Reeb dynamics in dimension 3 and broken book decompositions

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Introduction

M orientable closed 3-manifold.

> **E. Giroux**
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> Every contact structure ξ on *M* is carried by a rational open book decomposition, *i.e.* for some contact form α defining ξ, the Reeb vector field R_α is tangent to the binding *K* and transverse to the pages $\pi^{-1}(t)$.
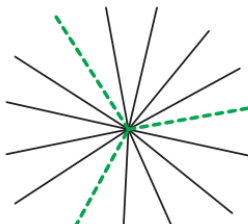
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Definition (Rational open book decomposition)

A rational open book decomposition of *M* is a pair (K, π) with *K* a link (the *binding*), $\pi : M \setminus K \to \mathbb{S}^1$ a fibration and $\pi^{-1}(t)$ is the interior of a compact surface (a *page*) whose boundary is *K*.

Introduction

Note: we drop orientations hypothesis along K and embeddedness of the pages along their boundary.

Definition

A Birkhoff section for R is a compact surface S with ∂S tangent to R, $int(S)$ embedded and transverse to R, and intersecting every orbit in backward and forward time.

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Question

Is every Reeb vector field carried by a rational open book? Or equivalently, does it have a Birkhoff section?

Dynamics of the Reeb vector field is then described by its first return map on a fiber (Poincaré's strategy).

Theorem 1

Every nondegenerate Reeb vector field R is carried by a broken book decomposition of M.

Results 1

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Every nondegenerate Reeb vector field R is carried by a broken book decomposition of M.

Theorem 2

Every nondegenerate Reeb vector field on M has either two or infinitely many periodic orbits (and two are possible only on the tight 3-sphere or lens spaces).

Note (Cristofaro-Gardiner, Hutchings, Pomerleano): In particular, if ξ is overtwisted, every nondegenerate Reeb vector field has infinitely many periodic orbits.

Results 2

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If M is not graphed (e.g. hyperbolic), every nondegenerate Reeb vector field on M has positive topological entropy. In particular, the number of its periodic orbits of period $< L$ grows exponentially with L.

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Theorem 4

Every strongly nondegenerate Reeb vector field without homoclinic orbit (and in particular with vanishing topological entropy) is carried by a rational open book decomposition: it has a Birkhoff section.

Previous results

- Every Reeb vector field has at least two periodic orbits (Taubes 2007 for one periodic orbit, Cristofaro-Gardiner and Hutchings 2016).

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- Results on positive entropy by Macarini-Schlenk, Alves, Alves-Colin-Honda (no nondegeneracy hypothesis).
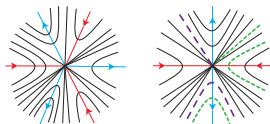
Broken books

Definition

A broken book decomposition of M is a pair $(K, \mathcal{F})$ where K is a link and

- $\mathcal{F}$ is a cooriented foliation of $M \setminus K$ whose leaves are properly embedded in $M \setminus K$ and become compact after adding some components of K;

- $K = K_r \sqcup K_b$; near K_r the foliation is transversally radial; near K_b the foliation is transversally
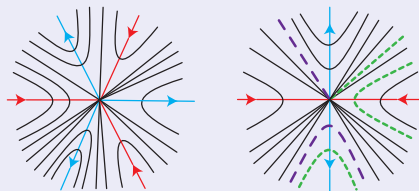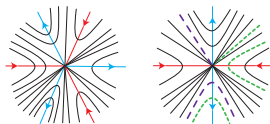
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A Reeb vector field is carried by a broken book $(K, \mathcal{F})$ if it is tangent to the binding K and transverse to the leaves of $\mathcal{F}$.

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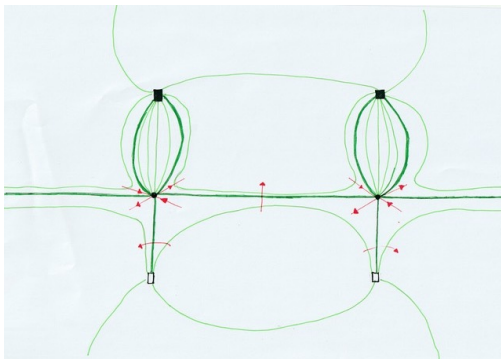## Definition

A Reeb vector field is carried by a broken book $(K, \mathcal{F})$ if it is tangent to the binding K and transverse to the leaves of $\mathcal{F}$.

Remarks

A periodic orbit of a nondegenerate R that belongs to K_r can be elliptic or hyperbolic with respect to the dynamics of the flow. A periodic orbit that belongs to K_b has to be dynamically hyperbolic.

- The nondegeneracy assumption implies that transversally near K_b there are exactly 4 sectors foliated by hyperbolas (right hand figure).
- If $K_b = \emptyset$, the broken book is a rational open book.

Broken books

Definition

A leaf of $\mathcal{F}$ is *rigid* (or *broken*) if it doesn't belongs to the interior of an $\mathbb{R}$-family of diffeomorphic leaves.

There are finitely many rigid pages. The complement of their union fibers over $\mathbb{R}$.

Sketch of the proof of Theorem 1

1. ECH (in particular the U-map) together with desingularizing the projected pseudoholomorphic curves, provide a section through every point $z \in M$.

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3. The complement of this system of sections fibers over $\mathbb{R}$, which gives the broken book structure.

Sketch of the proof of Theorem 1

Choosing an ECH class Γ with $U(\Gamma) \neq 0$, whose existence is granted *via* the isomorphism of ECH and Seiberg-Witten Floer homology (Taubes), let $\mathcal{P}$ be the finite set of periodic orbits of the Reeb vector field R of action less than $\mathcal{A}(\Gamma)$.

The main input from ECH-holomorphic curve theory is the following.

Lemma

For every z in $M \setminus \mathcal{P}$, there exists an embedded pseudo-holomorphic curve u in the symplectization $\mathbb{R} \times M$ asymptotic to periodic orbits of R in $\mathcal{P}$ and whose projection to M contains z in its interior.
If z belongs to $\mathcal{P}$, it is either in the interior of the projection of a curve or in a boundary component of its closure.

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The projection of such a pseudo-holomorphic curve to *M*

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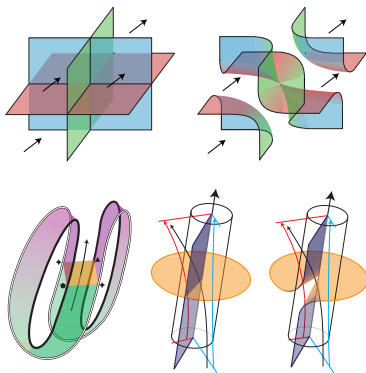
Getting rid of singularities

Surround each singularity *p* of the projected curve in *M* with a Reeb flow box $B = D_\epsilon \times [-1, 1]$, $p = ((0, 0), 0)$ such that:

- $\{pt\} \times [-1, 1]$ is tangent to the flow;
- the projected curve does not intersect $D_\epsilon \times \{\pm 1\}$.

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Desingularize (resolve intersections) outside of *B*. The new surface induces embedded circles on $(\partial D_\epsilon) \times [-1, 1]$, transverse to *R* (in particular not contractible in $(\partial D_\epsilon) \times [-1, 1]$).

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Desingularize (resolve intersections) outside of B. The new surface induces embedded circles on $(\partial D_\epsilon) \times [-1, 1]$, transverse to R (in particular not contractible in $(\partial D_\epsilon) \times [-1, 1]$).

Extend the surface inside B by disks transverse to the flow.

Sketch of the proof of Theorem 1

Lemma

There exists a finite number of sections with disjoint interiors, intersecting all orbits of R, and such that:

- *if an orbit of R is not asymptotically linking these sections, it has to converge to one of their boundary components, which is a hyperbolic orbit of the flow;*

- *in this case, each one of the four (or two) sectors delimited by the stable and unstable manifolds of the hyperbolic orbit is intersected by at least one section having the orbit as a boundary component.*

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> - *in this case, each one of the four (or two) sectors delimited by the stable and unstable manifolds of the hyperbolic orbit is intersected by at least one section having the orbit as a boundary component.*

When we cut *M* open along these sections, the remaining part is essentially an *I*-bundle, i.e. a product on which the broken book is taken to be the product foliation.

Sketch of the proof of Theorem 2

Theorem

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The key are the hyperbolic periodic orbits in K_b of a supporting broken book.

If $K_b = \emptyset$, the broken book decomposition is a rational open book decomposition.

Let S be a page. If S is a disk or an annulus, then R has either 2 or infinitely many periodic orbits (Franks). If not, we prove that the first return map to S has infinitely many periodic points, a generalization of a result of Franks and Handel in the flux zero, possibly not isotopic to the identity, case (the result for homeomorphisms was established by Le Calvez and Sambarino).

Sketch of the proof of Theorem 2

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Strongly nondegenerate: The stable and unstable manifolds of hyperbolic orbits intersect transversally.

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The stable (unstable) manifold of an orbit $k \in K_b$ intersects the unstable (stable) manifold of another orbit in K_b.

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The stable (unstable) manifold of an orbit $k \in K_b$ intersects the unstable (stable) manifold of another orbit in K_b.

Proof of the Lemma. Let $W^s(k)$ be half of the stable manifold of k and assume it does not intersect the unstable manifold of any orbit in K_b. It is an injectively immersed cylinder, that has to intersect one rigid page P infinitely many times. This intersection is then a infinite collection C of embedded circles in P.

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Lemma

The stable (unstable) manifold of an orbit $k \in K_b$ intersects the unstable (stable) manifold of another orbit in K_b.

Continuation of the Proof of the Lemma. There are finitely many of these circles that bound a disc in P. Indeed, take two, c_1 and c_2, bounding two discs D_1 and D_2 in P.

Let $A \subset W^s(k)$ be the annulus bounded by c_1 and c_2, then

$$0 = \int_{D_1 \cup A \cup D_2} d\alpha = \int_{D_1} d\alpha - \int_{D_2} d\alpha$$

implying that the discs have the same $d\alpha$-area. They have to be disjoint and since $\overline{P}$ is compact, we conclude.

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Continuation of the Proof of the Lemma. Hence there are infinitely many circles not bounding a disc, and we can find 2 of them bounding an annulus A' in P that does not contain other circles in C. Let now c_3 and c_4 be the bounding circles and $A'' \subset W^s(k)$ be the annulus bounded by c_3 and c_4. Then

$$0 = \int_{A' \cup A''} d\alpha = \int_{A'} d\alpha,$$

a contradiction.

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If all the intersections in it (or at least one) are transversal, we obtain a homoclinic intersection providing infinitely many periodic orbits of R and even positive topological entropy.

The proof without the strongly nondegenerate hypothesis requires more argumentation:

1. We first prove the result if all the stable/unstable manifolds coincide pairwise. We conclude that in this case there are infinitely many periodic orbits.

2. We then prove that a stable/unstable manifold that does not coincide with another unstable/stable manifold, contains a crossing intersection.

About Theorem 3

Essentially, if the entropy is vanishing, there is no broken component of the binding (true if strongly nondegenerate), and there is a supporting rational open book. The monodromy cannot have a pseudo-Anosov component, and so is decomposed into periodic ones: the manifold is graphed.

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Some extra:

- If the Reeb vf is strongly nondegenerate and if there is only one broken component in the binding, a construction by Fried allows to obtain a supporting rational open book.
- We in fact can prove that our results are valid for an open set containing nondegenerate Reeb vector fields.

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- We in fact can prove that our results are valid for an open set containing nondegenerate Reeb vector fields.

Conjecture

Every nondegenerate Reeb vector field is carried by a rational open book decomposition, i.e. has a Birkhoff section.