Zoll contact forms are local maximizers of the systolic ratio

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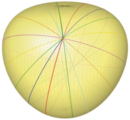

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Systolic ratio of (M, α):

$$\rho_{\mathrm{sys}}(M, \alpha) := \frac{T_{\mathsf{min}}(\alpha)^n}{\mathrm{vol}(M, \alpha)},$$

$T_{\mathsf{min}}(\alpha) :=$ minimum of all periods of closed orbits of R_α.

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Boothby & Wang (1958): α_0 Zoll on $M \Rightarrow$ Basis B of circle bundle $\pi : M \to B$ induced by S^1-action of R_{α_0} has integral symplectic form ω such that $d\alpha_0 = T_{\min}(\alpha_0)\pi^*\omega$, and hence $\rho_{\mathrm{sys}}(M, \alpha_0) = \frac{1}{N}$, where $N := \langle [\omega]^{n-1}, [B] \rangle \in \mathbb{N}$ is the Euler number.

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Main example: S^{2n-1} with standard contact form α_0, whose Reeb orbits are the fibers of the Hopf fibration $\pi : S^{2n-1} \to \mathbb{CP}^{n-1}$, and $\rho_{\mathrm{sys}}(S^{2n-1}, \alpha_0) = 1$.

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- Any contact form that is a local maximizer of ρ_{sys} must be Zoll.
- α_t smooth path of contact forms with α_0 Zoll. Then either $t \mapsto \rho_{\mathrm{sys}}(M, \alpha_t)$ has a strict local maximum at $t = 0$, or α_t is tangent up to every order to the space of Zoll contact forms.

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C^3-local maximality of Zoll contact forms in dimension 3: For $M = S^3$: A. A., B. Bramham, U. Hryniewicz & P. Salomão (2018). For any closed 3-manifold: G. Benedetti & J. Kang.

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The systolic ratio is unbounded from above on the space of contact forms supporting any given contact structure: closed 3-manifolds (ABHS, 2019), contact manifolds of arbitrary dimension (M. Săglam).

Metric systolic geometry, I

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Corollary 1 answers a question of M. Berger (1970).

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Conjecture (C. Viterbo, 2000). Let c be a normalized symplectic capacity on $(\mathbb{R}^{2n}, \omega_0)$. For every convex body $K \subset \mathbb{R}^{2n}$ we have

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EHZ-capacities: $\lambda_0 := \frac{1}{2} \sum_{j=1}^n (x_j \; dy_j - y_j \, dx_j)$ primitive of ω_0, K smooth convex body with $0 \in \mathrm{int}(K)$, so that λ_0 restricts to a contact form on ∂K.

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$$c(K) = T_{\mathsf{min}}(\lambda_0|_{\partial K}).$$

We denote by c_{EHZ} one of them. Viterbos' conjecture for c_{EHZ} reads:

$$T_{\mathsf{min}}(\lambda_0|_{\partial K})^n \leq \mathrm{vol}(\partial K, \lambda_0|_{\partial K}), \qquad \text{i.e.} \qquad \rho_{\mathrm{sys}}(\partial K, \lambda_0|_{\partial K}) \leq 1,$$

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Corollary 2. There exists a C^3-neighborhood $\mathcal{U}$ of the ball in the space of smooth convex bodies in $\mathbb{R}^{2n}$ such that

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Case $n = 2$: ABHS (2018).

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Characterization of the equality: Need to show that if the Reeb flow on ∂K is Zoll then K is symplectomorphic to a closed ball.

Shadows of symplectic balls, I

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Gromov's non-squeezing theorem (1985): V symplectic 2-plane in $(\mathbb{R}^{2n}, \omega_0)$, P_V symplectic projector onto V, B unit ball in $\mathbb{R}^{2n}$. Then

$$\mathrm{area}(P_V \varphi(B), \omega_0|_V) \geq \pi$$

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A. A. & R. Matveyev (2013): If V is a symplectic $2k$-plane with $1 < k < n$ and $\epsilon > 0$, then there exists a symplectomorphism $\varphi : B \hookrightarrow \mathbb{R}^{2n}$ such that

$$\mathrm{vol}(P_V \varphi(B), \omega_0^k|_V) < \epsilon.$$

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$$\mathrm{vol}(P_V\Phi(B), \omega_0^k|_V) = \frac{\pi^k}{w(\Phi^{-1}(V))},$$

where

$$w(X) := \frac{|\omega_0^k[u_1, u_2, \ldots, u_{2k}]|}{k!|u_1 \wedge u_2 \wedge \cdots \wedge u_{2k}|}, \ u_1, u_2, \ldots, u_{2k} \text{ basis of } X \in \mathrm{Gr}_{2k}(\mathbb{R}^{2n}).$$

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Corollary 3. There exists a C^3_{loc}-neighborhood $\mathcal{U}$ of the set of linear symplectorphisms in the space of all smooth symplectomorphisms of $\mathbb{R}^{2n}$ such that for every symplectic $2k$-plane $V \subset \mathrm{R}^{2n}$ we have

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Therefore:

$$\mathrm{vol}(M, \alpha) = \int_M S^n \, \alpha_0 \wedge d\alpha_0^{n-1}$$

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and we conclude as in the simple case treated before. $\square$

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[Our proof uses ideas of E. Kerman (1999)]

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