Concours Putnam
Atelier de Pratique
Le lundi, 2 octobre 12h30-13h30 (Salle: Pavillon André-Aisenstadt 5448)

Suites et séries

1. Let \( u \) be a real number with \( 0 < u < 1 \). Let \( u_0 = u \), and for \( n \geq 1 \) define \( u_n \) recursively by

\[
u_n = \frac{1}{u_{n-1}} + u.
\]

Prove that the sequence \( \{u_n\}_{n \geq 1} \) converges and find its limit.

2. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence such that \( \lim_{n \to \infty} (x_n - x_{n-2}) = 0 \). Show that

\[
\lim_{n \to \infty} \frac{x_n - x_{n-1}}{n} = 0.
\]

3. Does the series

\[
\sum_{n=0}^{\infty} \frac{n^n}{2^n}
\]

converge?

4. Decide if the series

\[
\sum_{n=2}^{\infty} \frac{1}{n \ln n}
\]

converges

5. Let \( a_n \) be a sequence of positive reals satisfying \( a_n \leq a_{2n} + a_{2n+1} \) for all \( n \). Prove that \( \sum a_n \) diverges.

6. The sequence \( a_n \) is monotonic and \( \sum a_n \) converges. Show that \( \sum n(a_n - a_{n+1}) \) converges.

7. Does \( \sum_{n=0}^{\infty} \frac{n!k^n}{(n+1)^n} \) converge or diverge for \( k = \frac{19}{7} \)?

8. The real sequence \( a_n \) satisfies \( a_n = \sum_{k=0}^{\infty} a_k^2 \). Show \( \sum a_n \) does not converge unless all \( a_n \) are zero.

9. The series \( \sum a_n \) of non-negative terms converges and \( a_i \leq 100a_n \) for \( i = n, n + 1, n + 2, \ldots, 2n \). Show that \( \lim_{n \to \infty} na_n = 0 \).