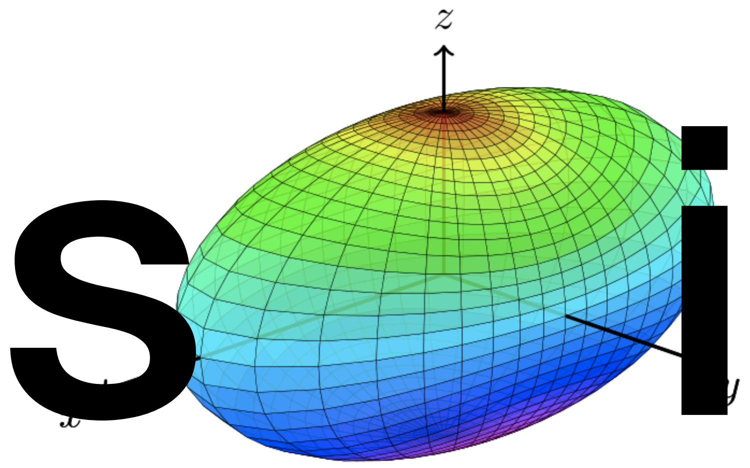
Welcome to ellipsid day!

Now: Ana Rita Pires - Infinite staircases and reflexive polygons
(mostly on joint work with Dan Cristofaro-Gardiner, Tara Holm, Alessia Mandini)

In 1h45m: Roger Casals - Sharp ellipsoid embeddings and toric mutations
(on joint work with Renato Vianna)

In 4h45m: Dan Cristofaro-Gardiner - Obstructing infinite staircases
(partly on joint work with Dan Cristofaro-Gardiner, Tara Holm, Alessia Mandini)

2v1 [math.SG] 27 Apr 2020

INFINITE STAIRCASES AND REFLEXIVE POLYGONS

DAN CRISTOFARO-GARDINER, TARA S. HOLM, ALESSIA MANDINI, AND ANA RITA PIRES

Abstract. We explore the question of when an infin
ellipsoid embedding function of a convex toric domain. F
four dimensions, we conjecture a complete answer to thi
that are distinguished by the fact that their moment p
better when infinite staircases occur, we prove that any in
accumulation point given as the solution to an explicit qu
a uniform proof of the existence of infinite staircases for o
the first, we use recursive families of almost toric fibrati
into closed symplectic manifolds. In order to establish
domains, we prove a result of potentially independent in
embeds into a closed symplectic toric four-manifold if an
a corresponding convex toric domain. For the second
convex lattice paths that provide obstructions to embedd
of Usher, who finds infinite families of infinite staircases

Contents

Sharp Ellipsoid Embeddings and Toric Mutations

ROGER CASALS
RENATO VIANNA

ABSTRACT: This article introduces a new method to construct volume-filling symplectic embeddings of 4-dimensional ellipsoids by employing polytope mutations in toric and almost-toric varieties. The construction uniformly recovers the sharp sequences for the Fibonacci Staircase of McDuff-Schlenk, the Pell Staircase of Frenkel-Müller and the Cristofaro-Gardiner-Kleinman's Staircase, and adds new infinite sequences of sharp ellipsoid embeddings. In addition, we initiate the study of symplectic tropical curves for almost-toric fibrations and emphasize the connection to quiver combinatorics.

1 Introduction

The central novel contribution of the a
infinite staircases for symplectic ellips
explanation for the sharp embeddings
Staircase [52, 53] and the Frenkel-Mü
Cristofaro-Gardiner-Kleinman [12]. I
(X,ω) with a sharp infinite staircase in
turally all such domains with this prop
the first part of the article, presented in

The manuscript also develops a new u
tions, incorporating the works of M. S

Special eccentricities of rational four-dimensional ellipsoids

Dan Cristofaro-Gardiner

April 29, 2020

Abstract

A striking result of McDuff and Schlenk asserts that in determining when a four-dimensional symplectic ellipsoid can be symplectically embedded into a four-dimensional symplectic ball, the answer is governed by an "infinite staircase" determined by the odd-index Fibonacci numbers and the Golden Mean. Here we study embeddings of one four-dimensional symplectic ellipsoid into another, and we show that if the target is rational, then the infinite staircase phenomenon found by McDuff and Schlenk is quite rare. Specifically, in the ratio-

THE EMBEDDING CAPACITY OF 4-DIMENSIONAL SYMPLECTIC ELLIPSOIDS

DUSA MCDUFF AND FELIX SCHLENK

ABSTRACT. This paper calculates the function $c(a)$ whose value at a is the infimum of the size of a ball that contains a symplectic image of the ellipsoid $E(1,a)$. (Here $a \geq 1$ is the ratio of the area of the large axis to that of the smaller axis.) The structure of the graph of $c(a)$ is surprisingly rich. The volume constraint implies

$$c(a) := c_{B^4}(a) := \inf\left\{\lambda | E(1,a) \hookrightarrow B^4(\lambda)\right\}$$

A **symplectic embedding** is a map $\psi : (M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$ such that $\psi^*\omega_2 = \omega_1$.

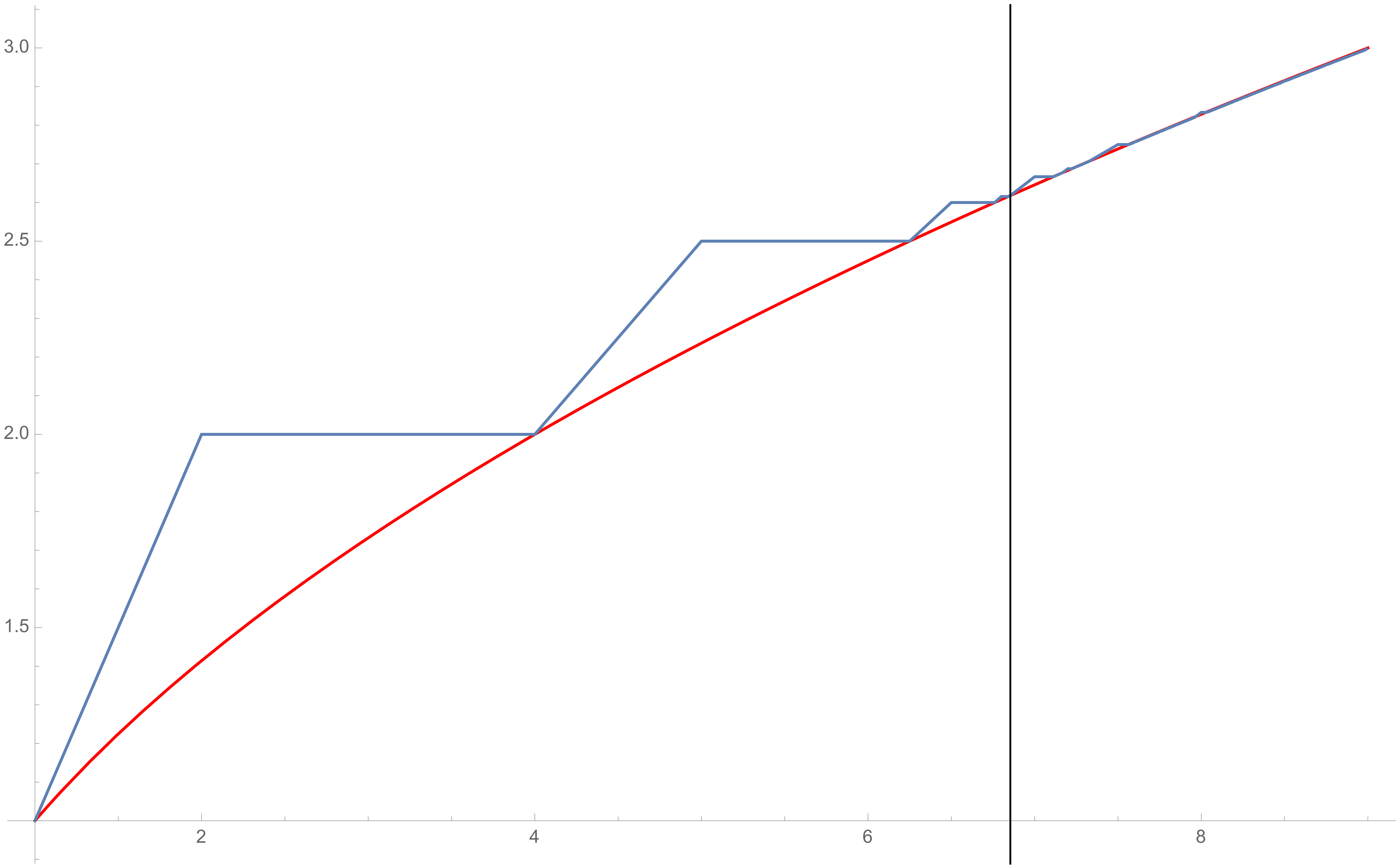
$$E(a_1, a_2) := \left\{(z_1, z_2) \in \mathbb{C}^2 | \pi\left(\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2}\right) < 1\right\}$$

$$B^4(\lambda) := E(\lambda, \lambda)$$

Ellipsoid embedding function of the 4-ball

$$c_{B^4}(a) = \inf\left\{\lambda \mid E(1,a) \hookrightarrow B^4(\lambda)\right\} \geq \sqrt{a}$$

Volume constraint



Q: Why look at this function?

- Gromov non-squeezing: A ball embeds symplectically into a cylinder if and only if it embeds via inclusion.
- Gromov width of a symplectic manifold M: What is the size of the biggest ball which embeds symplectically into M?
- Packing stability (Biran, Buse-Hind): there is no obstruction beyond volume to embedding a collection of small enough balls into (some) M.
- Hofer conjecture on ellipsoids (McDuff): E(a,b) embeds into E(c,d) if and only if …..(a certain combinatorial relation between a,b,c,d)…..

Q: Is this “infinite staircase” behaviour a characteristic of this function or a characteristic of the ball?

- $c_X(a) = \inf\{\lambda | E(1,a) \hookrightarrow \lambda X\}$, where $\lambda(X,\omega) = (X, \lambda\omega)$
- What does c_X(a) look like for other targets X?

> In general, c_X(a) is piecewise linear when not equal to the volume constraint.
> We say that it has an **infinite staircase** if it has infinitely many non-smooth points.

- Does it always an infinite staircase? If yes, WHY? If not, when does it and when does it not, and WHY?
- Other work on infinite staircases since then:
  - X = polydisk D(1)xD(1) — Frenkel-Muller
  - X = E(1,1), E(1,2), E(2,3) — Cristofaro-Gardiner—Kleinman
  - X = certain polydisks D(1)xD(b) — Usher
  - X = certain Hirzebruch surfaces — Bertozzi, Holm, Maw, McDuff, Mwakyoma, P., Weiler
  - full embeddings for X = ball, polydisk, and more — Casals - Vianna

Theorem: (Cristofaro-Gardiner, Holm, Mandini, P.) [Staircase obstruction theorem]

Let X be a closed toric symplectic manifold (or more generally a convex toric domain with finite blowup vector).
If the ellipsoid embedding function $c_X(a)$ has an infinite staircase then it accumulates at a_0, a real solution of the quadratic equation

$$a^2 - \left(\frac{\mathrm{per}^2}{\mathrm{vol}} - 2\right) a + 1 = 0.$$

Furthermore, at a_0 the ellipsoid embedding function touches the volume curve:

$$c_X(a_0) = \sqrt{\frac{a_0}{\mathrm{vol}}}.$$

1. What are convex toric domains and how do they generalize closed toric symplectic manifolds in this context?
2. Why does the staircase have to accumulate at a finite point (rather than going off to infinity)?
3. This equation usually has two solutions, what about the other one?
4. What are per and vol?
5. What is the “volume curve”?

How good is this as an obstruction to the existence of infinite staircases?

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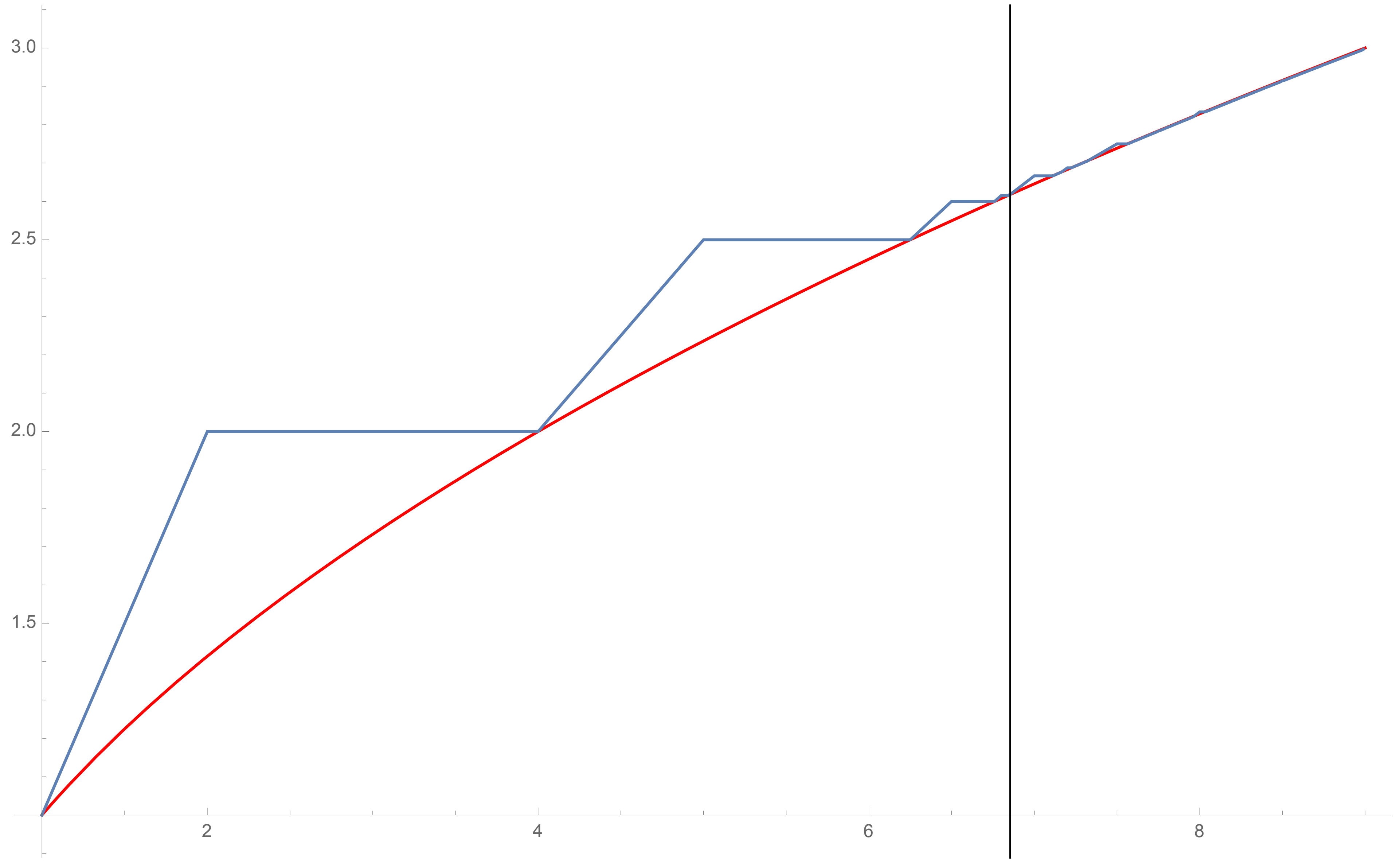
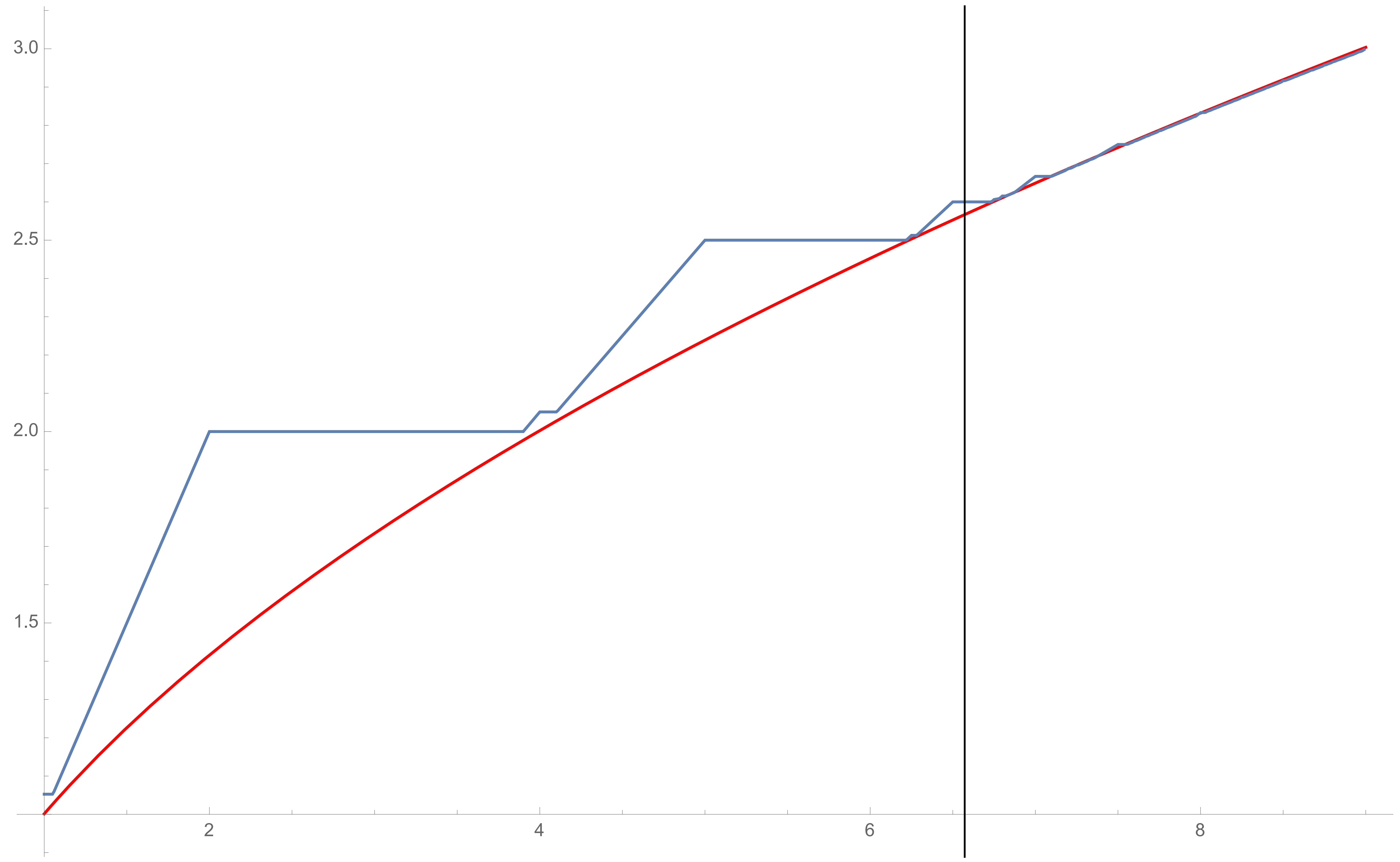


(1) = ball



(1;0.05)

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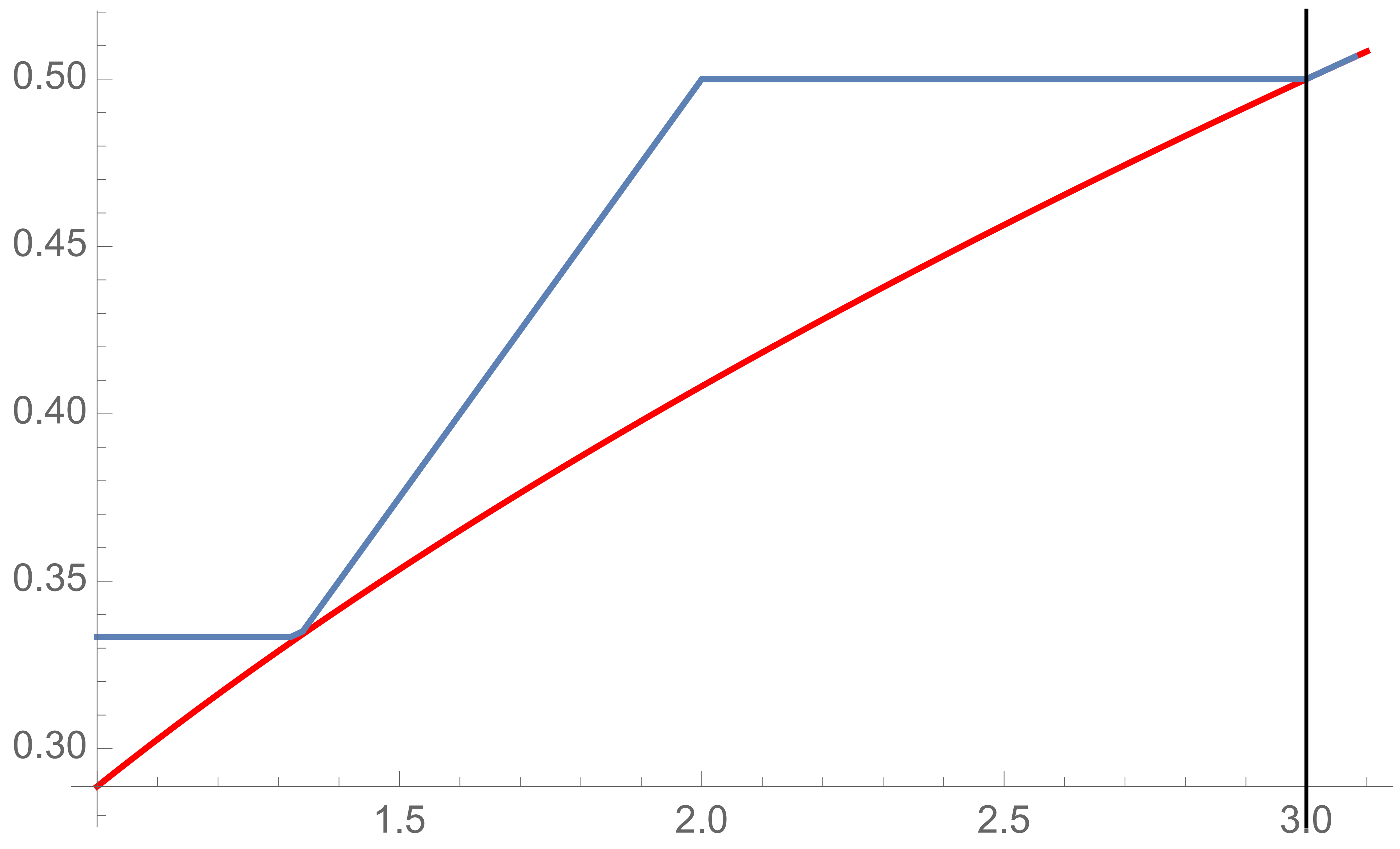
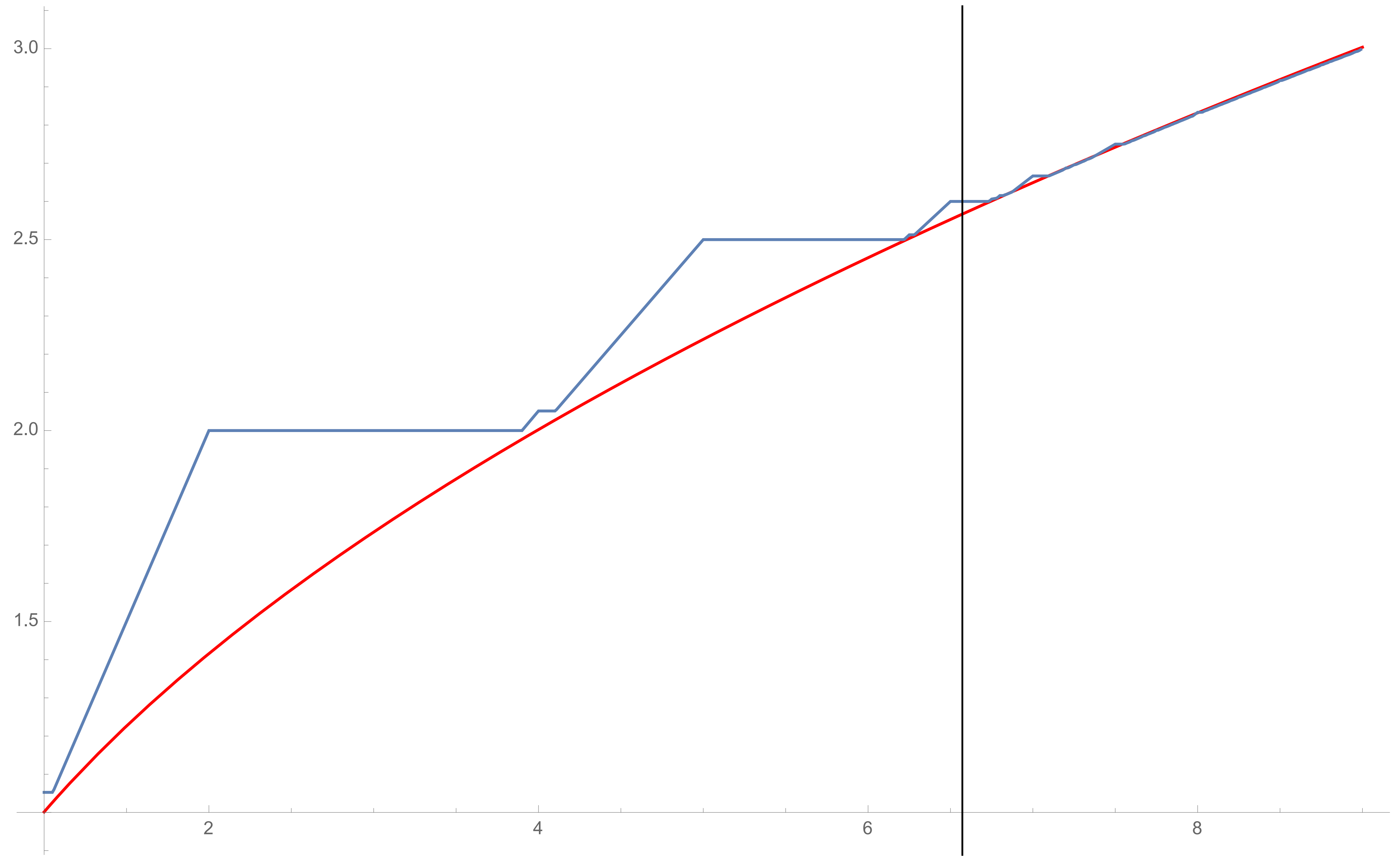


E(4,3)

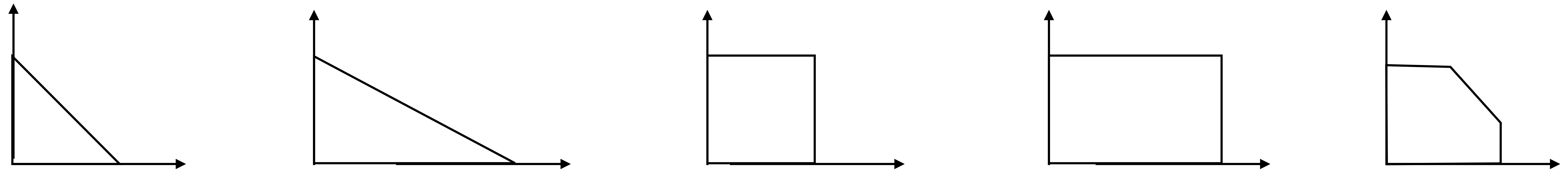


(1;0.05)

1. What are convex toric domains and how do they generalize closed toric symplectic manifolds?

Let $\mu : \mathbb{C}^2 \longrightarrow \mathbb{R}^2$ be given by $(z_1, z_2) \mapsto \pi(|z_1|^2, |z_2|^2)$.

Convex toric domain: for a convex region $\Omega \subset \mathbb{R}^2_{\geq 0}$, define $X_\Omega = \mu^{-1}(\Omega)$.



- A convex toric domain can be described in terms of a blowup vector: (b; b1, b2, …..).
- Karshon, Kessler, Pinsonnault: Any closed toric symplectic 4-manifold is symplectomorphic to CP2 or CP1xCP1 blown-up a finite number of times.
- Symplectomorphic manifolds have the same ellipsoid embedding function.

Theorem: (Cristofaro-Gardiner, Holm, Mandini, P.)

Let $\Omega \subset \mathbb{R}^2_{\geq 0}$ be a convex region that is also a Delzant polygon for a closed toric symplectic manifold M. Then

$$E(a,b) \hookrightarrow M \iff E(a,b) \hookrightarrow X_\Omega.$$

Theorem: (Cristofaro-Gardiner, Holm, Mandini, P.) [Staircase obstruction theorem]

Let X be a closed toric symplectic manifold (or more generally a convex toric domain with finite blowup vector).
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3. A quadratic equation usually has two solutions, what about the other one?
4. What are per and vol?
5. What is the "volume curve"?

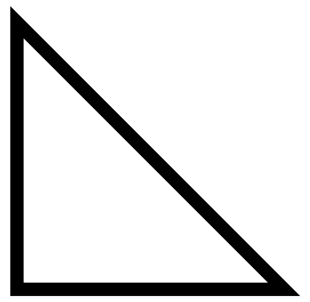
2. McDuff; Cristofaro-Gardiner: for X a convex toric domain, $E(1,a) \hookrightarrow X \iff \sqcup_i B(a_i) \hookrightarrow X$
Bute, Hind, Opshtein: all closed symplectic 4-manifolds have the packing stability property, i.e., there are no obstructions beyond volume to embedding a collection of sufficiently small balls.

3. The solutions are a0 and 1/a0. There is an involution of c_X(a) about a=1 because E(1,1/a) is essentially a rescaling of E(1,a).
We only consider c_X(a) with a>1.

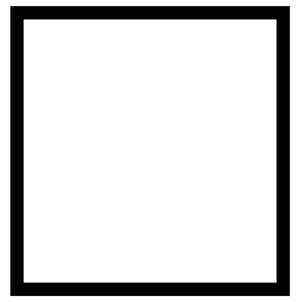
4. If X_Ω has moment image Ω and blowup vector $(b; b_1, \ldots, b_n)$, then
per $=$ affine perimeter of $\Omega = 3b - \sum b_i$
vol $= 2 \times$ area of $\Omega = b^2 - \sum b_i^2$

5. $E(1,a) \hookrightarrow tX \implies \mathrm{volume}(E(1,a)) \leq \mathrm{volume}(tX) \implies a \leq t^2\mathrm{vol} \implies t \geq \sqrt{\frac{a}{\mathrm{vol}}}$

A beautiful part of this story is when the blowup sizes are all rational.
More generally / using scaling, we look at $X = (b; b_1, \ldots, b_n)$ with $b, b_1, \ldots, b_n \in \mathbb{N}$.



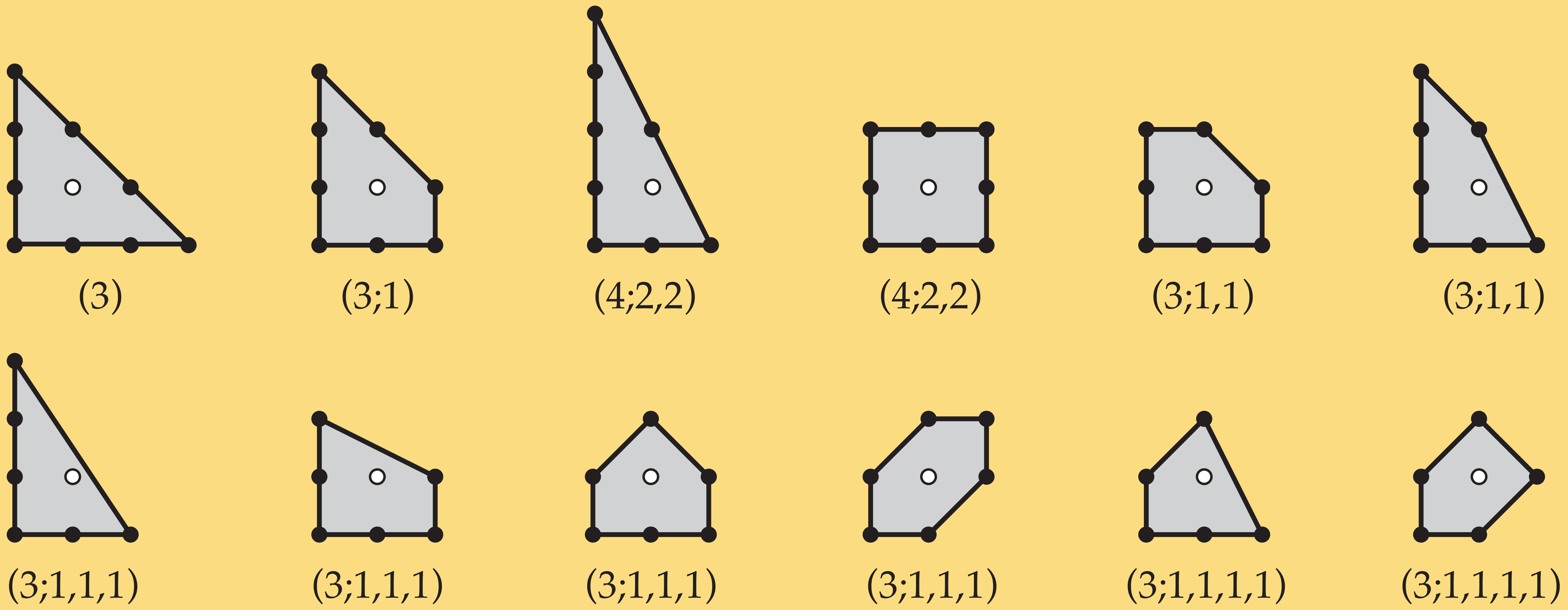
$\mathbb{C}P^2$/ball



$\mathbb{C}P^1 \times \mathbb{C}P^1$/polydisk

Theorem: (Cristofaro-Gardiner, Holm, Mandini, P.)

There is an infinite staircase in the ellipsoid embedding function $c_X(a)$ for the following convex toric domains:



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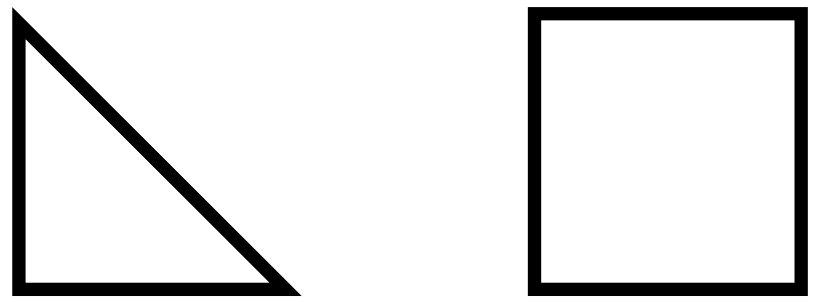

$\mathbb{C}P^2$/ball $\mathbb{C}P^1 \times \mathbb{C}P^1$/polydisk

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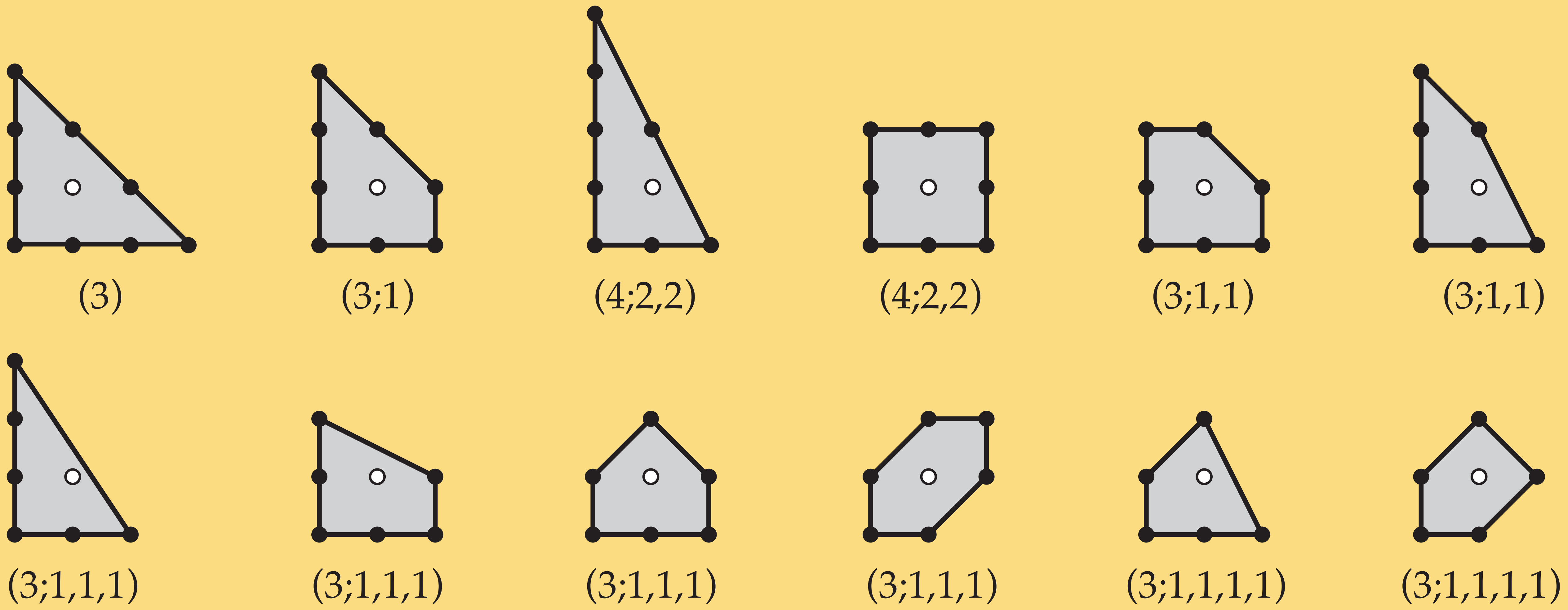

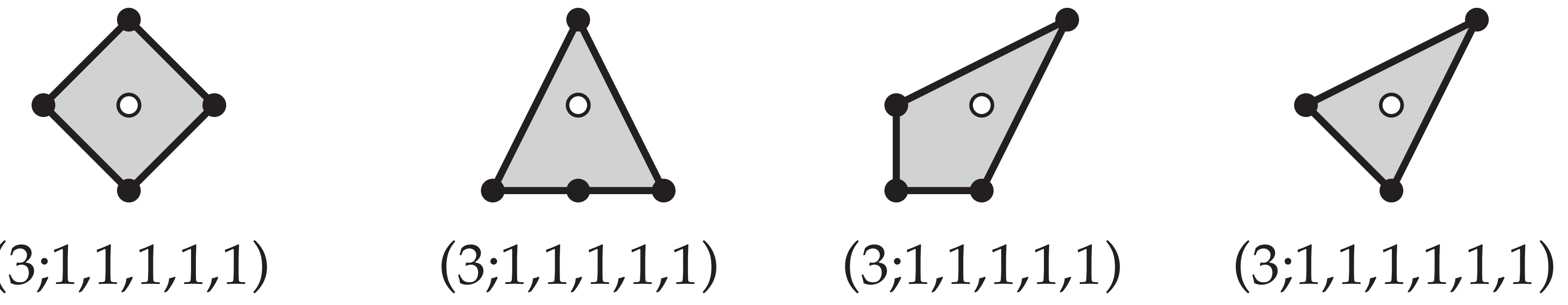

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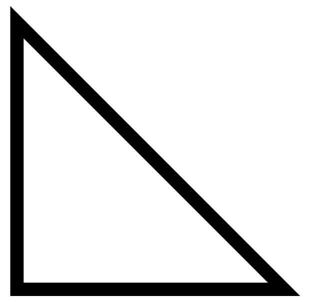
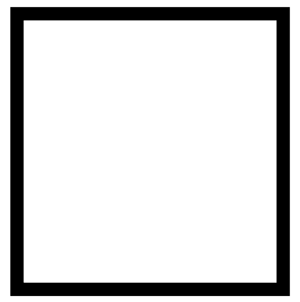


$\mathbb{C}P^2$/ball



$\mathbb{C}P^1 \times \mathbb{C}P^1$/polydisk

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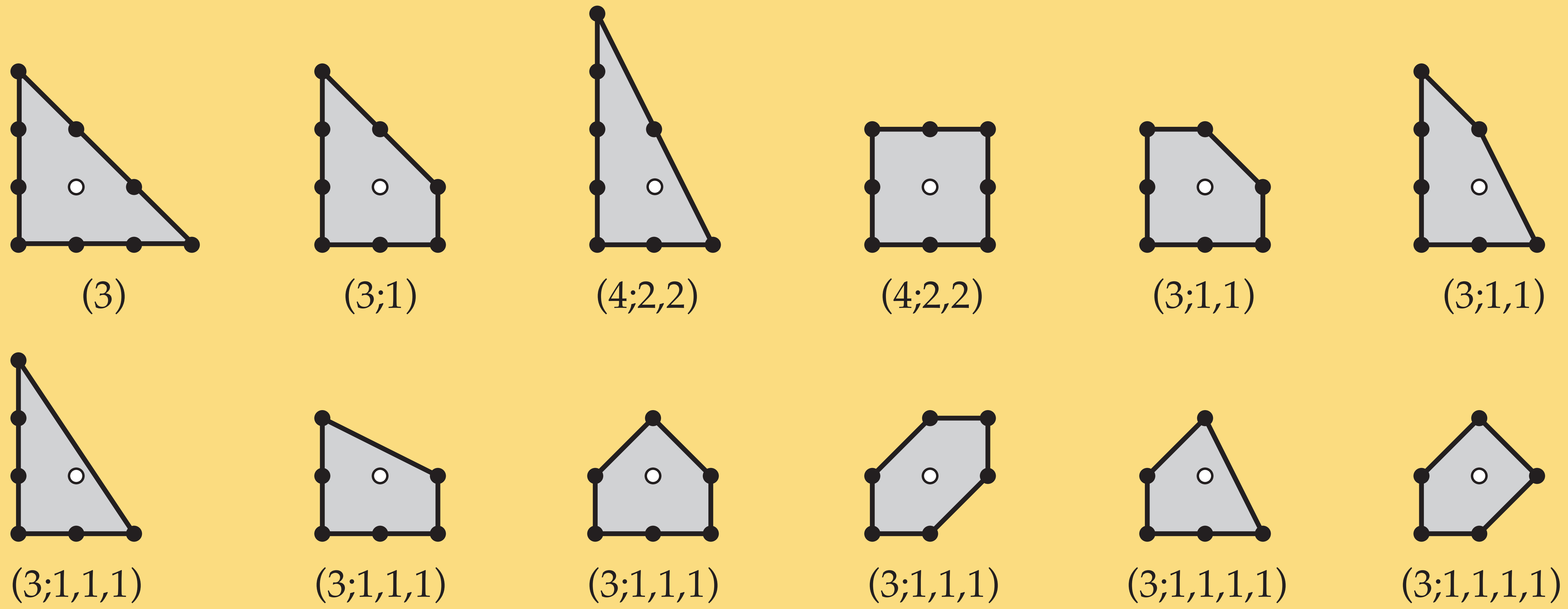
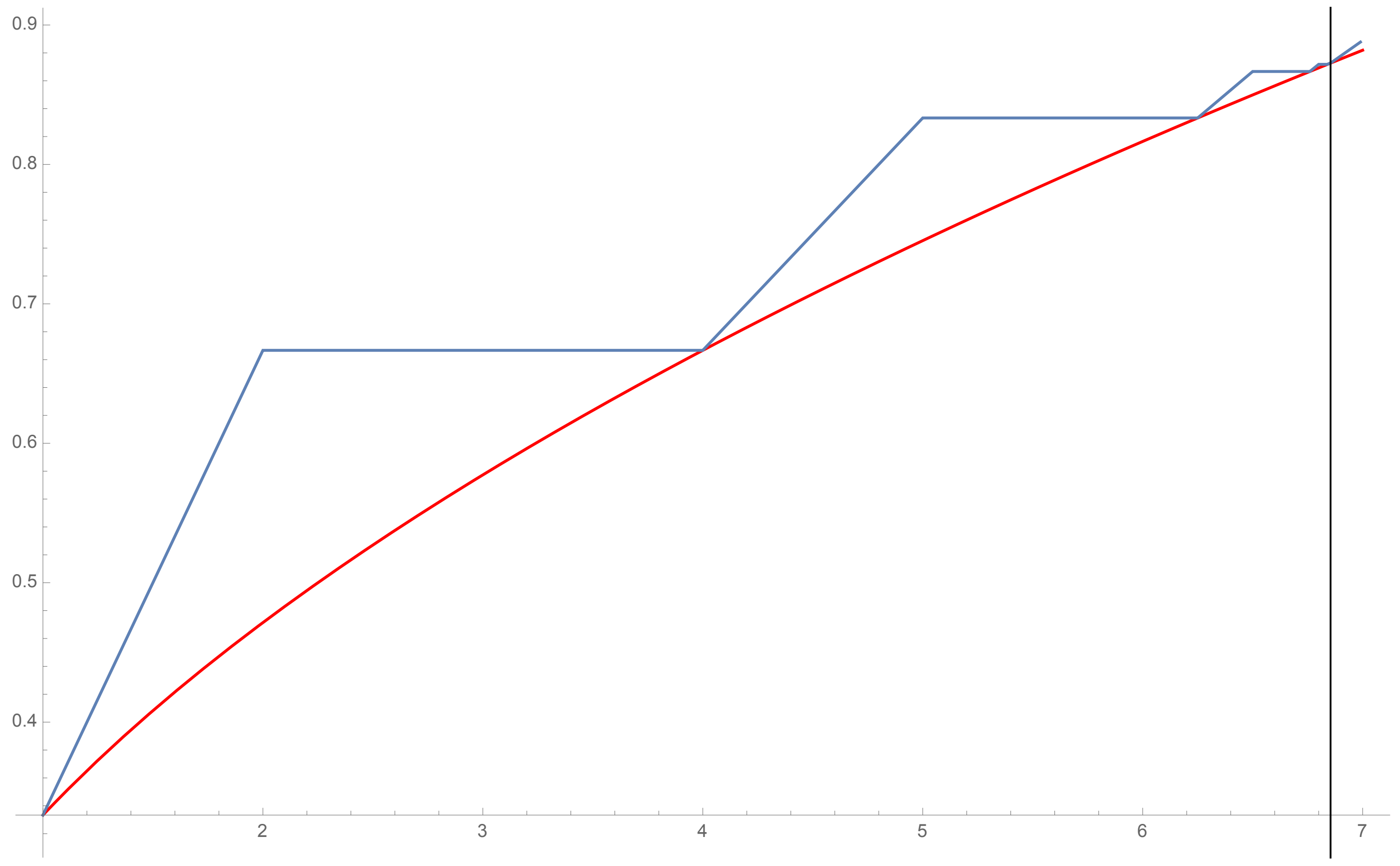


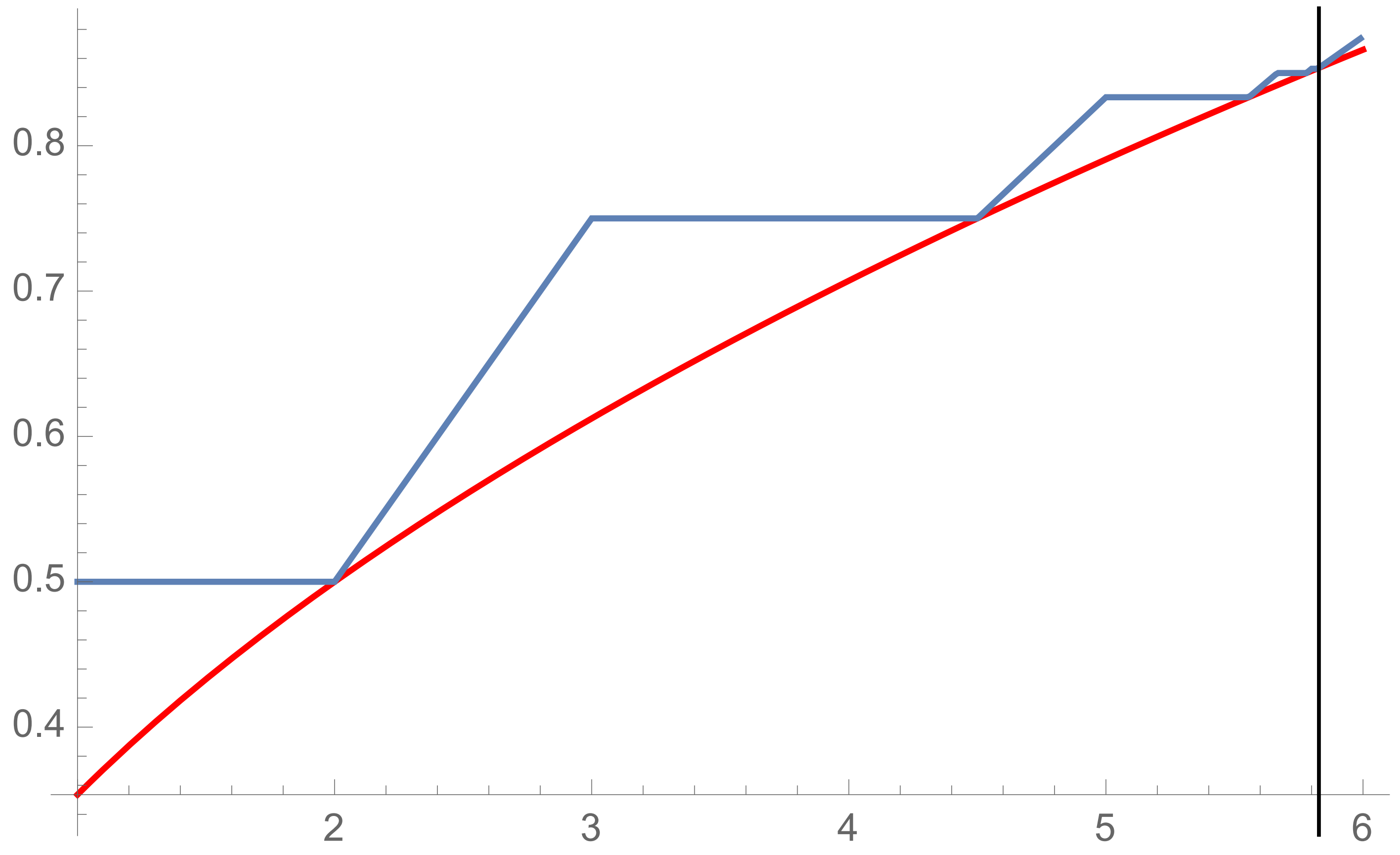
Conjecture: If $c_X(a)$ has an infinite staircase, then the moment polygon of X is a reflexive polygon.

In particular, the 12 examples above are the only (rational) ones with infinite staircases (up to scaling).

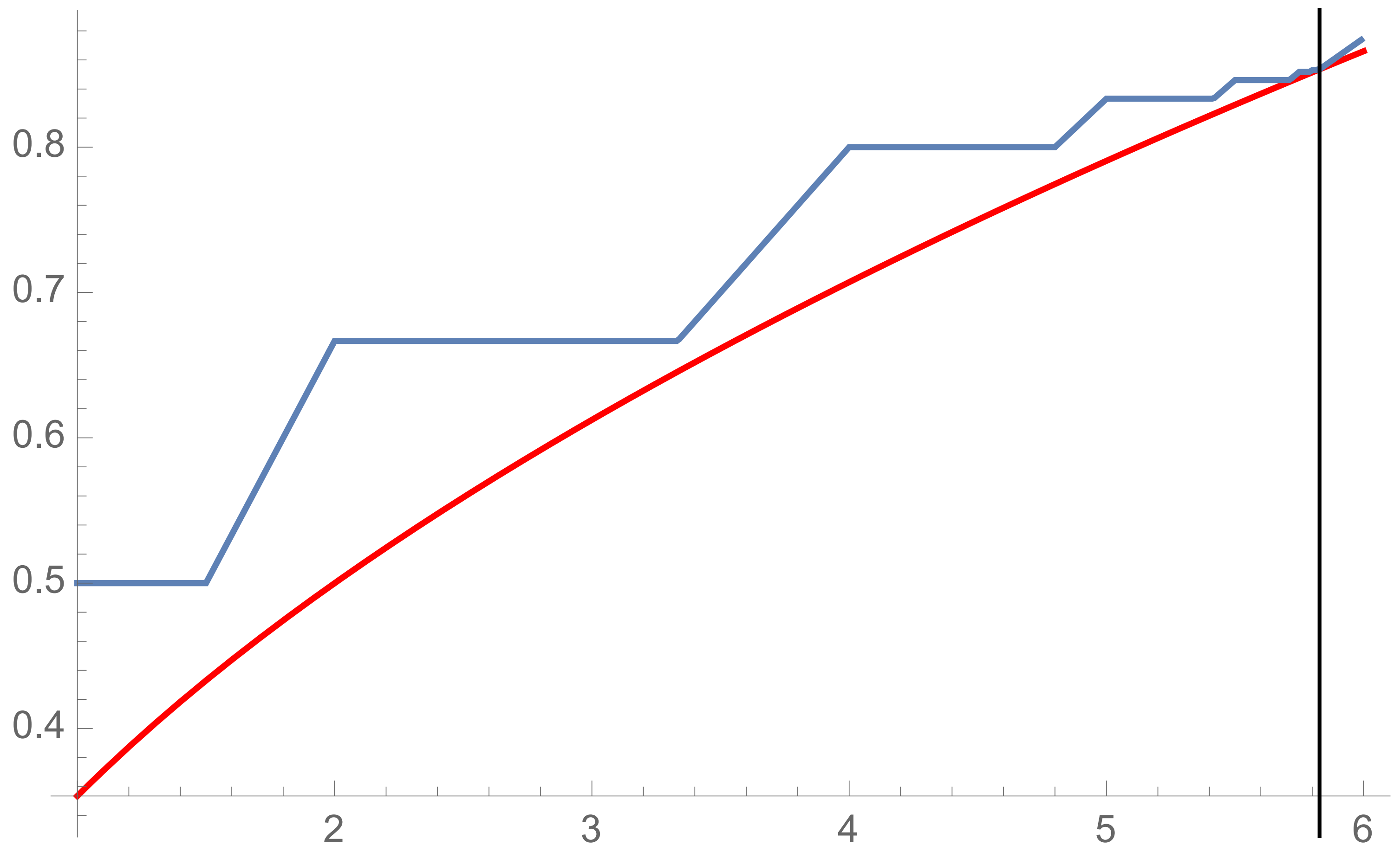
What do these staircases look like?



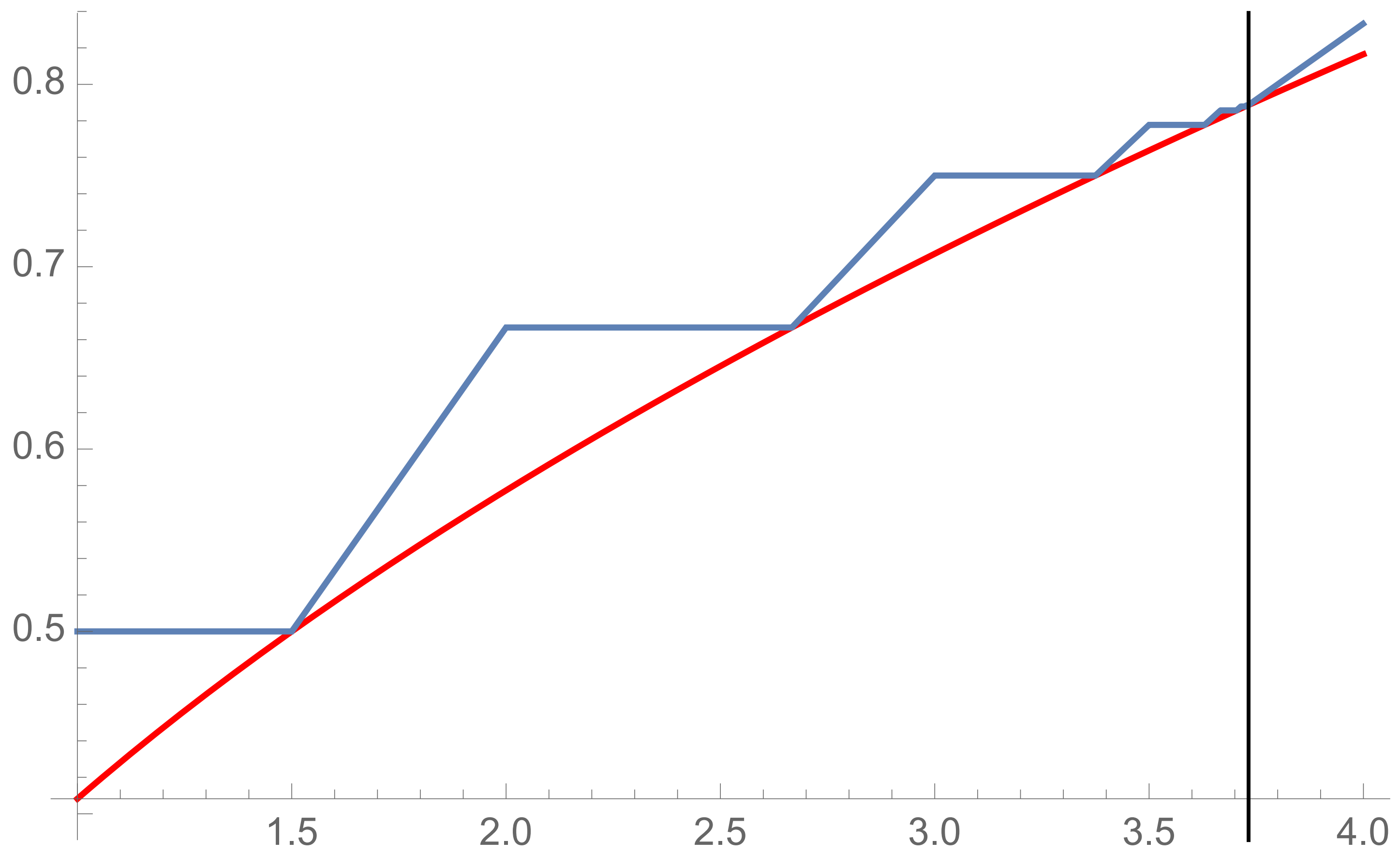
(3)
ball



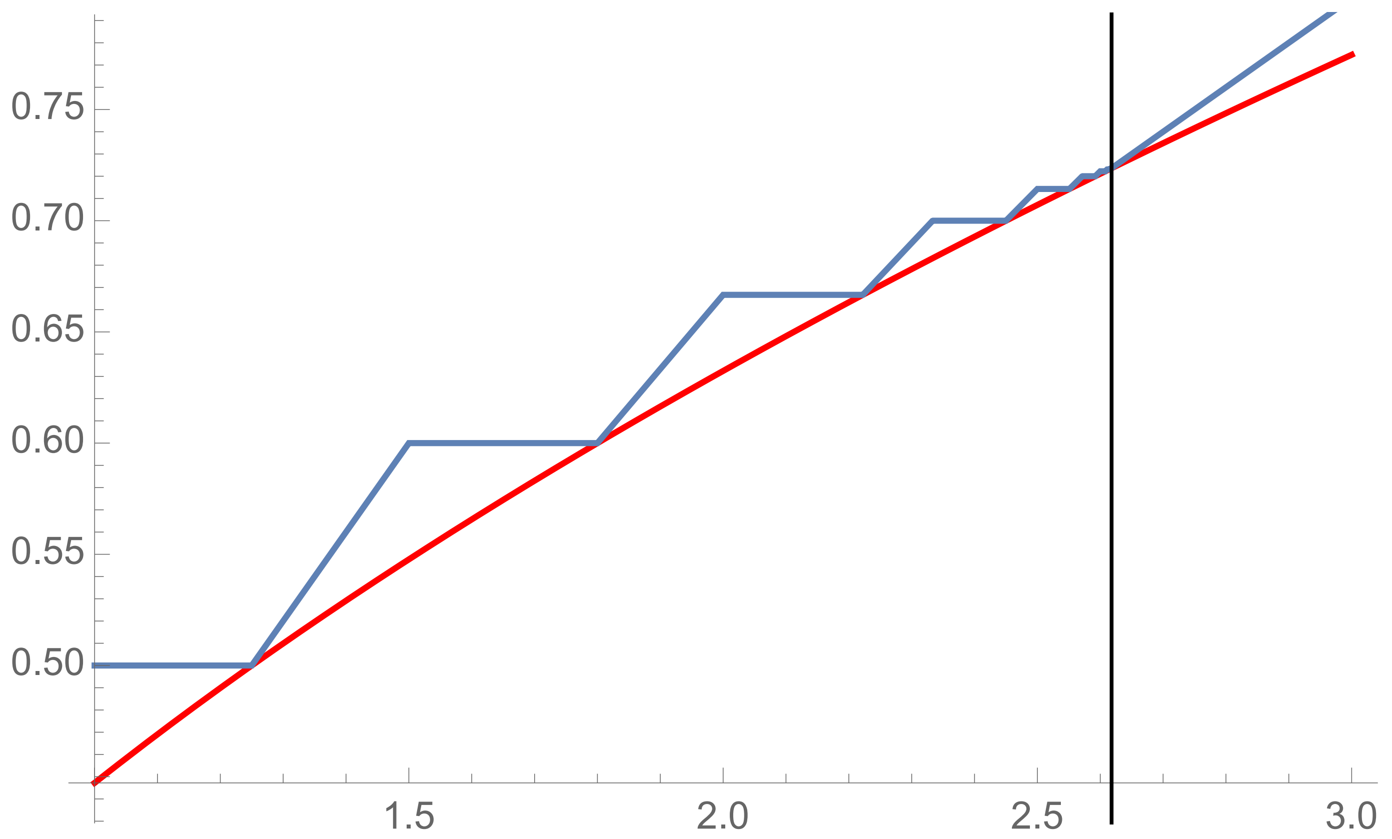
(4;2,2)
polydisk



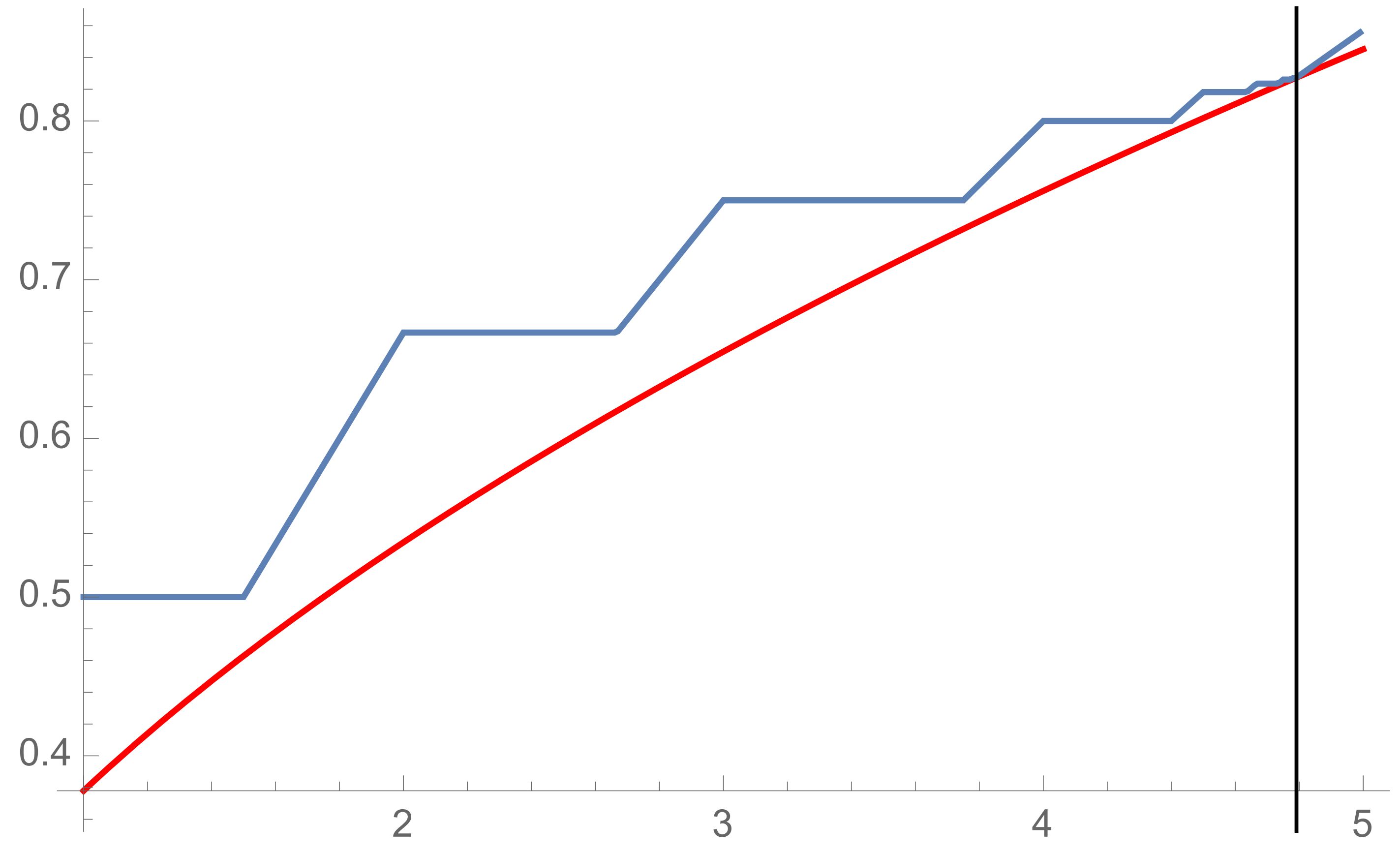
(3;1)



(3;1,1,1)
E(2,3)



(3;1,1,1,1)

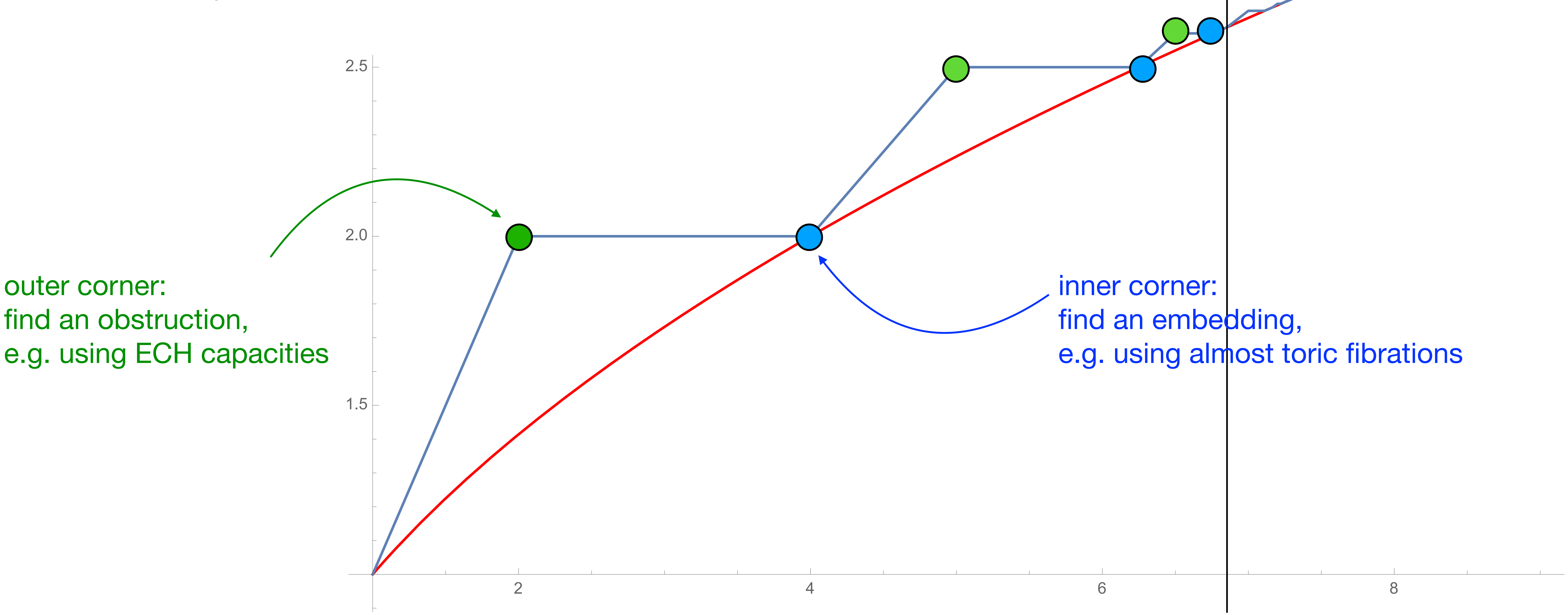


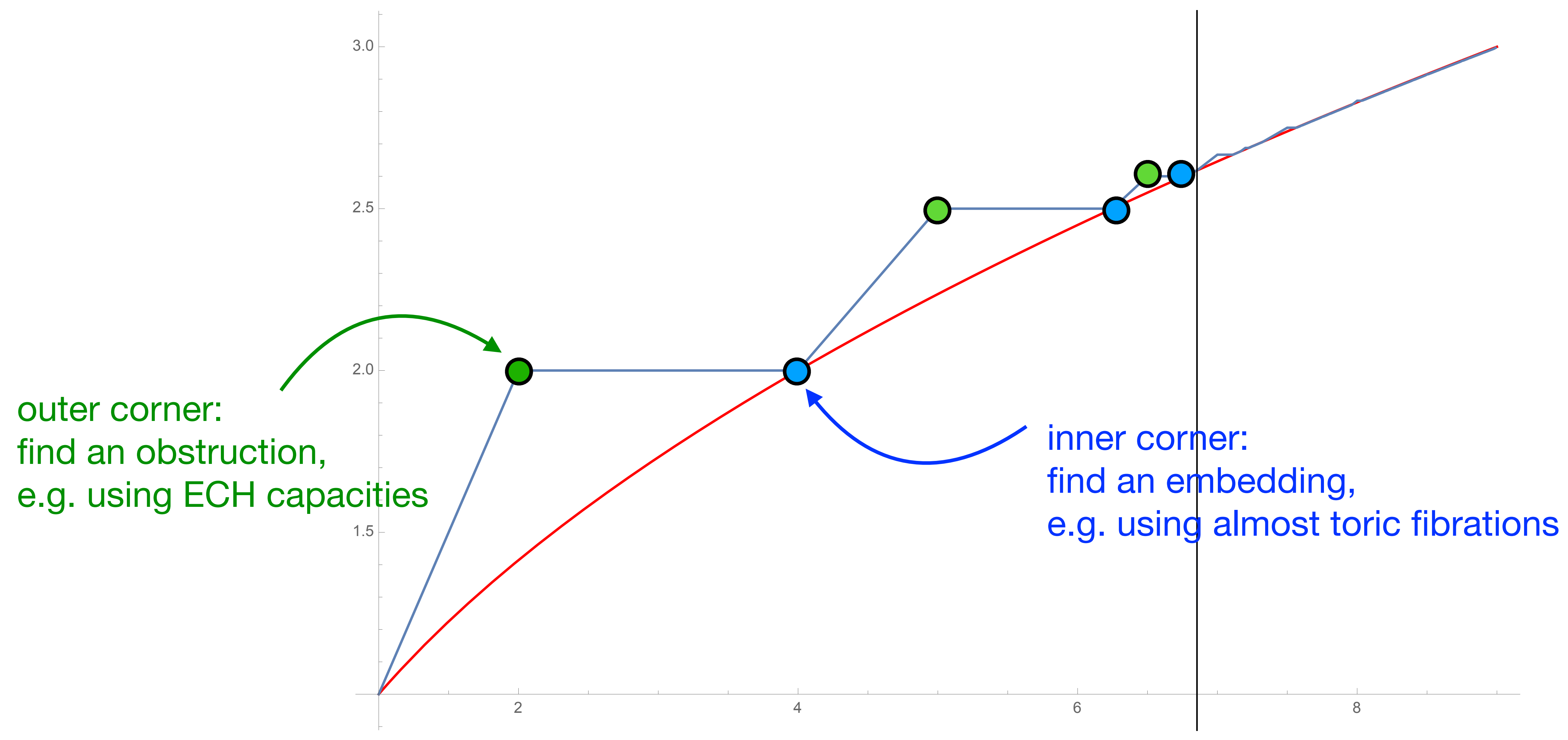
(3;1,1)

How does one prove that an infinite staircase exists?

For a convex toric domain with finite blowup vector, the ellipsoid embedding function is:

- continuous
- non-decreasing
- has the following scaling property: $c_X(t\,a) \le t\,c_X(a)$ for all $t \ge 1$
- equal to the volume curve for a>>1
- piecewise linear when not equal to the volume constraint





[Cristofaro-Gardiner] In the problem of embedding a concave toric domain into a convex toric domain, ECH capacities are sharp.

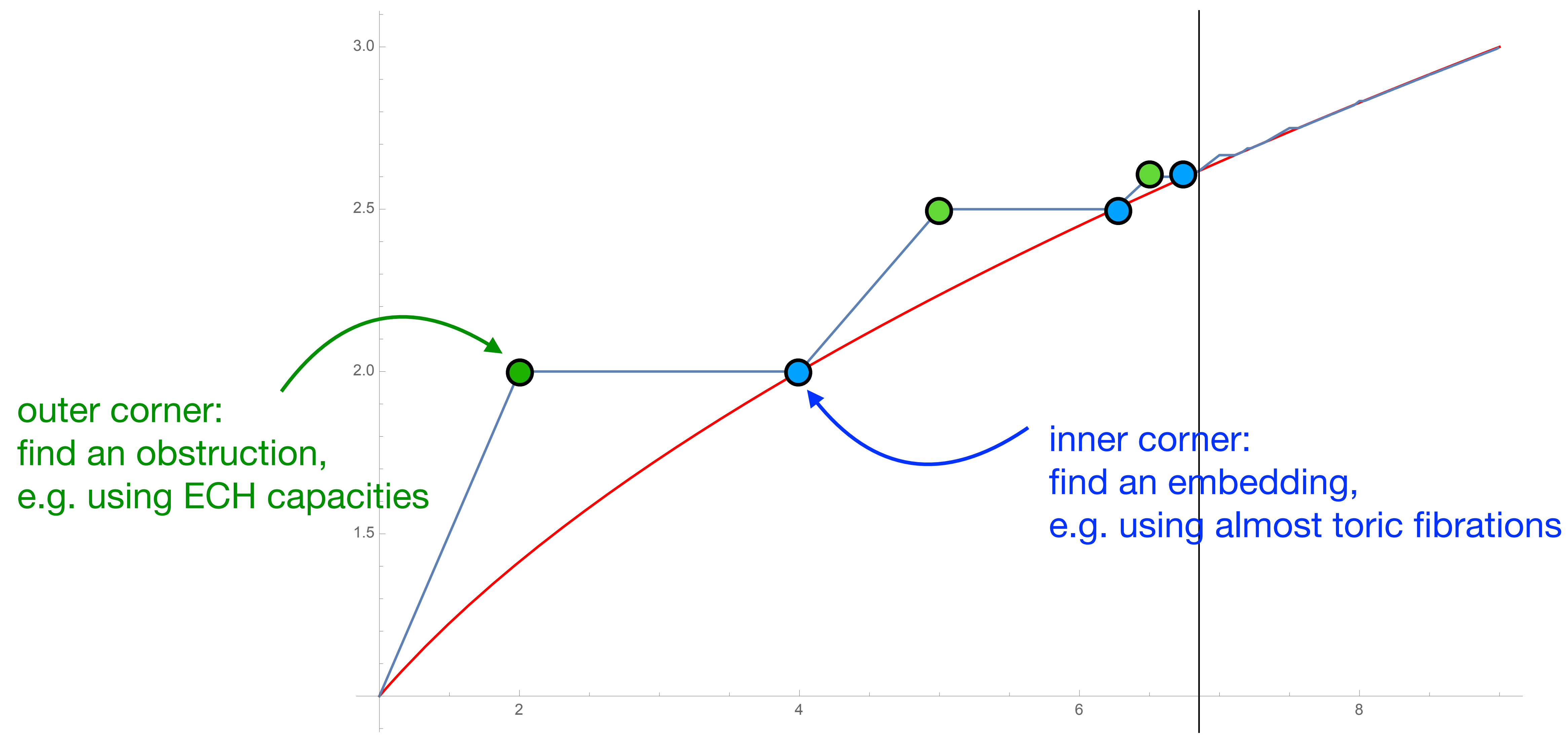
Therefore: $$E(1,a) \hookrightarrow \lambda X \iff C_k\left(E(1,a)\right) \le C_k\left(\lambda X\right) \text{ for all } k \in \mathbb{N}$$

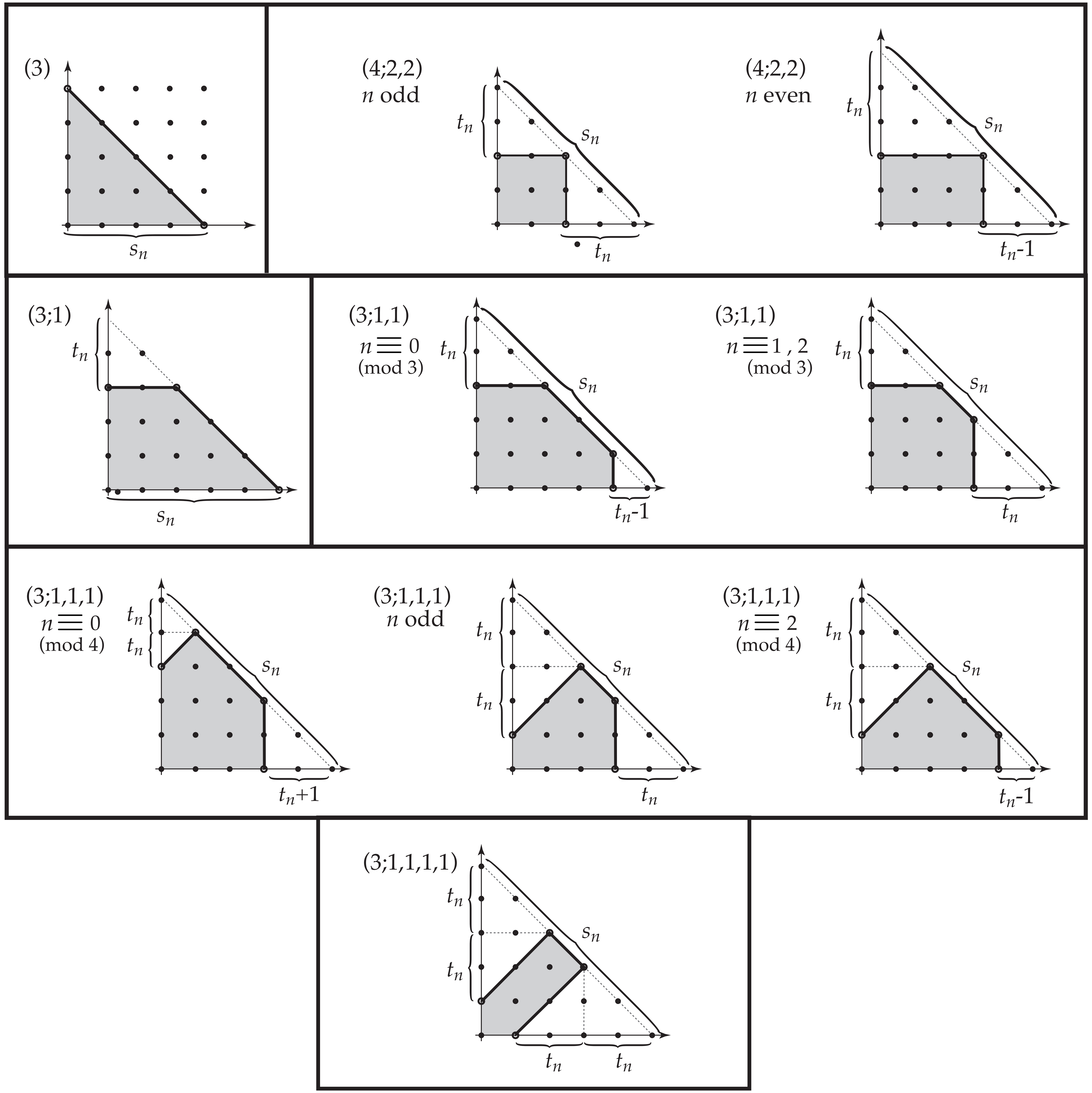
$$\iff \lambda \ge \frac{C_k\left(E(1,a)\right)}{C_k\left(X\right)} \text{ for all } k \in \mathbb{N}$$

Therefore: $$c_X(a) = \sup_{k\in\mathbb{N}} \frac{C_k(E(1,a))}{C_k(X)}$$

easy

hard





[Cristofaro-Gardiner] In the problem of embedding a concave tc

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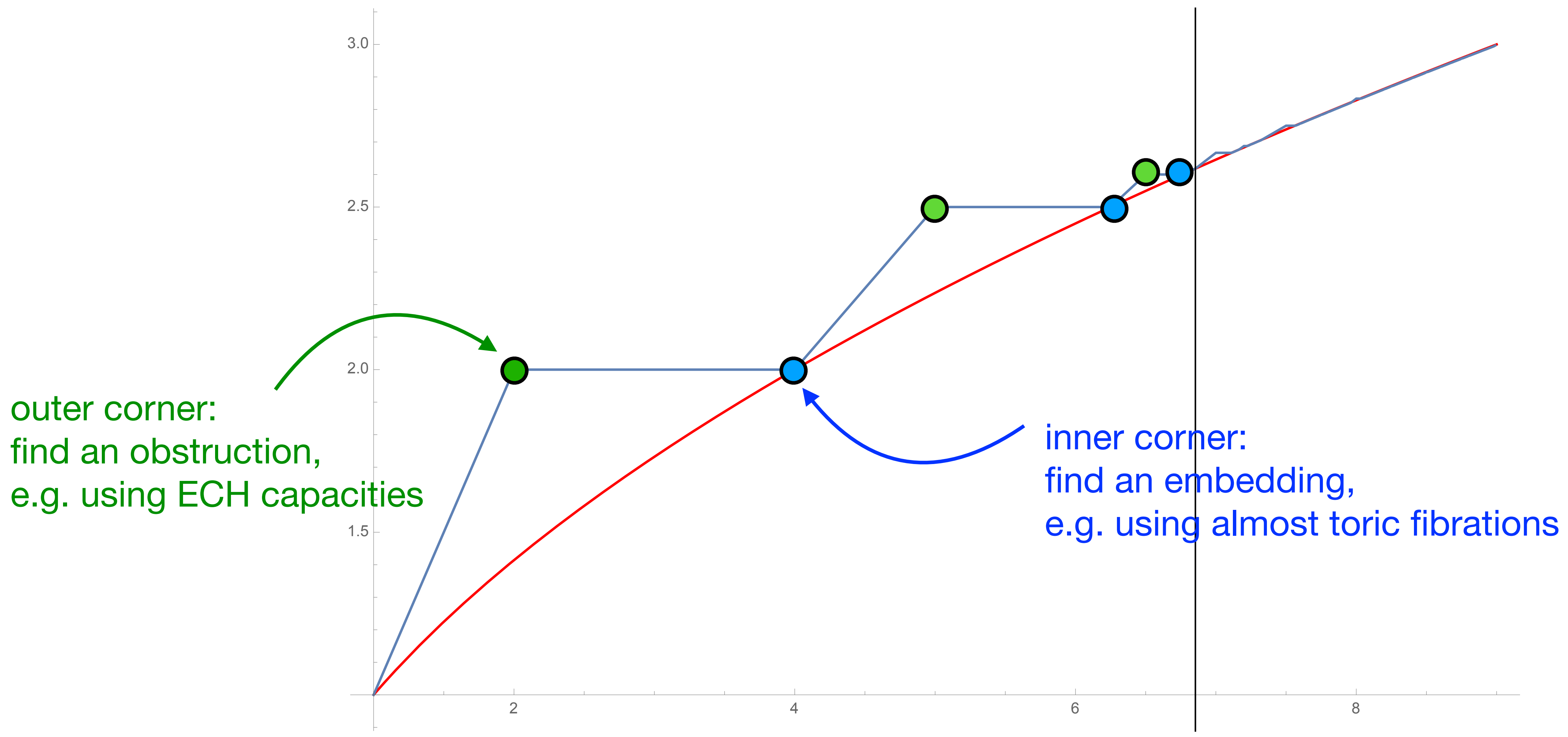
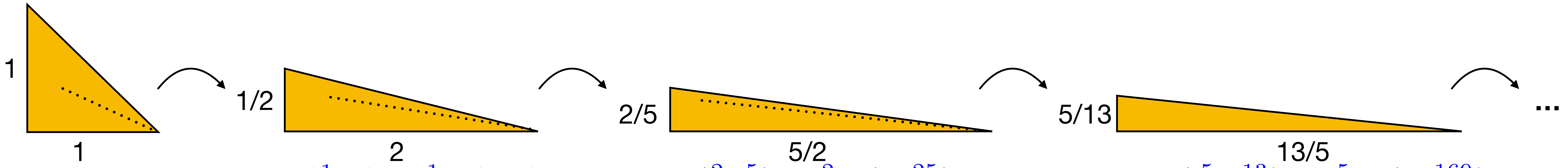
easy

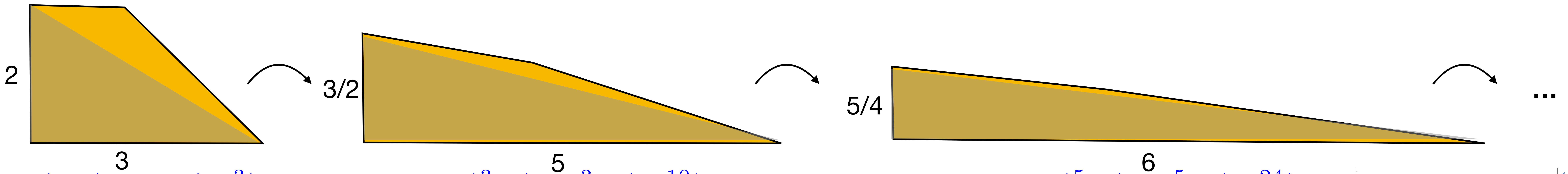
hard



(Note that $E(1,a) \hookrightarrow \lambda X \iff \frac{1}{\lambda}E(1,a) \hookrightarrow X$.)

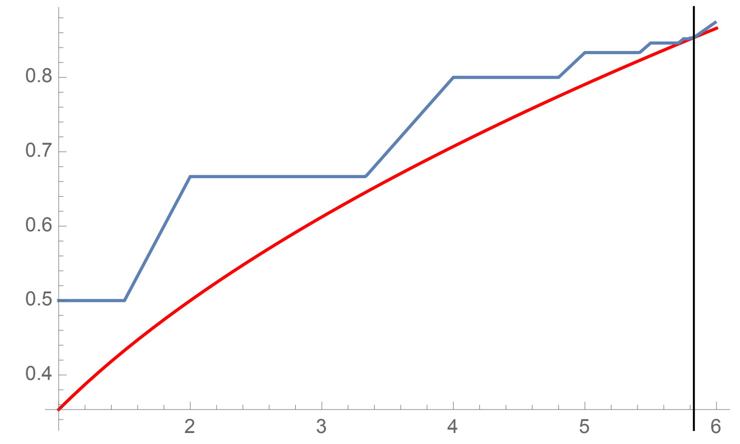


$\mathrm{E}(1{,}1)$ $\quad$ $\mathrm{E}(\frac{1}{2},2) = \frac{1}{2}\,\mathrm{E}(1,4)$ $\quad$ $\mathrm{E}(\frac{2}{5},\frac{5}{2}) = \frac{2}{5}\,\mathrm{E}(1,\frac{25}{4})$ $\quad$ $\mathrm{E}(\frac{5}{13},\frac{13}{5}) = \frac{5}{13}\,\mathrm{E}(1,\frac{169}{25})$



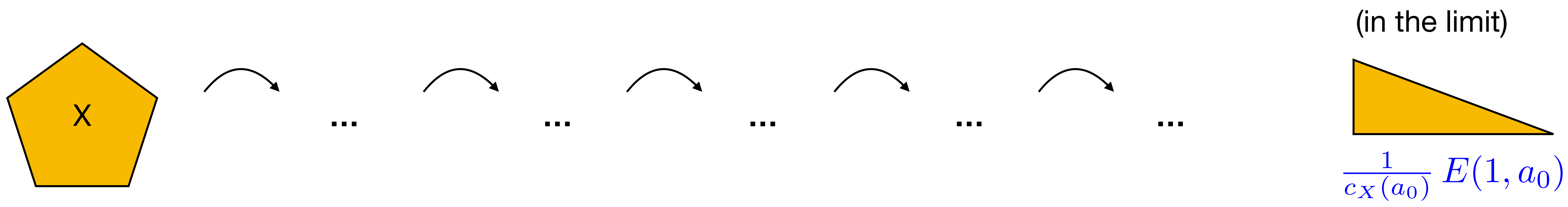
$\mathrm{E}(2,3) = 2\,\mathrm{E}(1,\frac{3}{2})$ $\quad$ $\mathrm{E}(\frac{3}{2},5) = \frac{3}{2}\,\mathrm{E}(1,\frac{10}{3})$ $\quad$ $\mathrm{E}(\frac{5}{4},6) = \frac{5}{4}\,\mathrm{E}(1,\frac{24}{5})$

Must start with closed
toric symplectic manifolds!



Short proof of location of a0, using the following **assumptions**:

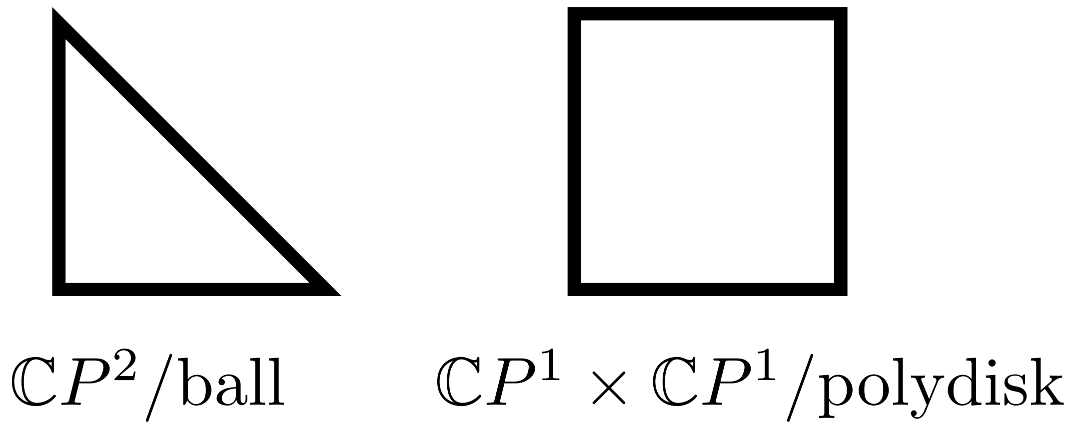
- the interior corners of the infinite staircase are given by ATF's,
- the accumulation point a0 is irrational,
- E(1,a0) gives a full filling.



This iterative process does not change the area and affine perimeter of the shapes, so:

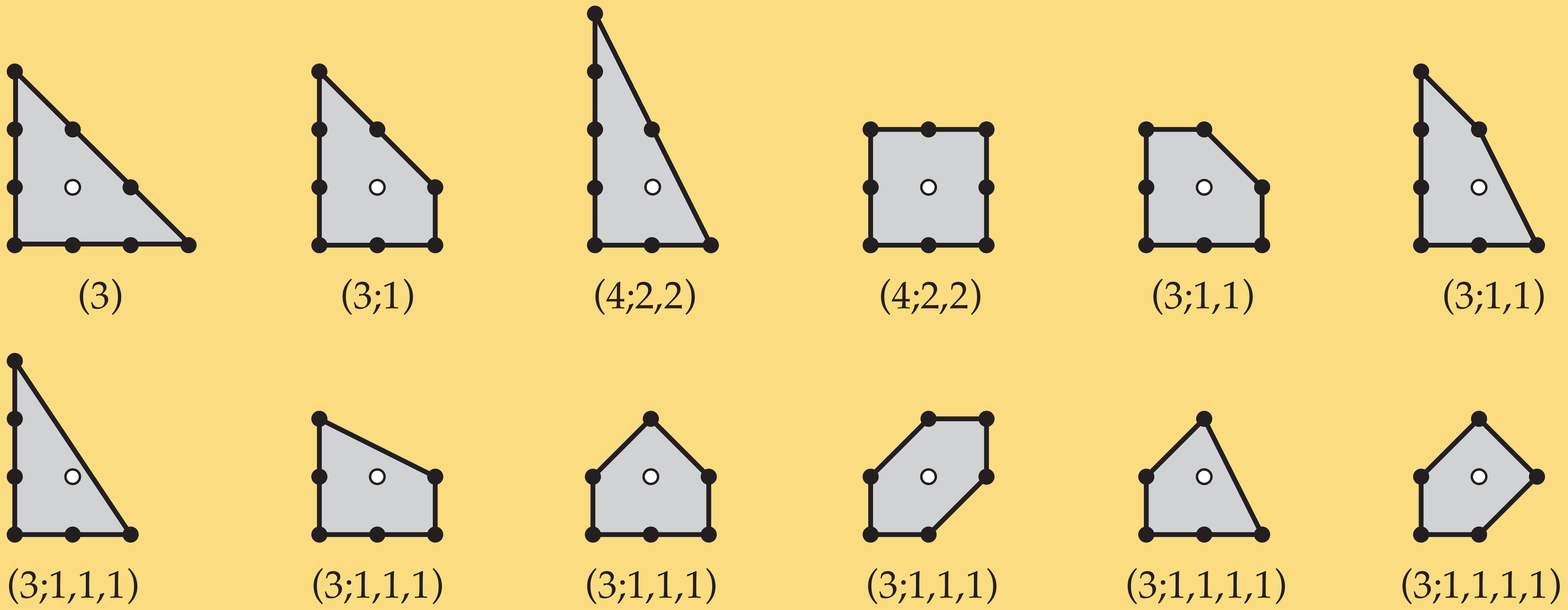
$$\begin{cases} \mathrm{vol} = \left(\frac{1}{c_X(a_0)}\right)^2 (1 \times a_0) \\ \mathrm{per} = \left(\frac{1}{c_X(a_0)}\right)(1 + a_0) \end{cases} \iff \begin{cases} c_X(a_0) = \sqrt{\frac{a_0}{\mathrm{vol}}} \\ a_0^2 - \left(\frac{\mathrm{per}^2}{\mathrm{vol}} - 2\right) a_0 + 1 = 0 \end{cases}$$

A beautiful part of this story is when the blowup sizes are all rational.
More generally / using scaling, we look at $X = (b; b_1, \ldots, b_n)$ with $b, b_1, \ldots, b_n \in \mathbb{N}$.



Theorem: (Cristofaro-Gardiner, Holm, Mandini, P.)

There is an infinite staircase in the ellipsoid embedding function $c_X(a)$ for the following convex toric domains:

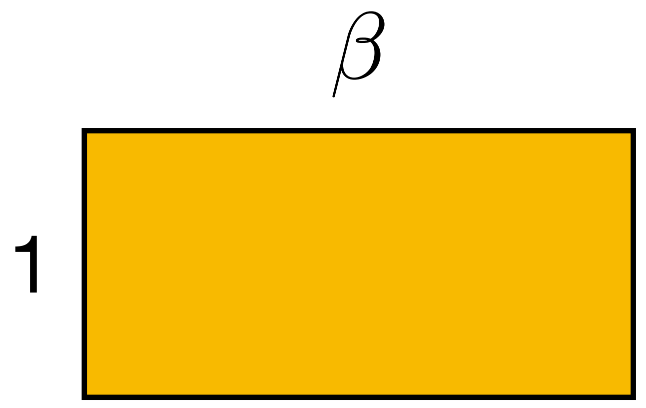


Conjecture: If $c_X(a)$ has an infinite staircase, then the moment polygon of X is a reflexive polygon.

In particular, the 12 examples above are the only (rational) ones with infinite staircases (up to scaling).

BUT!!!!!

There are infinitely many infinite staircases for irrational convex toric domains!



Polydisks

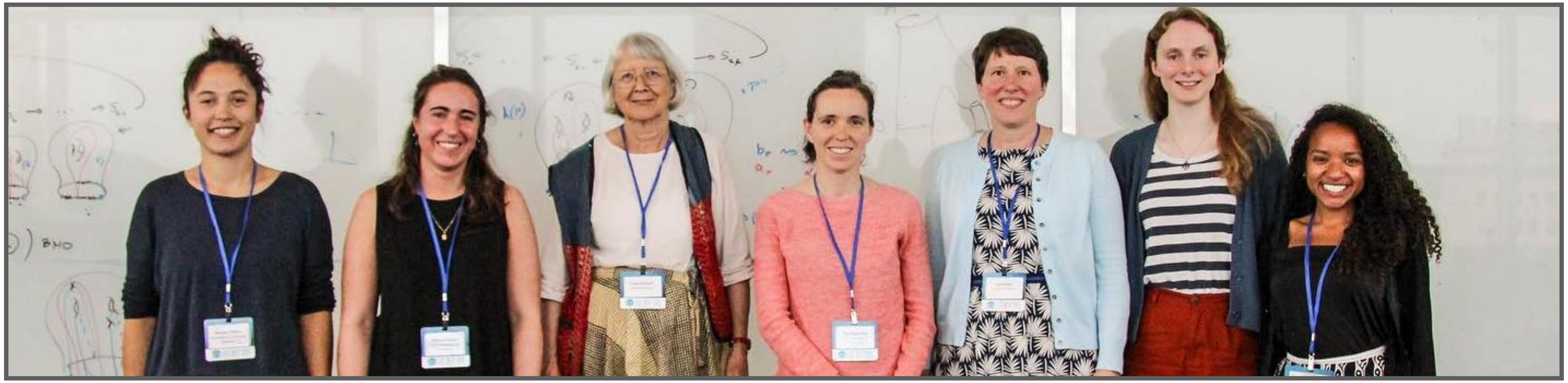
INFINITE STAIRCASES IN THE SYMPLECTIC EMBEDDING PROBLEM FOR FOUR-DIMENSIONAL ELLIPSOIDS INTO POLYDISKS

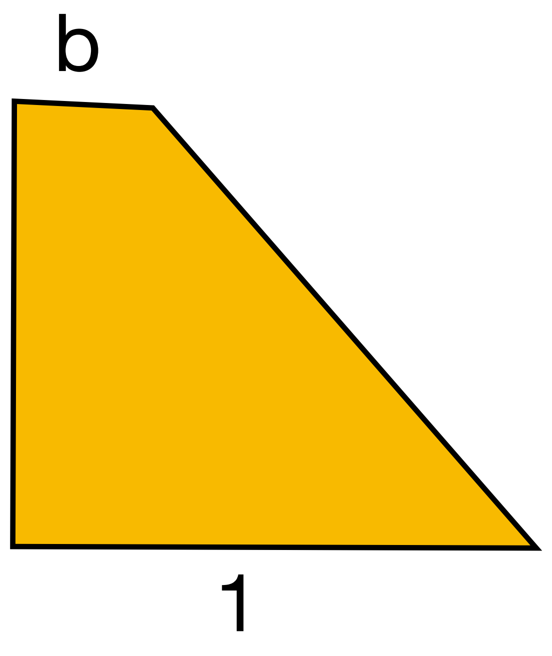
MICHAEL USHER

ath.SG] 21 Jan 2018

ABSTRACT. We study the symplectic embedding capacity function C_β for ellipsoids $E(1,\alpha) \subset \mathbb{R}^4$ into dilates of polydisks $P(1,\beta)$ as both α and β vary through $[1,\infty)$. For $\beta = 1$ results of [FM15] show that C_β has an infinite staircase accumulating at $\alpha = 3+2\sqrt{2}$, while for *integer* $\beta \geq 2$ [CFS17] found that no infinite staircase arises. We show that, for arbitrary $\beta \in (1,\infty)$, the restriction of C_β to $[1, 3+2\sqrt{2}]$ is determined entirely by the obstructions from [FM15], leading C_β on this interval to have a finite staircase with the number of steps tending to ∞ as $\beta \to 1$. On the other hand, in contrast to [CFS17], for a certain doubly-indexed sequence of irrational numbers $L_{n,k}$ we find that $C_{L_{n,k}}$ has an infinite staircase; these $L_{n,k}$ include both numbers that are arbitrarily large and numbers that are arbitrarily close to 1, with the cor-

[Br28] *Brahmagupta's Brāhmasphuṭasiddhānta* (628), edited by Acharyavara Ram Swarup Sharma, Indian Institute of Astronomical and Sanskrit Research, New Dehli, 1965, vol. 1.





Hirzebruch surfaces

WiSCon Project 9:

Morgan Weiler, Maria Bertozzi, Dusa McDuff, Ana Rita Pires, Tara Holm, Emily Maw, Grace Mwakyoma

BUT!!!!!

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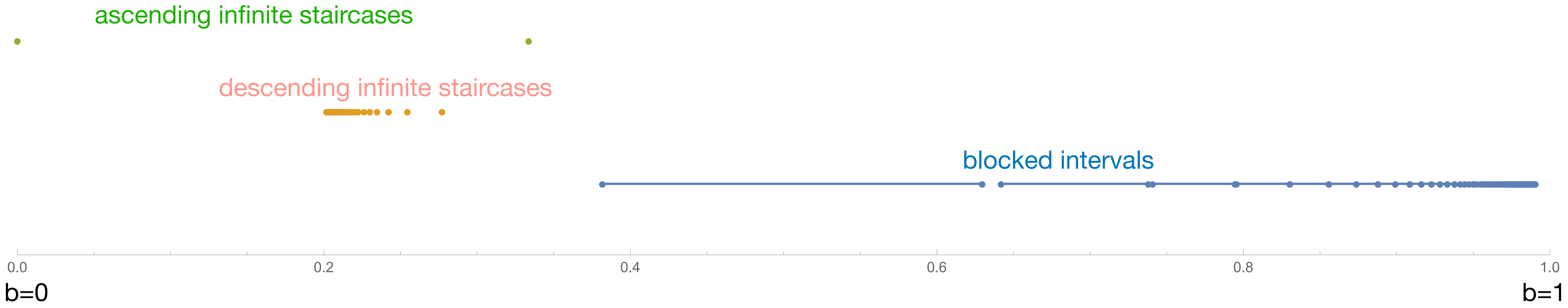

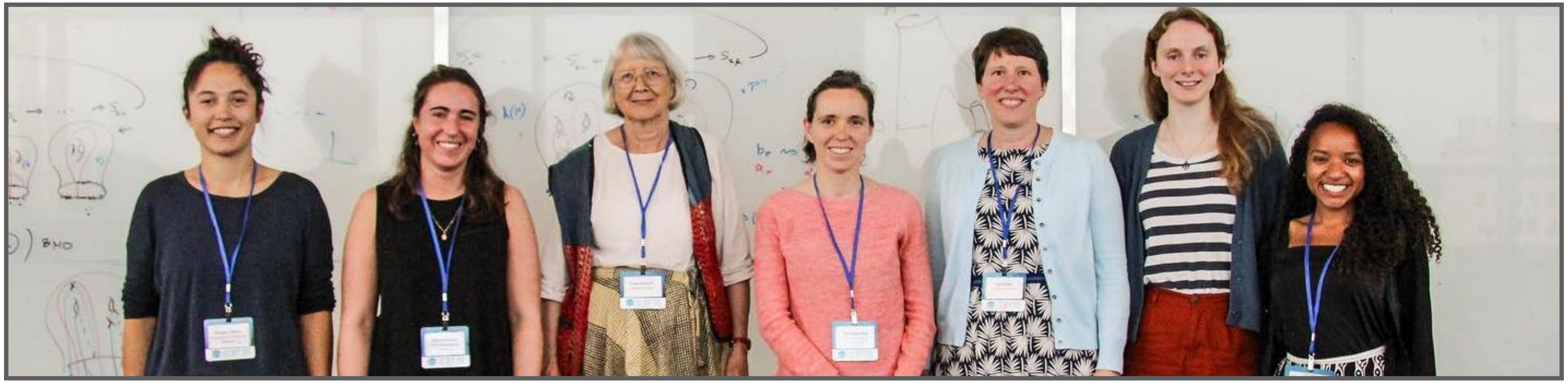

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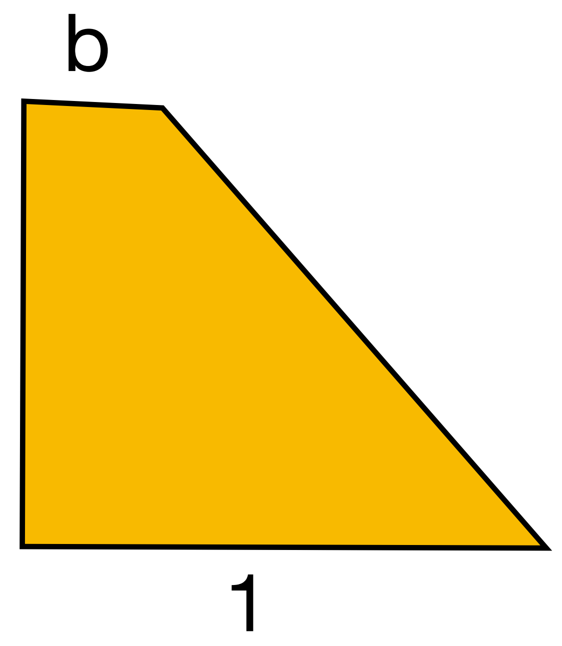
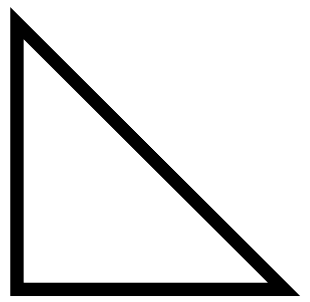
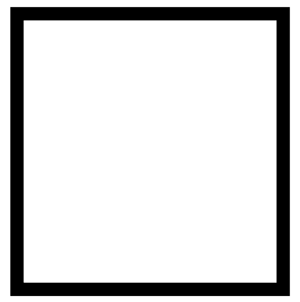


Hirzebruch surfaces

A beautiful part of this story is when the blowup sizes are all rational.
More generally / using scaling, we look at $X = (b; b_1, \ldots, b_n)$ with $b, b_1, \ldots, b_n \in \mathbb{N}$.



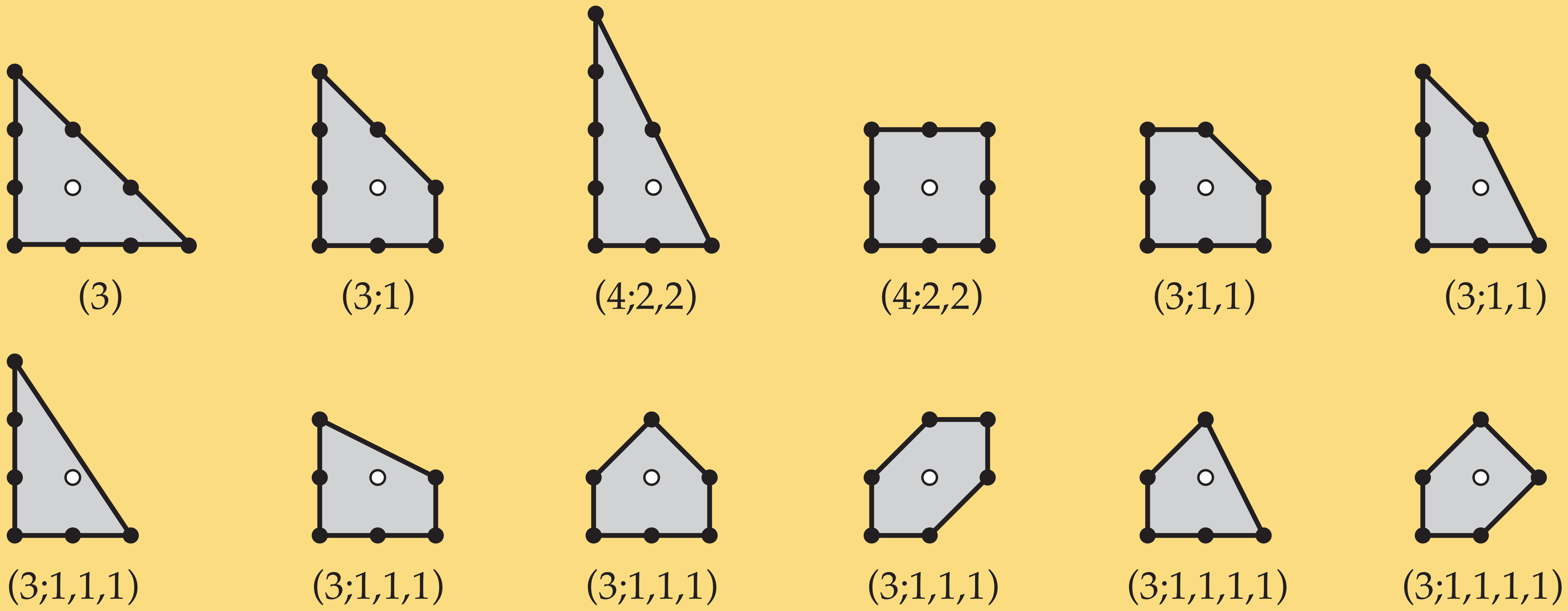
$\mathbb{C}P^2$/ball



$\mathbb{C}P^1 \times \mathbb{C}P^1$/polydisk

Theorem: (Cristofaro-Gardiner, Holm, Mandini, P.)

There is an infinite staircase in the ellipsoid embedding function $c_X(a)$ for the following convex toric domains:



Conjecture: If $c_X(a)$ has an infinite staircase, then the moment polygon of X is a reflexive polygon.

In particular, the 12 examples above are the only (rational) ones with infinite staircases (up to scaling).

$X = X_\Omega$ with integral blowup vector $(b; b_1, \dots, b_n)$. (Assume $a_0 \notin \mathbb{Q}$.)

If X has an infinite staircase then $E(1, a_0) \hookrightarrow \sqrt{\frac{a_0}{\mathrm{vol}}} X$

$\iff C_k(E(u_0, v_0)) \leq C_k(X)$ for all $k \in \mathbb{N}$ $\quad$ $\left(u_0 = \sqrt{\frac{\mathrm{vol}}{a_0}}\right.$ and $\left.v_0 = \sqrt{\frac{a_0}{\mathrm{vol}}}\right)$

$\iff \mathrm{cap}_{E(u_0,v_0)}(T) \geq \mathrm{cap}_X(T)$ for all $T \in \mathbb{N}$ $\quad$ $(\mathrm{cap}_M(T) := \#\{k | C_k(M) \leq T\})$

$\iff \mathrm{ehr}_{\Delta_{u_0,v_0}}(T) \geq \mathrm{cap}_X(T)$ for all $T \in \mathbb{N}$ $\quad$ $(\mathrm{ehr}_{\Delta_{u,v}}(T) = \#\{\mathbb{Z}^2 \cap T \cdot \Delta_{u,v}\})$

$\iff \frac{1}{2\mathrm{vol}}T^2 + \frac{\mathrm{per}}{2\mathrm{vol}}T + d(T) \geq \frac{1}{2\mathrm{vol}}T^2 + \frac{\mathrm{per}}{2\mathrm{vol}}T + \Gamma_{r(T)}$ for all $T \in \mathbb{N}$ $\quad$ $(T \equiv r(T) \pmod{\mathrm{vol}})$

$\iff d(T) \geq \Gamma_{r(T)}$ for all $T \in \mathbb{N}$

Hardy and Littlewood (1920) showed that for certain a_0's, $d(x) = d_{a_0}(x)$ is "optimally $\pm\mathcal{O}(\log x)$" for $x \in \mathbb{R}$.

Experimentally we observe that $d_{a_0}(T)$ is either periodic or "optimally $\pm\mathcal{O}(\log T)$" for $T \in \mathbb{N}$.

if Ω is a scaling of a reflexive polygon

otherwise

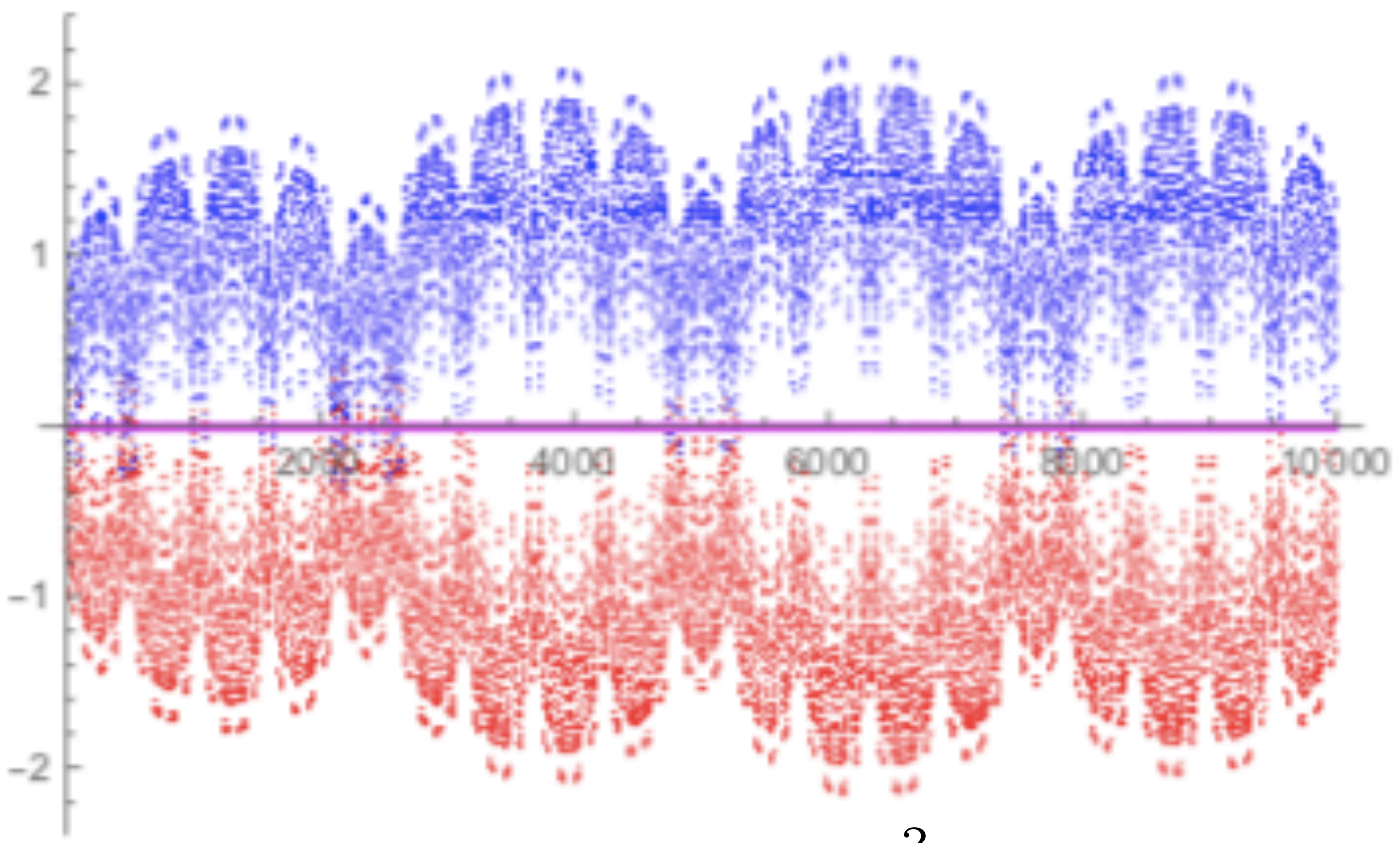
$\mathcal{O}(\log x)$ plus
$\exists$ constant K and increasing sequences x_i, x_j such that
$d_{a_0}(x_i) > K\log(x_i)$ and $d_{a_0}(x_j) < -K\log(x_j)$.

Experimentally we observe that $d_{a_0}(T)$ is either periodic or "optimally $\pm\mathcal{O}(\log T)$" for $T \in \mathbb{N}$.

if Ω is a scaling of
a reflexive polygon

otherwise

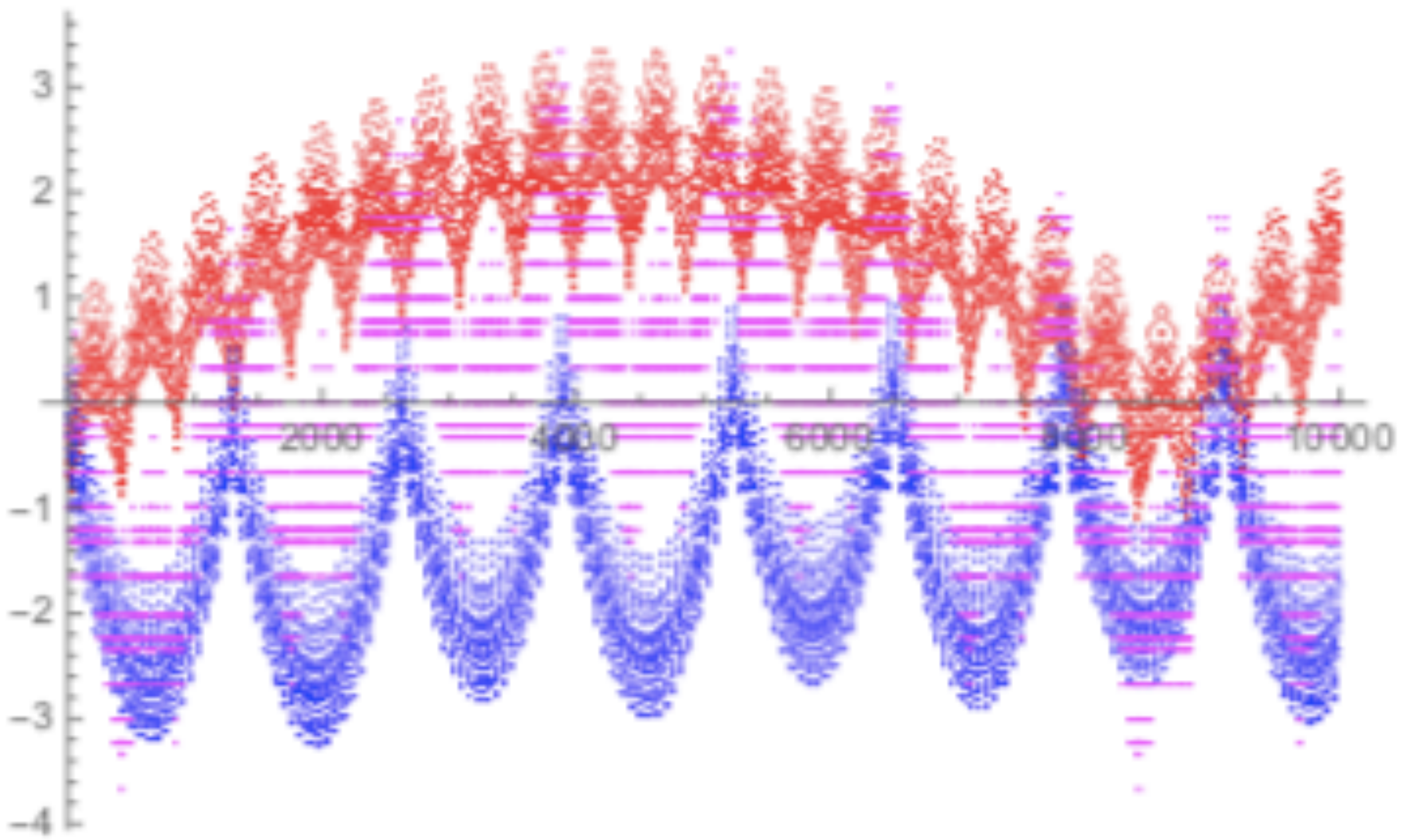
Some evidence: when $\mathrm{per}|T$, we have $d(T) = S_{a_0}(\frac{T}{\mathrm{per}}) + S_{\frac{1}{a_0}}(\frac{T}{\mathrm{per}})$, where $S_\theta(n) = \sum_{k=1}^{n} \left(\{k\theta\} - \frac{1}{2}\right)$.



$$a_0 + \frac{1}{a_0} = \frac{\mathrm{per}^2}{\mathrm{vol}} \in \mathbb{N}$$

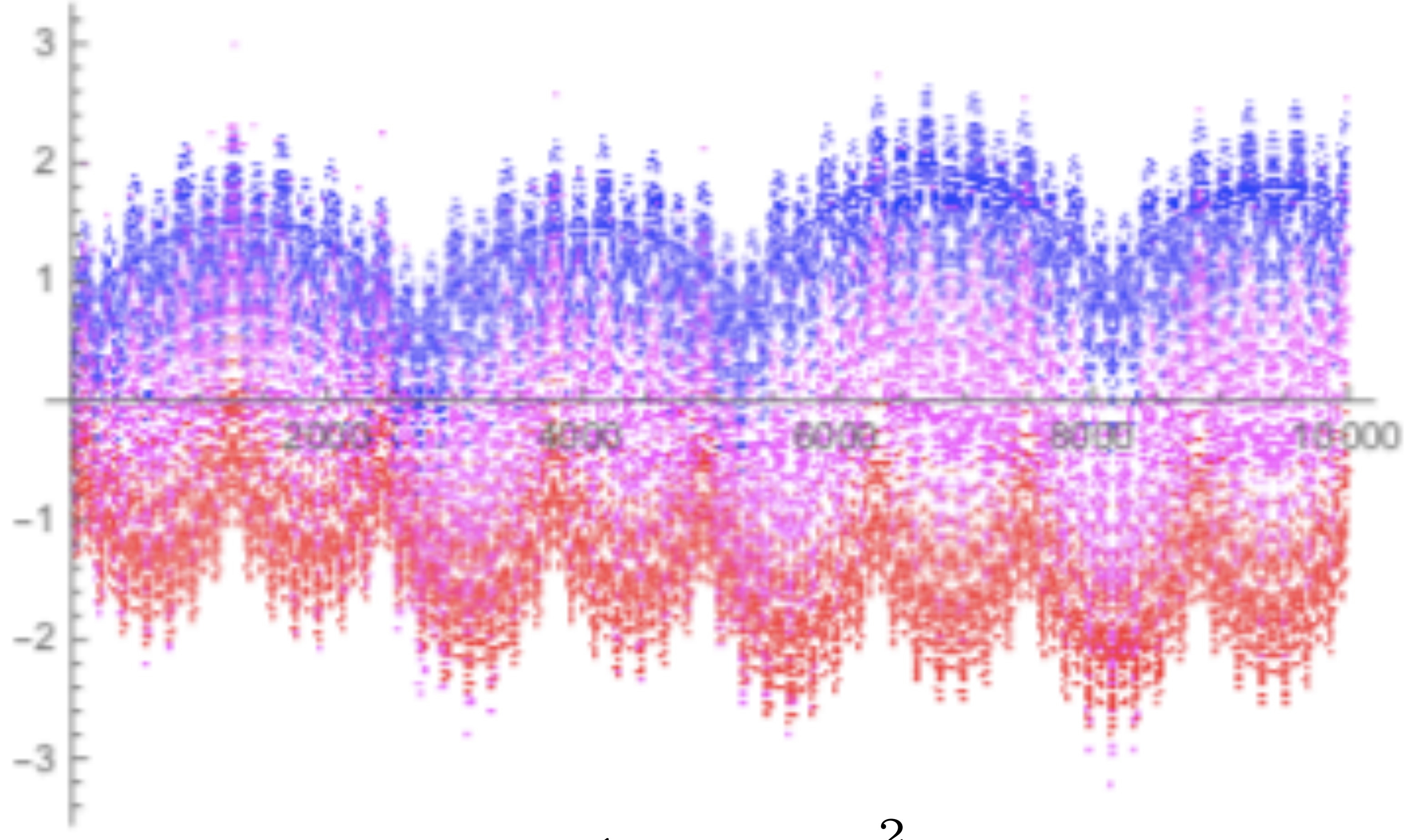
lattice polygon that is a
scaling of a reflexive polygon

$d(T) = 0$



$$a_0 + \frac{1}{a_0} = \frac{\mathrm{per}^2}{\mathrm{vol}} \in \mathbb{Q} \setminus \mathbb{N}$$

lattice polygon that is not a
scaling of a reflexive polygon



$$a_0 + \frac{1}{a_0} = \frac{\mathrm{per}^2}{\mathrm{vol}} \notin \mathbb{Q}$$

not a lattice polygon
(one of Usher's infinite staircases)

(Assume $a_0 \notin \mathbb{Q}$.)

$X = X_\Omega$ with integral blowup vector $(b; b_1, \ldots, b_n)$.

If X has an infinite staircase then $\mathrm{ehr}_{\Delta_{u_0,v_0}}(T) \geq \mathrm{cap}_X(T)$ for all $T \in \mathbb{N}$ $\quad$ $\left(\mathrm{ehr}_{\Delta_{u,v}}(T) = \#\left\{\mathbb{Z}^2 \cap T \cdot \Delta_{u,v}\right\}\right)$

$$\iff \tfrac{1}{2\mathrm{vol}}T^2 + \tfrac{\mathrm{per}}{2\mathrm{vol}}T + d(T) \geq \tfrac{1}{2\mathrm{vol}}T^2 + \tfrac{\mathrm{per}}{2\mathrm{vol}}T + \Gamma_{r(T)} \text{ for all } T \in \mathbb{N}$$

Cristofaro-Gardiner, Li, Stanley on $\mathrm{ehr}_{\Delta_{u,v}}(T)$ for $T \in \mathbb{N}$ and $\frac{u}{v} \notin \mathbb{Q}$:

Even though this is not the case in general, for exactly certain u, v's the function $\mathrm{ehr}_{\Delta_{u,v}}(T)$ is a quasipolynomial!

$$\underbrace{\tfrac{1}{u} + \tfrac{1}{v}}_{= \frac{\mathrm{per}}{\mathrm{vol}}} \text{ and } (u+v)\underbrace{\left(\tfrac{1}{u} + \tfrac{1}{v}\right)}_{= \frac{\mathrm{per}^2}{\mathrm{vol}}} \in \mathbb{N}.$$

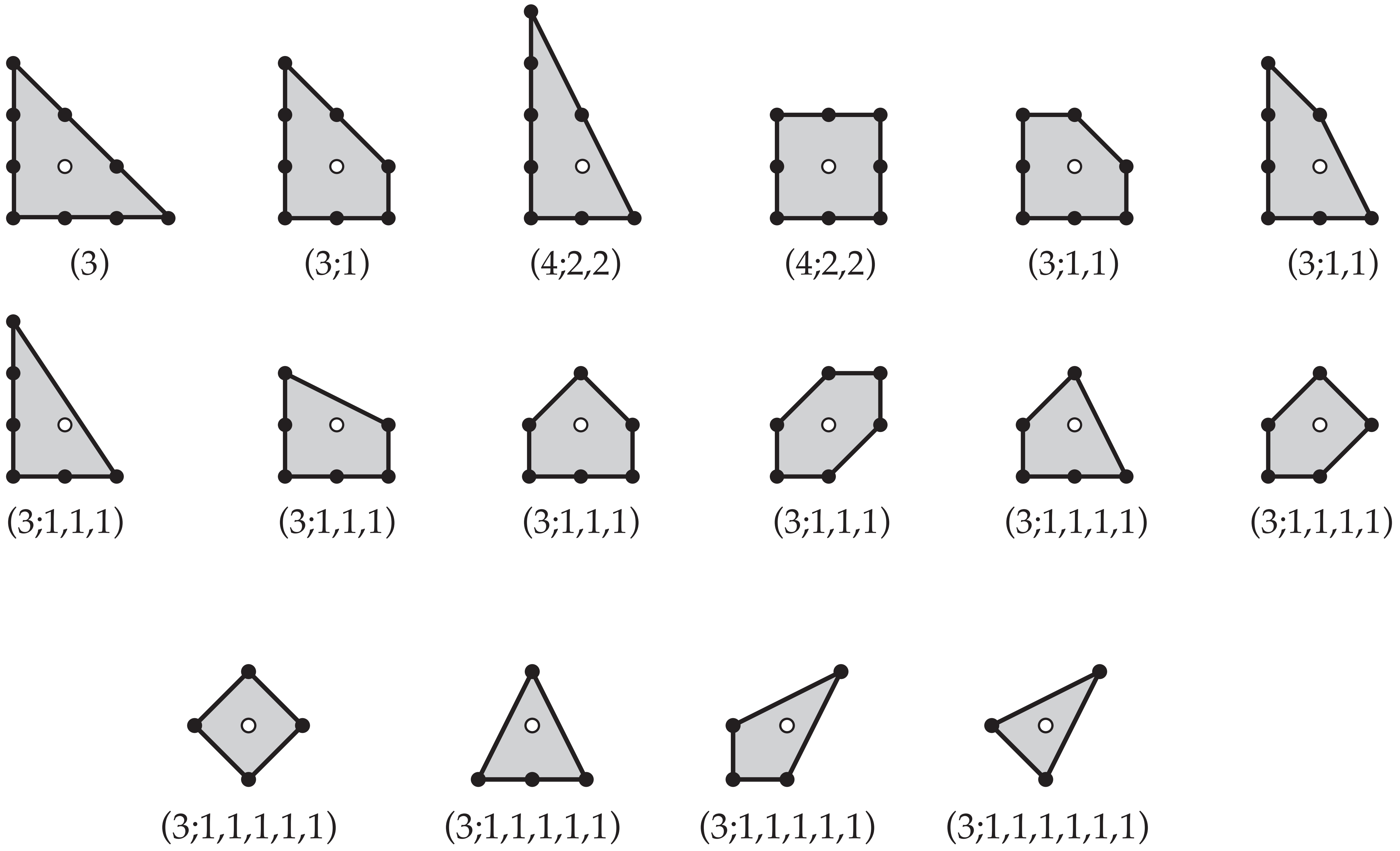
Now: Ω is lattice polygon $\implies$ scaling $\tilde{\Omega} = \frac{\mathrm{per}}{\mathrm{vol}} \cdot \Omega$ is also a lattice polygon. So can use Pick's Theorem on $\tilde{\Omega}$:

$$\text{area} = \#\text{ interior lattice points} + \frac{\#\text{ boundary lattice points}}{2} - 1$$

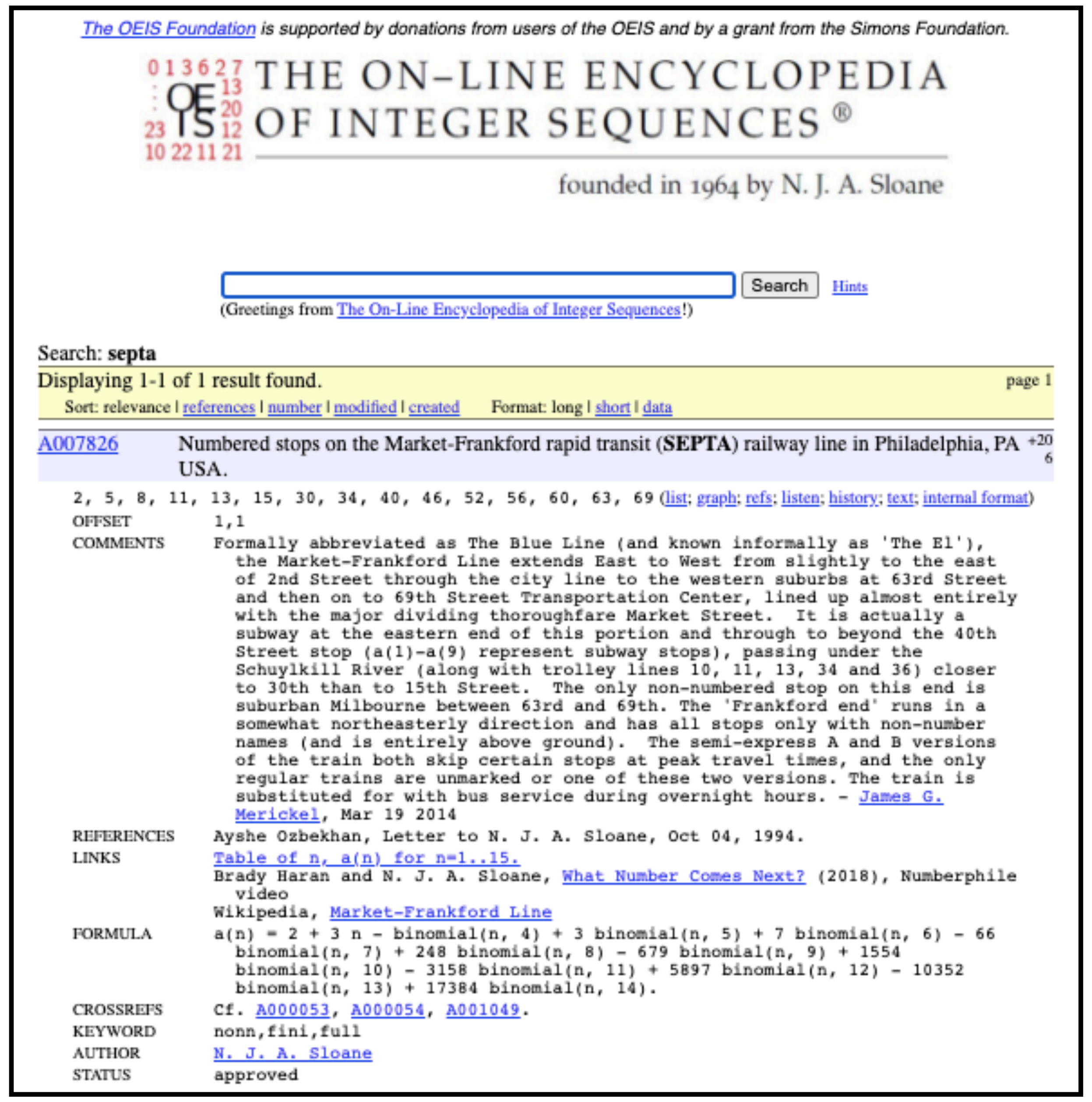
$$\iff \frac{\widetilde{\mathrm{vol}}}{2} = \#\text{ interior lattice points} + \frac{\widetilde{\mathrm{per}}}{2} - 1$$

$$\iff \#\text{ interior lattice points} = 1$$

$$\left(\widetilde{\mathrm{per}} = \widetilde{\mathrm{vol}} = \frac{\mathrm{per}^2}{\mathrm{vol}}\right)$$



If you don’t have any other questions, ask me about this!



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THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **septa**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A007826 Numbered stops on the Market-Frankford rapid transit (**SEPTA**) railway line in Philadelphia, PA USA. +20 6

2, 5, 8, 11, 13, 15, 30, 34, 40, 46, 52, 56, 60, 63, 69 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS Formally abbreviated as The Blue Line (and known informally as 'The El'), the Market-Frankford Line extends East to West from slightly to the east of 2nd Street through the city line to the western suburbs at 63rd Street and then on to 69th Street Transportation Center, lined up almost entirely with the major dividing thoroughfare Market Street. It is actually a subway at the eastern end of this portion and through to beyond the 40th Street stop (a(1)-a(9) represent subway stops), passing under the Schuylkill River (along with trolley lines 10, 11, 13, 34 and 36) closer to 30th than to 15th Street. The only non-numbered stop on this end is suburban Milbourne between 63rd and 69th. The 'Frankford end' runs in a somewhat northeasterly direction and has all stops only with non-number names (and is entirely above ground). The semi-express A and B versions of the train both skip certain stops at peak travel times, and the only regular trains are unmarked or one of these two versions. The train is substituted for with bus service during overnight hours. - James G. Merickel, Mar 19 2014

REFERENCES Ayshe Ozbekhan, Letter to N. J. A. Sloane, Oct 04, 1994.

LINKS Table of n, a(n) for n=1..15.
Brady Haran and N. J. A. Sloane, What Number Comes Next? (2018), Numberphile video
Wikipedia, Market-Frankford Line

FORMULA a(n) = 2 + 3 n - binomial(n, 4) + 3 binomial(n, 5) + 7 binomial(n, 6) - 66 binomial(n, 7) + 248 binomial(n, 8) - 679 binomial(n, 9) + 1554 binomial(n, 10) - 3158 binomial(n, 11) + 5897 binomial(n, 12) - 10352 binomial(n, 13) + 17384 binomial(n, 14).

CROSSREFS Cf. A000053, A000054, A001049.

KEYWORD nonn,fini,full

AUTHOR N. J. A. Sloane

STATUS approved

The end.

For $\mathbb{C}P^2_3\#4\overline{\mathbb{C}P}^2_1$, $J = 2$ but the manifold is not toric. We begin by using Vianna's trick [41, §3.2] to find an appropriate ATF on this manifold. Specifically, we begin with the ATF on $\mathbb{C}P^2_3\#3\overline{\mathbb{C}P}^2_1$ given in Figure B.4(e). This ATF has a smooth toric corner at the origin where we may perform a toric blowup of symplectic size 1. In terms of the base diagram, this corresponds to chopping off a 1×1 triangle at the origin. This results in a quadrilateral with two nodal rays representing an ATF on $\mathbb{C}P^2_3\#4\overline{\mathbb{C}P}^2_1$, shown in Figure B.5(b). There is then a sequence of ATF moves that achieves a triangle with two nodal rays. See Figure B.5.

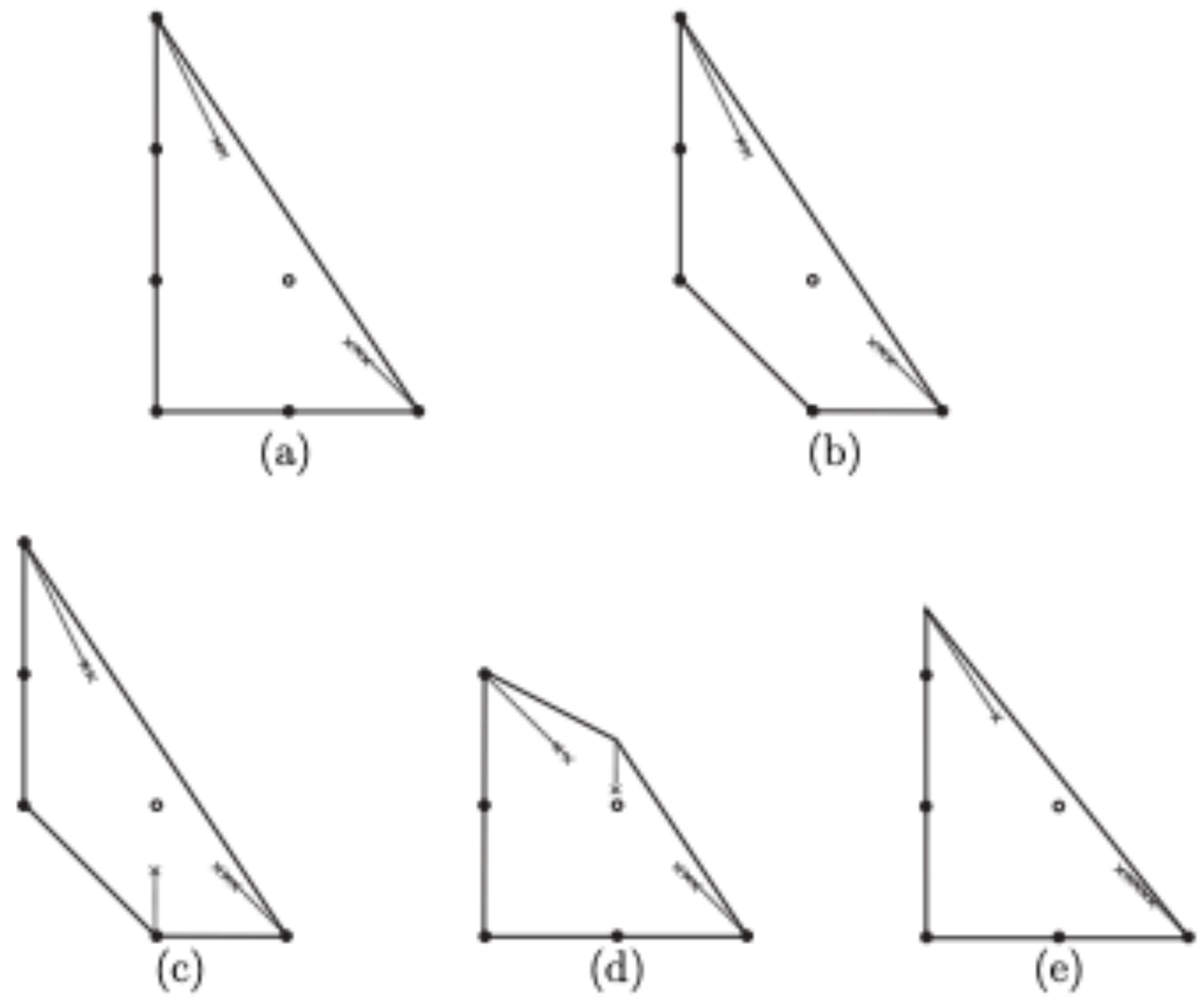

Figure B.5. In (a), we see the base diagram for an ATF on $\mathbb{C}P^2_3\#3\overline{\mathbb{C}P}^2_1$. From (a) to (b), we have applied a toric blowup of size 1 at the origin, resulting in an almost toric fibration on $\mathbb{C}P^2_3\#4\overline{\mathbb{C}P}^2_1$. From (b) to (c), we apply one nodal trade. From (c) to (d), we apply mutation, with resulting base diagram a quadrilateral with three nodal rays. Finally, from (d) to (e), we perform a second mutation, with resulting base diagram the desired triangle with two nodal rays. In (e), one of the nodal rays has one singular fiber and the other has five singular fibers.