

# Pontryagin - Thom for orbifold bordism

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Definition: An orbispace is a "space" (a separated topological stack) which is locally  $\cong Y/\Gamma$  where  $\Gamma$  is a finite group acting continuously on a topological space  $Y$ .

## Examples:

- An orbifold is an orbispace.
- The moduli stack  $M_g$  (of Riemann surfaces of genus  $g$ ) is an orbifold (hence also an orbispace).
- Moduli stacks of pseudo-holomorphic curves are orbispaces.
- $B\Gamma := * / \Gamma$  is an orbispace for any finite group  $\Gamma$  (not to be confused with the topological space  $B\Gamma = E\Gamma / \Gamma$ )

Features of the orbispace context include:

- For a pair of orbispaces  $X$  and  $Y$ , the collection of maps  $X \rightarrow Y$  forms a groupoid rather than a set.
- Every point  $x$  of an orbispace  $X$  has an isotropy group  $G_x$  (a finite group). The group  $G_x$  is the automorphism group of the map  $x: * \xrightarrow{\sim} X$ .
- An orbispace is a space iff all its isotropy groups are trivial.
- A map of orbispaces  $f: X \rightarrow Y$  induces maps on isotropy groups  $G_x \rightarrow G_{f(x)}$ . The map  $f$  is called representable iff these maps are all injective.
- For a representable map  $f: X \rightarrow Y$ , we may form the mapping cylinder  $cyl(f) := (X \times [0,1]) \cup_X Y$ .
- For a closed subspace  $A \subseteq X$ , we usually cannot form the quotient  $X/A$ .

An orbi-(W-complex) is built by attaching cells of the form  $(D^k, \partial D^k) \times \mathbb{B}\Gamma$  along representable maps.

$$X_{-1} := \emptyset \quad X_k := X_{k-1} \cup \bigsqcup_{\alpha} \frac{\sqcup D^k \times \mathbb{B}\Gamma_\alpha}{\sqcup \partial D^k \times \mathbb{B}\Gamma_\alpha} \quad \text{for } f: \partial D^k \times \mathbb{B}\Gamma_\alpha \rightarrow X_{k-1}$$

representable

and  $X = \bigcup_{i \geq 0} X_i$  (compare Gepner-Henniques)

Definition: For a "basepoint"  $p: \mathbb{B}G \rightarrow X$ , let  $\pi_k^G(X, p)$  denote the homotopy classes of maps  $S^k \times \mathbb{B}G \rightarrow X$  along with an isomorphism between the restriction to  $* \times \mathbb{B}G \cong S^k \times \mathbb{B}G$  and  $p$ .

As usual:

- $\pi_0^G(X, p)$  is a pointed set (independent, as a set, of  $p$ ).
- $\pi_1^G(X, p)$  is a group.
- $\pi_k^G(X, p)$  is an abelian group for  $k \geq 2$ .

Proposition: A map of orbi-(W-complexes) is a homotopy equivalence iff it induces isomorphisms on  $\pi_k^G$  for all finite groups  $G$  and all basepoints.

Theorem (P) ("Enough vector bundles"): For every finite dimensional orbi-(W-complex)  $X$  whose isotropy groups have bounded order, there exists a finite-dimensional vector bundle  $E$  over  $X$  whose isotropy group representations  $G_x \wr E_x$  are all faithful (equivalently, whose isotypic pieces  $(E_x)_\rho$  for  $\rho \in \widehat{G_x}$  are all nontrivial).

## Orbifold bordism

Let  $(X, A)$  be an orbispace-pair (ie.  $X$  is an orbispace and  $A \subseteq X$  is a subcomplex).

$\mathcal{S}\mathcal{U}_*(X, A) :=$  compact orbifolds with boundary  $(M, \partial M)$  together with a representable map  $(M, \partial M) \rightarrow (X, A)$  modulo bordism.

Remark:

- When  $(X, A)$  is a CW-pair,  $\mathcal{S}\mathcal{U}_*(X, A)$  is bordism classes of manifolds w/ over  $(X, A)$ .
- Removing the representability requirement results in no greater generality, since the functor  $\text{Rep OrbSpc} \rightarrow \text{OrbSpc}$  has a right adjoint  $R$ . So, for example, bordism classes of closed orbifolds =  $\mathcal{S}\mathcal{U}_*(R(*))$ .

Properties of  $\mathcal{S}\mathcal{U}_*$

(Excision)  $\mathcal{S}\mathcal{U}_*(P, P \cap A) \rightarrow \mathcal{S}\mathcal{U}_*(X, A)$  is an isomorphism for  $X = P \cup A$  (union of subcomplexes)

(Exactness)  $\mathcal{S}\mathcal{U}_*(Y, B) \rightarrow \mathcal{S}\mathcal{U}_*(X, A) \rightarrow \mathcal{S}\mathcal{U}_*(X, A \cup Y)$  is exact for subcomplexes  $A, Y \subseteq X$  and  $B = A \cap Y$

(Inverse Thom maps) For  $V$  a vector bundle over  $X$ , there is a natural map  
(terminology following Schwede)  $\mathcal{S}\mathcal{U}_*(X, A) \longrightarrow \mathcal{S}\mathcal{U}_{*-|V|}((X, A)^V)$

which is an isomorphism if  $V$  is coarse (all isotropy representations of  $V$  are trivial).

## Derived orbifold bordism

Definition: A derived orbifold chart is a triple  $(D, E, s)$  where  $D$  is an orbifold,  $E$  is a vector bundle over  $D$ , and  $s: D \rightarrow E$  is a section.

Operations on derived orbifold charts:

- Restriction: replace  $D$  with an open subset containing  $s^{-1}(0)$ .
- Stabilization: replace  $D$  with the total space of a vector bundle  $F$  over it, replace  $E$  with  $E \oplus F$ , and replace  $s$  with  $s \oplus \text{id}_F$ .

A derived orbifold chart is "compact" iff  $s^{-1}(0)$  is compact.

"Definition" A derived orbifold is an "object" with an "open cover" by derived orbifold charts modulo restriction and stabilization.

Corollary of enough vector bundles: every compact derived orbifold has a global chart!

Definition:  $\mathbb{S}^{\text{der}}_*(X, A) :=$  compact derived orbifold charts  $(D, E, s)$  with representable map  $(D, \partial D) \rightarrow (X, A)$  modulo restriction, stabilization, and bordism.

$\mathbb{S}^{\text{der}}_*$  satisfies excision (obvious) and exactness (requires enough vector bundles).

$\mathbb{S}^{\text{der}}_*$  has inverse Thom maps which are always isomorphisms. In fact,  $\mathbb{S}^{\text{der}}_*$  is the localization

of  $\mathbb{S}_*$  at all inverse Thom maps:  $\lim_{\substack{\longrightarrow \\ V/X}} \mathbb{S}_{*+|V|}(X, A) \xrightarrow{\sim} \lim_{\substack{\longrightarrow \\ V/X}} \mathbb{S}^{\text{der}}_{*+|V|}(X, A) \xleftarrow{\sim} \mathbb{S}^{\text{der}}_*(X, A)$ .  
 (parallel to a result of Schwede in global homotopy theory)

Orbifold bordism as derived orbifold bordism with tangential structure

For a finite-dimensional real representation  $V$  of a finite group  $G$ , we have a canonical decomposition into isotypic pieces  $V = \bigoplus_{\rho \in \widehat{G}} V_\rho$ . Let  $V_{\widehat{G}-1} := \bigoplus_{\rho \neq 1} V_\rho$ .

For a vector bundle  $E$  over an orbispace, let  $E_{\widehat{G}-1}^x \subseteq E$  be defined as

$$(E_{\widehat{G}-1}^x)_x := (E_x)_{\widehat{G}_x-1}^x \quad \text{(not a subbundle)}$$

Theorem (Wasserman 1969): Let  $E$  be a vector bundle over an orbifold  $X$ , and let  $\alpha: TX \rightarrow E$  be any map for which  $\alpha_{\widehat{G}-1}^x$  is surjective. Then there exists a section  $s: X \rightarrow E$  which is transverse to zero.

Corollary:  $\mathcal{S}_*(X, A)$  is derived orbifold charts  $(D, E, s)$  representable over  $(X, A)$  together with a vector bundle  $F$  over  $D$  and a stable isomorphism

$$TD - E = F - \underline{\mathbb{R}}^k$$

modulo restriction, stabilization,  $F \mapsto F \oplus \underline{\mathbb{R}}$ , and bordism.

Rmk: This result is crucial for extending orbifold bordism from orbispaces to orbispectra.

## Global Spectra

Schwede introduces a category of global spectra whose objects give rise to cohomology theories for orbispaces.

$\mathbb{O}$  := category whose objects are finite-dimensional real inner product spaces (positive definite) and in which the morphism space  $V \rightarrow W$  is the Thom space of the tautological vector bundle " $W/V$ " over the space of isometric embeddings  $V \hookrightarrow W$ .

An orthogonal spectrum is a continuous functor  $\mathbb{O} \rightarrow \text{Top}_*$

Def<sup>n</sup>: A map of orthogonal spectra  $F \rightarrow G$  is a global equivalence iff for every finite group ...

$\text{GloSp} := \text{OrthSp}[\text{global equivalences}]^{\wedge}$

Examples:

$S(V) := S^V = \text{one-point compactification of } V$

$mO(V) := \text{Gr}_{\mathbb{M}}(V \oplus \mathbb{R}^\infty)^T$

$MO(V) := \text{Gr}_{\mathbb{N}_1}(V \oplus V)^T$

(see Schwede for precise definitions)

Definition: For a global spectrum  $Z$ , let  $Z^0(X, A) := \lim_{E/X} \pi_0 T^*(X \text{ rel } A, Z(E))$   $\begin{pmatrix} E & \xrightarrow{\quad Z(E) \quad} \\ \downarrow & \rightsquigarrow \\ X & \xrightarrow{\quad Z \quad} \\ \downarrow & \\ X & \end{pmatrix}$

This satisfies excision and exactness.

Rmk: Viability of this definition depends on enough vector bundles, though there are other equivalent definitions which do not.

## Pontryagin-Thom isomorphism

Theorem (Pontryagin-Thom): For pointed spaces  $X$ , there are natural isomorphisms:

$$\mathcal{L}_*(X) = [S, X \wedge M_0] \quad \mathcal{L}^{fr}_*(X) = [S, X]$$

Recall Spanier-Whitehead duality  $D$  which is a contravariant involution of the category of finite spectra:

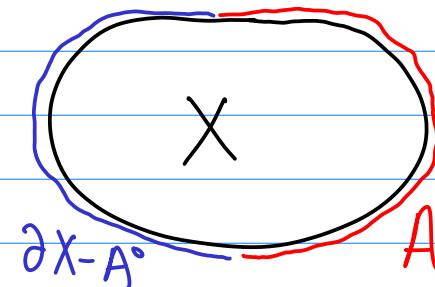
$$D: Sp^f \rightarrow (Sp^f)^{op} \quad D^2 = id$$

It is defined by the universal property:

$$[X, Y \wedge Z] = [X \wedge DY, Z]$$

If  $(X, A)$  is a manifold pair ( $X$  a compact mfld w/  $\partial$  and  $A \subseteq \partial X$  is a codim 0 submfld w/  $\partial$ ), then:

$$D(X, A) = (X, \partial X - A^\circ)^{-TX}$$



We may thus restate Pontryagin-Thom as:

$$\mathcal{L}_*(X) = [DX, M_0] \quad \mathcal{L}^{fr}_*(X) = [DX, S]$$

Theorem (P): For any finite orbispectrum  $X$ , there are natural isomorphisms

$$\mathcal{L}_*(X) = M_0^{-*}(DX) \quad \mathcal{L}^{der}_*(X) = M_0^{-*}(DX) \quad \mathcal{L}^{fr}_*(X) = S^{-*}(DX).$$

It remains to define the category of orbispectra and the involution  $D$ .

## Categories of orbispaces and orbispectra

$Spc = CW\text{-complexes and continuous maps}$

$OrbSpc = \text{orbi-}CW\text{-complexes and maps thereof}$

$RepOrbSpc = \text{orbi-}CW\text{-complexes and representable maps}$

$Spc_* = \text{pointed } CW\text{-complexes and pointed maps}$

$OrbSpc_* = \text{orbi-}CW\text{-pairs and "relative maps"}$

$RepOrbSpc_* = \text{orbi-}CW\text{-pairs and representable relative maps}$  Conj:  $RepOrbSpc$  (resp.  $RepOrbSpc_*$ ) is the category of (pointed) representable fibrations over  $R(*)$ .

$S_p^f = \varinjlim (Spc_*^f \xrightarrow{\Sigma} Sp_{*+}^f \xrightarrow{\Sigma} \dots)$  Concretely, this means an object of  $S_p^f$  is a formal symbol  $\sum^{-k} X$  for  $X \in Sp_{*+}^f$ , and the morphism space from  $\sum^{-k} X$  to  $\sum^{-l} Y$  is the direct limit over  $n$  of the morphism space from  $\sum^{n-k} X$  to  $\sum^{n-l} Y$  in  $Sp_{*+}^f$ .

$\varinjlim (OrbSpc_*^f \xrightarrow{\Sigma} OrbSpc_{*+}^f \xrightarrow{\Sigma} \dots)$  exists and is similar to  $S_p^f$ , but it is not what we want.

Definition:  $RepOrbSp^f := \varinjlim_{E/R(*)} RepOrbSpc_*$  (direct limit over vector bundles on  $R(*)$ , with transition functors given by passing to Thom spaces)

Conjecture:  $RepOrbSp$  is the category of parameterized spectra over  $R(*)$ .

Question: What is the relationship between  $RepOrbSp$ ,  $OrbSp$ , and  $GloSp$ ?

(combined work of Gepner-Henry and Schwede gives an equivalence in some sense between  $OrbSp$  and  $GloSp$ )

has a right adjoint  $X \mapsto \tilde{X}$  ("classifying space")  
and a left adjoint  $X \mapsto |X|$  ("coarse space")

$Spc \longrightarrow RepOrbSpc \longrightarrow OrbSpc$

$\uparrow$

has a right adjoint  
 $X \mapsto R(X)$

Rmk: The space of representable maps  $X \rightarrow R(*)$   
is contractible for all  $X \in RepOrbSpc$ .

Theorem (P): There exists a contravariant involution  $\mathcal{D}$  of  $\text{RepOrbSpt}$  defined by taking  $\mathcal{D}(X, A) := (X, \partial X - A^\circ)^{-TX}$  for any orbifold pair  $(X, A)$ .

Rmk: The proof depends crucially on enough vector bundles. I do not know how one might try to prove it without.

Problem: Find a universal property characterization of  $\mathcal{D}$ .

Definition: For any global spectrum  $E$ , define  $E_*(X) := E^{-*}(\mathcal{D}X)$ .

Rmk:  $E^*$  is functorial under all maps, but  $E_*$  is functorial only under representable maps!

Rmk: In usual stable homotopy theory,  $[\mathcal{D}X, E] = [S, X \wedge E]$ , but in the orbi context, both sides give distinct reasonable notions of " $E$ -homology of  $X$ ".

Theorem (P): For orbi- $(W)$ -pairs  $(X, A)$  and  $(Y, B)$ , the space of morphisms

$$\mathcal{D}((X, A)^{-\xi}) \rightarrow (Y, B)^{-\xi}$$

in  $\text{OrbSpt}$  (resp.  $\text{RepOrbSpt}$ ) is in natural bijection with the set of compact derived orbifolds  $(C, \partial C)$  with

- a representable map  $f: C \rightarrow X \quad \} \text{ with } \partial C \subseteq f^{-1}(A) \cup g^{-1}(B)$
- a (representable) map  $g: C \rightarrow Y \quad \}$
- a stable isomorphism  $T C = f^* \xi + g^* \xi$ .

Question: What category do we get if both  $f$  and  $g$  are not required to be representable??