mathrm{m}(\mathcal{A}_r^*)$ "on average".
- Pollington–Vaughan: enough to show $\displaystyle\prod_{\substack{p \mid qr/\gcd(q,r)^2 \\ p > D(q,r)}} (1 + 1/p) \ll 1$ "on average".
- When $\mathcal{S} = \mathrm{supp}(\Delta)$ is "dense or regular enough", we may use facts about the "anatomy of integers" to prove this (theorems of Duffin–Schaeffer and Erdős–Vaaler).

- Assume for contradiction $0 < \mathrm{m}(\mathcal{A}^*) < 1$ and let p be a prime.
- The maps $\psi_0(\alpha) = p\alpha \pmod 1$ and $\psi_1(\alpha) = p\alpha + 1/p \pmod 1$ are ergodic.
- We have $\psi_j(r\mathcal{A}_j^*) \subseteq pr\mathcal{A}_j^*$ for $j = 0, 1$, where

$$r\mathcal{A}_j^* := \left\{\alpha \in [0,1] : |\alpha - a/q| < r\Delta_q \text{ i.o. with } \gcd(a,q) = 1,\ p^j \| q\right\}$$

- Thus $\mathrm{m}(\mathcal{A}_j^*) \in \{0,1\}$ for $j = 0, 1$, whence $\mathrm{m}(\mathcal{A}_j^*) = 0$ for $j = 0, 1$.
- Conclusion: $\mathrm{m}(\mathcal{A}^*) = \mathrm{m}(\mathcal{A}_{\geqslant 2}^*)$, where

$$\mathcal{A}_{\geqslant 2}^* := \left\{\alpha \in [0,1] : |\alpha - a/q| < \Delta_q \text{ i.o. with } \gcd(a,q) = 1,\ p^2 | q\right\}$$

- But $\mathcal{A}_{\geqslant 2}^*$ is $\frac{1}{p}$-periodic, and p is arbitrary. Violates Lebesgue's density theorem.

- For simplicity, let $\Delta_q \in \left\{0, \frac{1}{q^2}\right\}$. We must show $\mathrm{m}(\mathcal{A}^*) > 0$ when

$$\sum_q \phi(q)\Delta_q = \infty \quad \Longleftrightarrow \quad \sum_{q\in\mathcal{S}} \frac{\phi(q)}{q^2} = \infty \quad \text{with} \quad \mathcal{S} = \mathrm{supp}(\Delta).$$

- To simplify further, assume $\exists\infty$-many $x \in \mathbb{N}$ s.t. $\sum_{q\in\mathcal{S}\cap[x,2x]} \frac{\phi(q)}{q} \asymp x$.
- Pollington–Vaughan: $\mathrm{m}(\mathcal{A}_q^* \cap \mathcal{A}_r^*) \ll \underbrace{\mathrm{m}(\mathcal{A}_q^*)\,\mathrm{m}(\mathcal{A}_r^*)}_{=\frac{\phi(q)\phi(r)}{qr}} \prod_{p \mid qr/\gcd(q,r)^2} (1 + 1/p)$.
- $\#\left\{n \leqslant x : \prod_{p\mid n} (1 + 1/p) > A\right\} \ll x/e^{e^A}$

Strategy to prove that $\mathrm{m}(\mathcal{A}^*)$ when $\sum_q \phi(q)\Delta_q = \infty$:

- Gallagher's $0-1$ law: enough to show $\mathrm{m}(\mathcal{A}^*) > 0$
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- Pollington–Vaughan: enough to show $\prod_{\substack{p \mid qr/\gcd(q,r)^2 \\ p > D(q,r)}} (1 + 1/p) \ll 1$ "on average".
- When $\mathcal{S} = \mathrm{supp}(\Delta)$ is "dense or regular enough", we may use facts about the "anatomy of integers" to prove this.

Question

What if $\mathcal{S}$ is supported on a sparse set of integers with lots of small prime factors?

Thank you for your attention