Exotic Symplectomorphisms and Contact Circle Actions

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Based on joint work with Dušan Drobnjak

Exotic morphisms

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Definition

An exotic symplectomorphism of the standard symplectic ball, $\mathbb{B}^{2n}$, is a symplectomorphism $\phi : \mathbb{B}^{2n} \to \mathbb{B}^{2n}$ that is not isotopic to the identity relative to the boundary through symplectomorphisms.

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$\phi : \mathbb{B}^{2n} \to \mathbb{B}^{2n}$ exotic symplectomorphism	$\longleftrightarrow$	$\nexists\, \phi_t : \mathbb{B}^{2n} \to \mathbb{B}^{2n}, t \in [0,1]$ compactly supported symplectomorphisms $\phi_0 = \mathrm{id}, \quad \phi_1 = \phi$

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Definition

An exotic diffeomorphism $\phi : \mathbb{B}^n \to \mathbb{B}^n$ is a diffeomorphism that is not isotopic to the identity relative to the boundary.

Diffeomorphisms vs symplectomorphisms

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n	1	2	3	4	5	6	7	8	9	10
$\mathbb{B}^n$										
How many?										

Table: Exotic diffeomorphisms?

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Theorem (Casals-Keating-Smith)

There exist an exotic diffeomorphism $\phi : \mathbb{B}^{4k} \to \mathbb{B}^{4k}$ *and a symplectic form* ω *on* $\mathbb{B}^{4k}$ *such that* $\phi^*\omega = \omega$ *whenever* $k \notin \{1, 3, 7, 15, 31\}$.

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Theorem (Seidel)

There exists a symplectomorphism $\phi : M \to M$ *that is smoothly isotopic to the identity but not symplectically.*

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- **A** *There exists an exotic symplectomorphism of the standard symplectic ball $\mathbb{B}^{2n}$.*

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Theorem (Drobnjak - U.)

For each $n \in \mathbb{N}$, at least one of the following statements is true.

- **A** *There exists an exotic symplectomorphism of the standard symplectic ball $\mathbb{B}^{2n}$.*
- **B** *Every free contact circle action on the standard contact sphere $\mathbb{S}^{2n-1}$ is topologically symmetric.*

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topological symmetry = a topological condition (to be defined shortly)

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$$(\forall k \in \mathbb{Z}) \quad \widetilde{H}_{m-k}(P; \mathbb{Z}_2) \cong \widetilde{H}_k(P; \mathbb{Z}_2).$$

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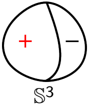

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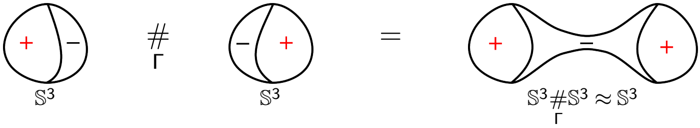

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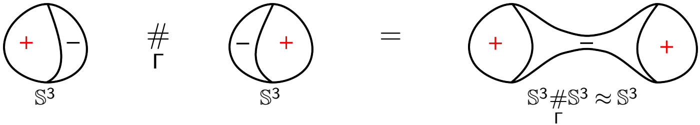

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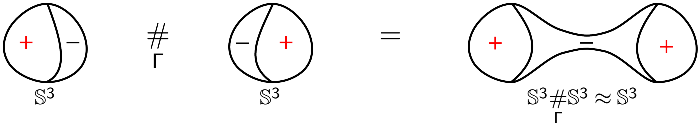

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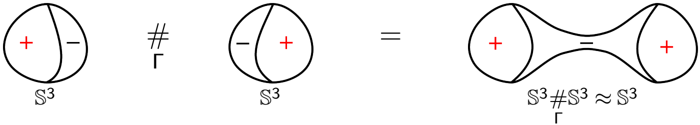

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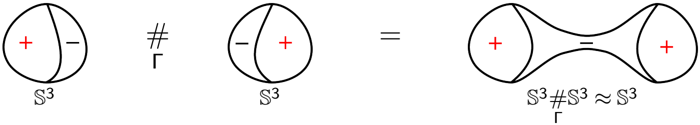

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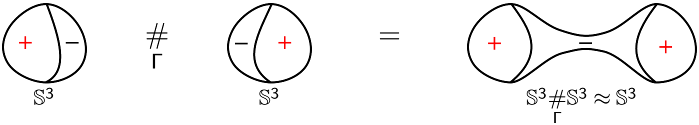

Gromov 1985 $\Longrightarrow$ no exotic symplectomorphisms on $\mathbb{B}^4$

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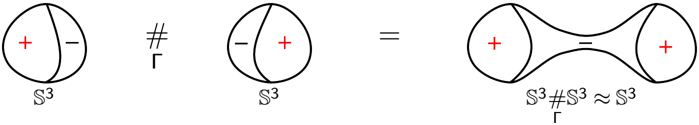

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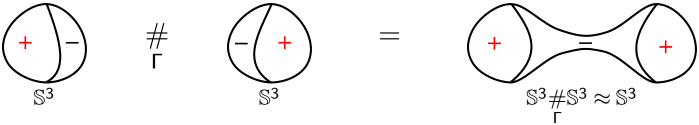

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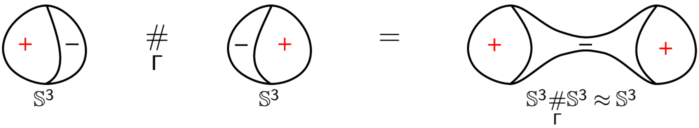

Gromov 1985	$\Longrightarrow$	no exotic symplectomorphisms on $\mathbb{B}^4$
	$\Longrightarrow$	$\mathbb{S}^3 \#_\Gamma \mathbb{S}^3$ is a non-standard contact sphere
Eliashberg 1989	$\Longrightarrow$	there is only one non-standard contact structure ξ_{ot} that is homotopic to the standard contact structure ξ_{st} on $\mathbb{S}^3$

Corollary

$$\left(\mathbb{S}^3, \xi_{st}\right) \#_\Gamma \left(\mathbb{S}^3, \xi_{st}\right) = \left(\mathbb{S}^3, \xi_{ot}\right)$$

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A free contact circle action $\varphi_t : \Sigma \to \Sigma$ with positive region $P \subset \Sigma$ is topologically symmetric with respect to W if there exists $m \in \mathbb{Z}$ such that

$$(\forall k \in \mathbb{Z}) \quad H_{m-k}(W, P; \mathbb{Z}_2) \cong H_k(W, P; \mathbb{Z}_2).$$

From loops of contactomorphisms to symplectomorphisms

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Biran-Giroux: $\exists$ a long exact sequence

$$\cdots \longrightarrow \pi_k \operatorname{Symp}_c W \longrightarrow \pi_k \operatorname{Symp} W \longrightarrow \pi_k \operatorname{Cont} \Sigma \xrightarrow{\Theta} \pi_{k-1} \operatorname{Symp}_c W \longrightarrow \cdots.$$

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Theorem (Drobnjak-U.)

Let W be a Liouville domain such that $c_1(W) = 0$. Let $\varphi_t : \partial W \to \partial W$ be a free contact circle action that is not topologically symmetric with respect to W. Then, $\Theta([\varphi_t])$ is a non-trivial element of $\pi_0 \operatorname{Symp}_c(W)$.

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Corollary

If a free contact circle action $\varphi_t : \Sigma \to \Sigma$ is not topologically symmetric with respect to some Liouville filling W of Σ, then φ_t gives rise to a non-contractible loop of contactomorphisms.

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$HF_*(h)$	independent of the choice of H does depend on the filling W

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- If f is autonomous, and if the flows of f and h commute, then

$$\mathcal{N}(F) : HF_*(h) \to HF_*(h - f),$$

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- $s \cdot h$ has no 1-periodic orbits for $s \in (-1, 0)$
- $\varepsilon - 1, -\varepsilon \in (-1, 0)$

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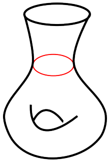

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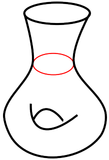

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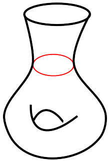

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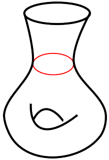

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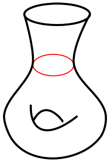

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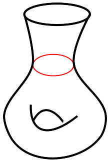

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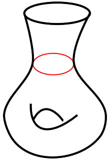

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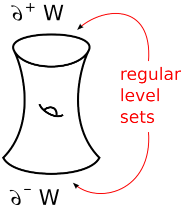

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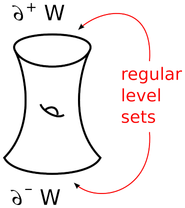

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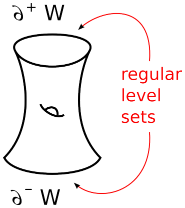

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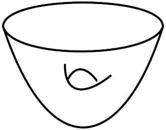

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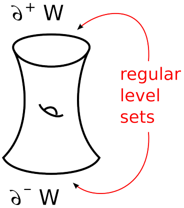
From Floer to Morse

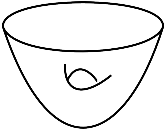
Floer homology for a C^2-small Hamiltonian reduces to Morse homology

$$h : \partial W \to \mathbb{R} \text{ constant} \implies HF_k(\varepsilon \cdot h) \cong \begin{cases} H_{k+n}(W, \partial W) & h > 0 \\ H_{k+n}(W) & h < 0 \end{cases}$$

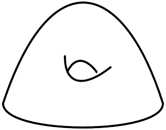


$\xrightarrow{HM_*} \quad H_*(W, \partial^- W)$

In particular,



$\xrightarrow{HM_*} H_*(W),$



$\xrightarrow{HM_*} H_*(W, \partial W).$

Morse theory on manifolds with boundary

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$h : \partial W \to \mathbb{R}$ a contact Hamiltonian with regular value 0

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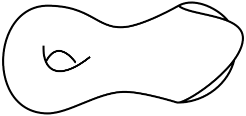

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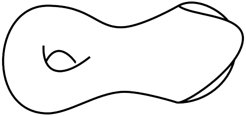

Claim: $H_k(W, P) \cong HM_k(-\varepsilon \cdot H)$

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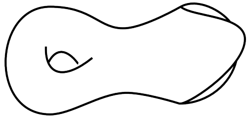

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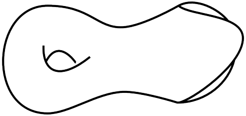

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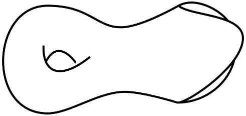
h not constant $\longrightarrow$ 

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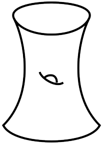
Idea: Reduce to the case of  !

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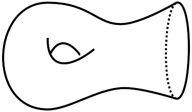
$W =$ 

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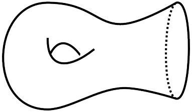
$W =$ 

Get rid of the (awkwardly positioned) boundary!

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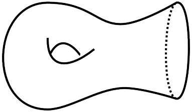
$W =$ 

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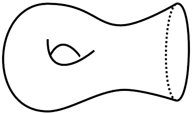
$W =$ 

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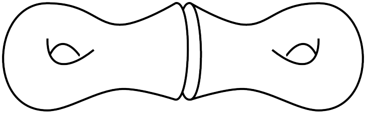

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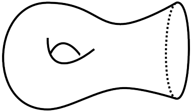
$W =$ 

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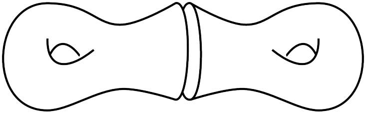

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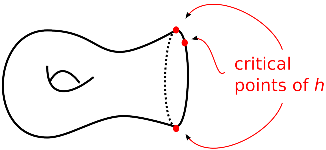
$W =$ 

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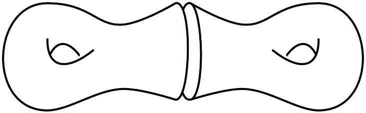

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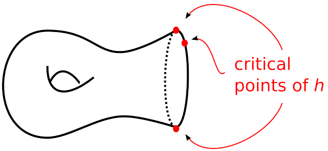
$W =$ 

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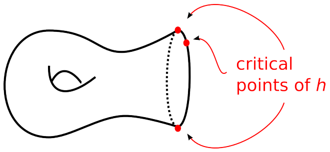
$W =$ 

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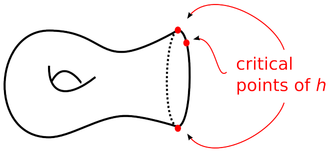
Extend W

Morse theory on manifolds with boundary

$W =$ 

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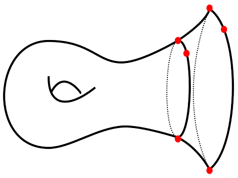

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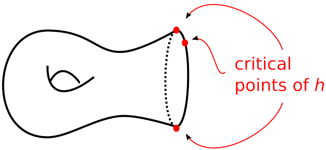
$W =$ 

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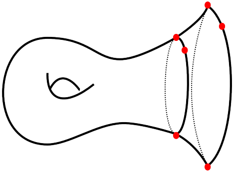

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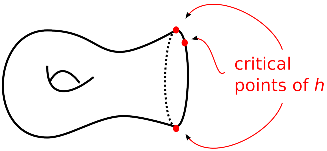
$W =$ 

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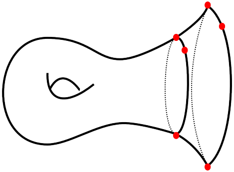

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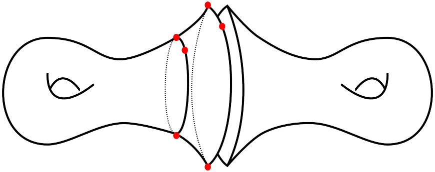

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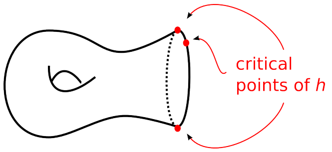
$W =$ 

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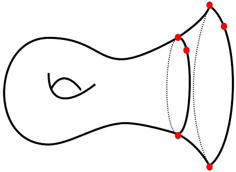

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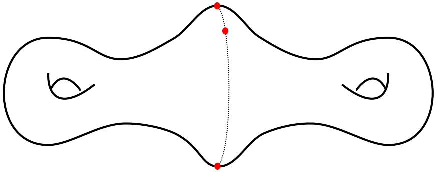

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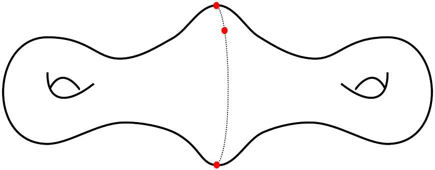
Cut-off the unwanted critical points



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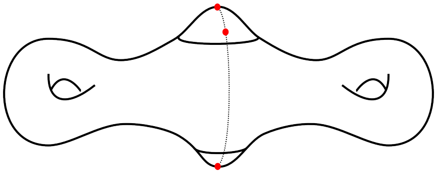

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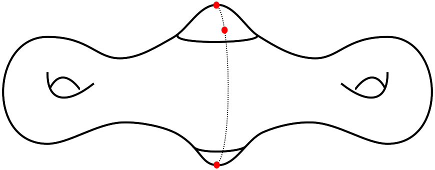

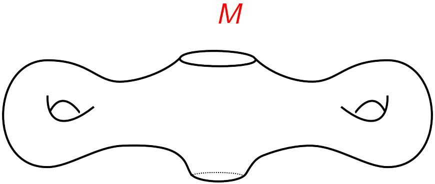

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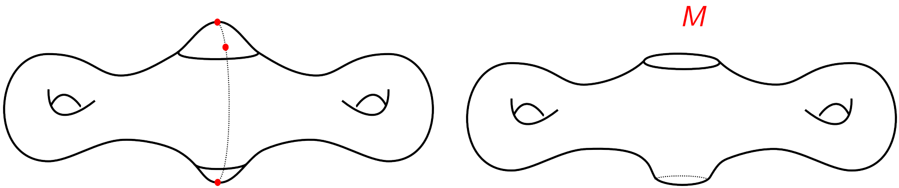

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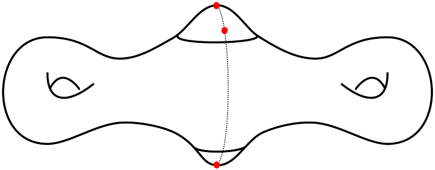

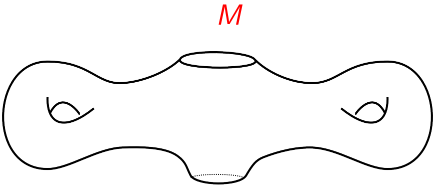

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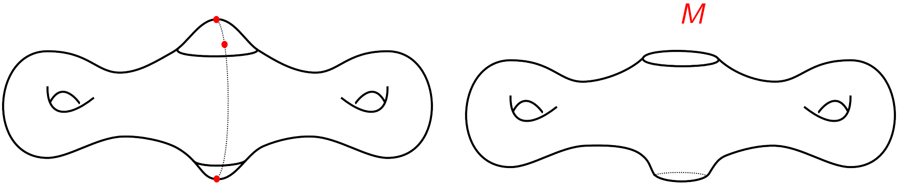

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Mayer-Vietoris (relative version)

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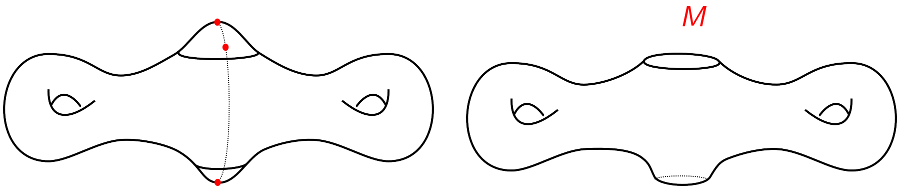

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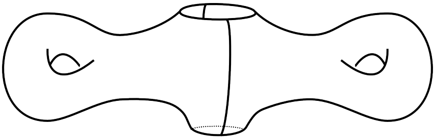

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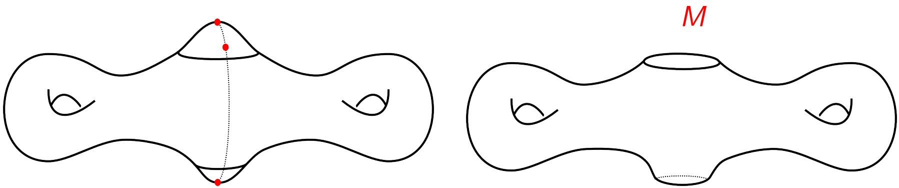

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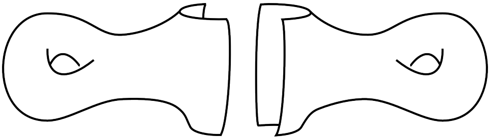

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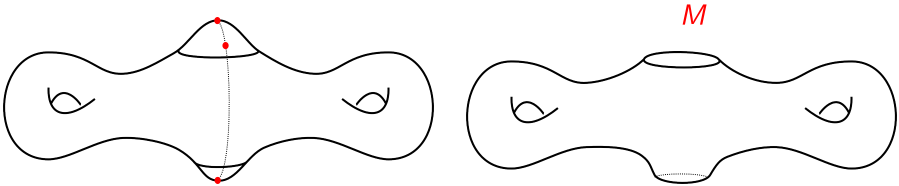

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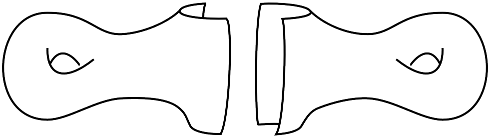

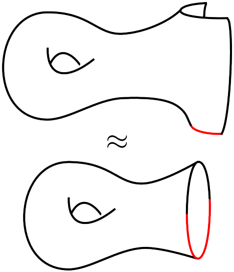

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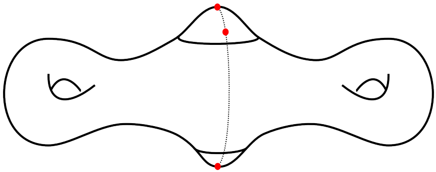

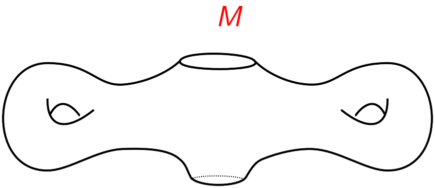

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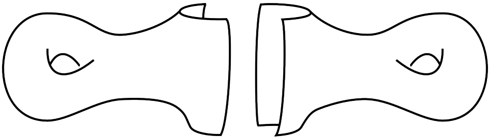

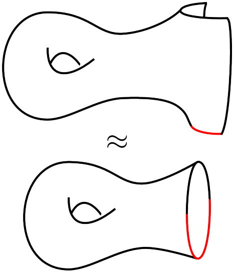

$\dim H_k(M, \partial^- M) = 2\dim H_k(W, P)$.

Thank you!