Formal Legendrian and Horizontal embeddings.

Javier Martínez-Aguinaga.

Joint work with E. Fernández and F. Presas.

Symplectic Zoominar (CRM-Montreal, Princeton/IAS, Tel Aviv, Paris).

Introduction.

Formal Legendrian and Horizontal embeddings.

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Applications.

$$\mathcal{L}eg(\mathbb{R}^3) \hookrightarrow \mathcal{F}\mathcal{L}eg(\mathbb{R}^3)$$

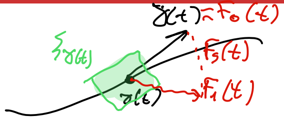
$$\pi_k(\mathcal{L}eg(\mathbb{R}^3)) \xrightarrow{i_*} \pi_k(\mathcal{F}\mathcal{L}eg(\mathbb{R}^3))$$

$$[\gamma^0] \longmapsto i_*[\gamma^0] \neq 0$$

$$\mathcal{H}or(\mathbb{R}^4) \xhookrightarrow{\text{whe}} \mathcal{F}\mathcal{H}or(\mathbb{R}^4)$$

Casals and del Pino

Formal Legendrian embeddings.



Definition

A formal Legendrian embedding in $\mathbb{R}^3$ is a pair (γ, F_s) satisfying the following two conditions:

(i) $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ is an embedding.

(ii) $F_s : \mathbb{S}^1 \to \gamma^*(T\mathbb{R}^3 \setminus \{0\})$ is a 1-parametric family, $s \in [0,1]$, such that $F_0 = \gamma'$ y $F_1(t) \in \xi_{\gamma(t)}$.

An auxiliary fibration.

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$\xi \sim \mathbb{R}^2 \setminus \{0\} \sim S^1$

Consider the space

$$\widehat{\mathfrak{FLeg}}(\mathbb{R}^3) = \{(\gamma, F_1) | \gamma \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3), F_1 \in \mathcal{M}\,\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^1)\}.$$

We have a natural fibration $\mathfrak{FLeg}(\mathbb{R}^3) \to \boxed{\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)}$.
The fiber over a point $(\gamma, \gamma') \in \widehat{\mathfrak{FLeg}}(\mathbb{R}^3)$ is

$\simeq \mathrm{Emb}(S^1, \mathbb{R}^3) \times S^1 \times \mathbb{Z}$

$$\mathcal{F}_{(\gamma,\gamma')} = \Omega_{\gamma'}(\mathcal{M}\,\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^2)).$$

Exact sequence associated to the fibration.

We have the following exact sequence of homotopy groups associated to the fibration:

$$
\begin{tikzcd}
 & \cdots \arrow[r] & \pi_2(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)) \arrow[dll] \\
\pi_2(\mathbb{S}^2) \oplus \pi_3(\mathbb{S}^2) \arrow[r] & \pi_1(\mathfrak{FLeg}(\mathbb{R}^3)) \arrow[r] & \pi_1(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)) \oplus \mathbb{Z} \arrow[dll] \\
\pi_2(S^2) \arrow[r] & \pi_0(\mathfrak{FLeg}(\mathbb{R}^3)) \arrow[r] & \pi_0(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)) \oplus \mathbb{Z}
\end{tikzcd}
$$

Classification Theorem for $\mathfrak{FLeg}(\mathbb{R}^3)$

Theorem (Folklore)

Formal Legendrian embeddings are classified by their topological type as parametrized knots, their rotation number and the Thurston-Bennequin invariant.

$$\pi_0(\mathfrak{FLeg}(\mathbb{R}^3)) \cong \pi_0(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)) \oplus \mathbb{Z} \oplus \mathbb{Z}$$

Computation of the fundamental group.

Theorem (Fundamental group of $\mathfrak{FLeg}(\mathbb{R}^3)$. FMP.)

The sequence

$$0 \to \underbrace{\mathbb{Z}} \oplus \underbrace{\mathbb{Z}_m} \longrightarrow \pi_1(\mathfrak{FLeg}(\mathbb{R}^3)) \longrightarrow \pi_1(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)) \oplus \overset{\mathrm{rot}_{\pi_1}}{\boxed{\underbrace{\mathbb{Z}}}} \to 0$$

is exact, where $m \geq 0$.

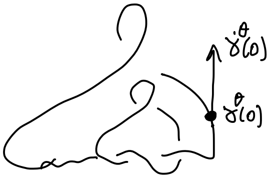
Theorem (Fundamental group of $\mathfrak{FHor}(\mathbb{R}^4)$. FMP.)

The sequence

$$0 \to \underbrace{\mathbb{Z}_2} \longrightarrow \pi_1(\mathfrak{FHor}(\mathbb{R}^4)) \longrightarrow \pi_1(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)) \oplus \underbrace{\mathbb{Z}} \to 0$$

is exact.

Geometric interpretation of formal invariants.



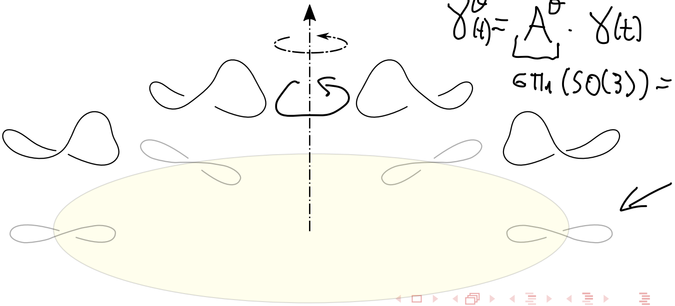
$$\mathrm{rot}_{\pi_1}(\gamma^\theta) = \deg\left(\theta \longmapsto \dot\gamma^\theta(0)\right)$$

Application: New examples of rigid loops.

Theorem (FMP)

For every knot type K and any Legendrian representative $\tilde{K}$, there exist infinitely many loops of Legendrian embeddings based at $\tilde{K}$ such that:

- *they are smoothly trivial.*
- *they are non trivial as loops of Legendrian embeddings.*



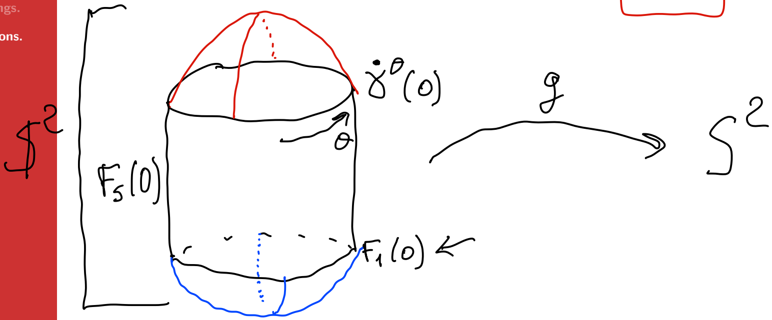
$$\gamma^{\theta}(t) = \underbrace{A^{\theta}}\cdot \gamma(t)$$

$$\in \pi_1(SO(3)) = \mathbb{Z}_2$$

Geometric interpretation of formal invariants.

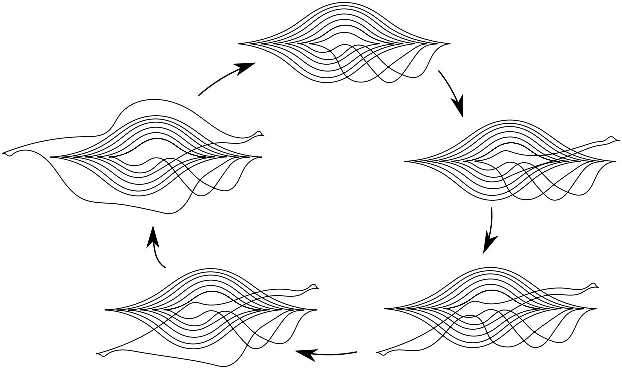
$$\mathbb{Z}_m \oplus \underbrace{\mathbb{Z}}_{\deg(g_{\gamma^\theta})} \longrightarrow \pi_1(\mathcal{F}Leg(\mathbb{R}^3)) \longrightarrow \underbrace{\pi_1(\mathrm{Emb}(S^1,\mathbb{R}^3))}_{0} \oplus \underbrace{\mathbb{Z}}_{0\ \mathrm{rot}_{\pi_1}}$$

$$[\gamma^\theta(t)] \longmapsto 0 \oplus 0$$

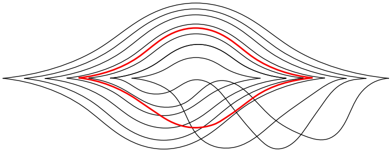


Application: Non-triviality of previous examples in the literature.

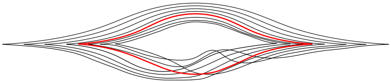
T. Kálmán [1] provided infinitely many examples of loops of Legendrians which are smoothly trivial but non-trivial in the space of Legendrians.



Scaling of the supporting knot

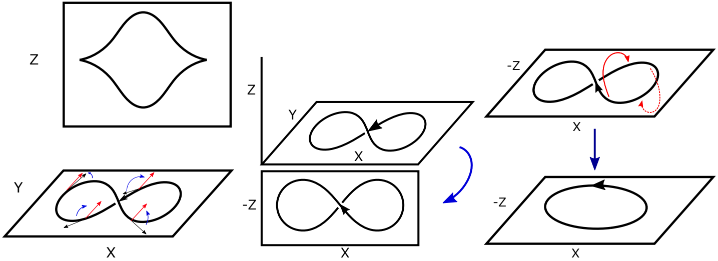


(a) Front projection of the knot and the core (in red).



(b) Knot C^1-close to the core.

C^1-approximation of the knot to the core, seen in the front projection.



(c) Step 2. (d) Step 3. (e) Step 4.

Construction of the path of loops into a simplified position.

Thank you very much for your attention!

Bibliography.

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