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Distribution-free option pricing

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Abstract

Nobody doubts the power of the Black and Scholes option pricing method, yet there are situations in which the hypothesis of a lognormal model is too restrictive. A natural way to deal with this problem consists of weakening the hypothesis, by fixing only successive moments and possibly the mode of the price process of a risky asset, and not the complete distribution. As a consequence of this generalization, the option price is no longer a unique value, but rather a range of possible values. In the present paper, we show how to find upper and lower bounds for this range, a range which turns out to be quite narrow in a lot of cases.

JEL classification: G130; C190; C650; E400

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Keywords: Black–Scholes; Option pricing; Limited information

1. Introduction

Since the famous paper of Black and Scholes (1973), the problem of how to determine prices for basic options has been solved at least to a certain extent. Many publications since then still deal with the same problem, mainly in two directions: firstly, the extension of their model to more complex classes of options, and secondly, alternatives or sophistications of their formula taking into account perceived deviations from the lognormal process, such as skewness, excess kurtosis, jumps and extreme events.

In the present paper, we want to make a contribution to this last category, by showing how to price options without imposing a complete model on the underlying price process – as is the case for the pricing formula of Black and Scholes. The main reason for our approach is the observation of several authors in the past, that – although the lognormal model, which makes up the foundation of the formula of Black and Scholes, is usually a rather good model for describing real price processes – there are some shortcomings of the model that can become important, and that can cause (more or less seriously) biased option prices. Without claiming any exhaustivity, we can cite e.g. interesting contributions of Teichmoeller (1971), Hull and White (1988), Becker (1991), Bakshi et al. (1997), Corrado and Su (1998), Gerber and Landry (1998), Sarwar and Krehbiel (2000), Backus et al. (2002), Kou (2002) and Gençay and Salih (2003).

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The method we present in this paper only utilizes successive moments of the underlying price process under a risk-neutral probability measure, and not the distribution itself. Thus, the hypothesis used to reach an option price is much weaker than is the case for the Black and Scholes formula. As a consequence, one unique value for the option price is not possible, since the knowledge of successive moments only gives limited information about the real process, such that expected pay-offs can only be calculated approximately. However, we will show that it is possible to construct close (and in some cases very close) absolute upper and lower bounds for the options prices, which can also be combined to yield one approximate price.

The paper is organized as follows. We start in Section 2 with a description of the problem and of the methodology. Afterwards, in Section 3, we present the results for bounds on European option prices when limited information about the price process is available. Proofs of these results are provided in the appendix. Section 4 is meant to illustrate our results numerically and graphically. Afterwards in Section 5, we show how the results can be extended to arithmetic Asian options. Section 6 concludes.

2. Description of the problem

2.1. The pricing of European call options

Consider a European call option on a risky asset with current price S, that matures at time T having exercise price K. In an arbitrage-free setting, the price of this option can be determined as

$$B_{EC}(T, K, S) = \mathrm{e}^{-rT}\ \mathbb{E}^{Q}\left[(S_T - K)_+\right], \tag{1}$$

where r denotes the risk-free interest rate, and where the stochastic process $\{S_t, t \geq 0\}$, starting at $S_0 = S$, describes the price process of the underlying risky asset. We assume that Q is the unique equivalent probability measure, such that the discounted price process is a martingale, or $\mathbb{E}^{Q}\left[\mathrm{e}^{-rt} S_t\right] = S$.

Following Merton (1973), we know that the option price must satisfy

$$\max(0, S - \mathrm{e}^{-rT} K) \leq B_{EC}(T, K, S) \leq S, \tag{2}$$

whatever the real price process of the underlying asset.

An exact option price can only be computed, if the distribution of the price process $\{S_t, t \geq 0\}$ is known for sure — which in reality is not the case. The most commonly used assumption is the Black and Scholes setting, where the price process is assumed to follow a geometric Brownian motion. This means that

$$S_t = S \cdot \mathrm{e}^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}, \tag{3}$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion. Thus, under the measure Q, the variables S_t/S are lognormally distributed with mean $(r - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. Under these assumptions, the price of a European call can be found according to the well-known Black and Scholes formula (see Black and Scholes (1973)), or

$$B_{EC}^{(B\&S)}(T, K, S) = S \cdot \Phi(d_1) - K\mathrm{e}^{-rT} \cdot \Phi(d_2), \tag{4}$$

with

$$\begin{aligned} d_1 &= \frac{1}{\sqrt{\sigma^2 T}}\left(\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right) T\right) \\ d_2 &= \frac{1}{\sqrt{\sigma^2 T}}\left(\ln(S/K) + \left(r - \frac{1}{2}\sigma^2\right) T\right), \end{aligned} \tag{5}$$

where $\Phi : \mathbb{R} \to [0, 1]$ denotes the cumulative distribution function of the standard normal distribution.

As mentioned in the introduction, the Black and Scholes pricing formula shows some imperfections in that the whole distribution of the price process is fixed by means of a lognormal model. Although this model performs well in a lot of cases (it is still the most widely used approach for the valuation of options), wrong prices can arise due to the strong assumptions, e.g. as the tails of the lognormal distribution are not as fat as in reality. In fact this problem is an aspect of the more general issue of model risk, a phenomenon which in the financial literature is often a subject of discussion, see e.g. Crouhy et al. (1998), Green and Figlewski (1999), Lhabitant et al. (2000), Rebonato (2001) and Giannetti et al. (2004).

In order to avoid this problem of model risk, we want to weaken the strong assumption of fixing the complete distribution of the price process, as has been done before by several authors. The first was Merton (1973), who formulates an upper and lower bound to option prices by means of general arbitrage considerations, requiring no specific information about the real price process. Levy (1985) extends these results. He works in a discrete setting, and he makes use of stochastic dominance criteria about the distributions of the return on investment, in particular an investment in the underlying security versus an investment in the call option, assuming the non-negativity of the consumption beta of the underlying security. He then shows that first degree stochastic dominance corresponds to the bounds of Merton, and he strengthens these bounds by proceeding to second degree stochastic dominance. His bounds are still not completely distribution-free, since concrete results depend on the distribution of the underlying stock. Lo (1987) derives upper bounds for option prices and for expected pay-offs that depend no longer on the distribution, but on the first and second moments of the price process. We will show that his result can be recovered by means of our methodology. In his paper, Lo notifies the difficulty that this bound works over a set of risk-neutral distributions with given moments, and not over the set of actual distributions, which has consequences for the interpretations. We will return to his considerations in Section 3, since we are faced with the same difficulty. Another contribution we think we ought to mention is Rodriguez (2003), where one can also find a brief overview of previous results. Rodriguez starts from a general expression for the call option price as a sum of three components, one of which contains the discount factor of the corresponding put option. By focusing on this discount factor, the author recognizes the former results of Merton, Levy and others; he extends the approach to a new tighter lower bound and he also derives an upper bound. We will compare our results numerically to the results of Rodriguez in Section 4.

In our paper we want to formulate new option bounds by making use of the knowledge of successive moments of the real price distribution instead of the complete distribution, in a risk-neutral setting. As a consequence of this milder hypothesis, we arrive at a range of acceptable prices instead of one single price, but the range turns out to be rather tight. We explain the method for European call options; however, it can be extended to other types of options, e.g. Asian arithmetic options, as will be shown later on.

2.2. Methodology

Suppose that we are dealing with a risk X for which we are interested in the expected value $\mathbb{E}[(X-K)_+]$ for a given value of K. Furthermore, suppose that the exact distribution of the risk X is not known, but that we have reliable estimates for the mean and for the variance, and preferably also for the mode or for the skewness. Although an exact calculation of the expected value $\mathbb{E}[(X-K)_+]$ in that case is not possible, we can derive boundary values that restrict the possible outcomes for the expected value, taken into account the information about the moments of the risk X.

This kind of problem has been treated in a rather fundamental way by De Vylder (1982), De Vylder and Goovaerts (1982) and Goovaerts et al. (1982). De Vylder transforms the basic problem into the "associated dual problem". Analytical solutions of this dual problem can be found in some cases, although in Goovaerts et al. (1982) the author admits that the method can only be analytically worked out in a limited number of cases. In Jansen et al. (1986) another reasoning is developed, which leads in fact directly to some kind of "dual problem" without mentioning it. Using this methodology, however, one can get more analytical results. Therefore, we will rely on this idea and its further applications in e.g. Heijnen (1989) to derive our results.

This means that in fact we are looking for

$$\sup_{F\in\mathcal{B}}\int_0^{+\infty}(x-K)_+\mathrm{d}F(x) \quad \text{and} \quad \inf_{F\in\mathcal{B}}\int_0^{+\infty}(x-K)_+\mathrm{d}F(x), \tag{6}$$

where $\mathcal{B}$ is the class of all distribution functions with domain $\mathbb{R}^+$ and with moments (and mode) as given.

Now, if $P(x)$ is a polynomial of degree 2 or less, the value of $\int_0^{+\infty}P(x)\mathrm{d}F(x)$ only depends on the first two moments of F. If $P(x)$ is a polynomial of degree 3, the value of the integral can also depend on the third moment of F. As a consequence, its value is the same for each distribution $F\in\mathcal{B}$ in the case where these moments are known. Therefore, the problem of finding the supremum in (6) can be reduced to the problem of finding such a polynomial

$P(x)$ greater than $(x-K)_+$ on $\mathbb{R}^+$ and such that for some distribution $G \in \mathcal{B}$ we have

$$\int_0^{+\infty} P(x)\mathrm{d}G(x) = \int_0^{+\infty} (x-K)_+ \mathrm{d}G(x). \tag{7}$$

When looking for the infimum, this polynomial should be smaller than $(x-K)_+$ on $\mathbb{R}^+$.

Note that the method is also valid if $(x-K)_+$ is replaced by a more general function $f : \mathbb{R} \to \mathbb{R}^+$. More details about this methodology can be found in Heijnen (1989).

3. Bounds for European option prices

Since the price of European call options depends on the expected pay-off

$$\mathbb{E}\left[(S_T - K)_+\right], \tag{8}$$

the methodology described in Section 2.2 can be applied straightforwardly. This results in absolute bounds that restrict the possible outcomes for the expected pay-off, taking into account the information about the moments of the risk S_T. By absolute bounds, we mean that the bounds hold for any distribution with given parameters, and that there exists at least one distribution for which the bounds are actually reached.

In order to be able to write the results as a function of the current asset price S, we will use information about moments of the relative future price S_T/S instead of the moments of S_T. We will use the notation μ_r for the non-central r-th moment, i.e.

$$\mu_r = \mathbb{E}\left[(S_T/S)^r\right]; \tag{9}$$

for the mode of S_T/S, if it exists, we will use the notation m.

When we talk about all possible distributions with given moments, we have to mention a difficulty we are faced with. In fact, the bounds are calculated over the set of risk-neutral distributions and not the actual distributions for the price process. As was also described in Lo (1987), it is possible to derive relations between the two types of distribution in the case where an equilibrium asset-pricing framework is specified. In general, unfortunately, such a relation is not always known, and as a consequence the bounds of the following subsections do not have immediate links to practice in that case. From a theoretical point of view, however, our results remain relevant; moreover, they can be transformed into real interpretable numbers in the case where more specifications are taken into account for the underlying price process. We cite Lo (1987) again for an illustration of this reasoning.

Note that, in order to guarantee the existence of a distribution function on $\mathbb{R}^+$ with given mode and moments, these "known" parameters cannot be chosen completely arbitrarily. This is summarized in the following lemma.

Lemma 1. *If a non-negative variable X has moments μ_1, μ_2 and μ_3 and mode m, then $\mu_2 \geq \mu_1^2$, $\mu_3 \geq \frac{\mu_2^2}{\mu_1}$ and $2\mu_1 \geq m$.*

Proof. If X is any variable satisfying the conditions mentioned in the lemma, then $(X-\mu_1)^2 \geq 0$ and $(X-\frac{\mu_2}{\mu_1})^2 \cdot X \geq 0$. Taking expectations of both inequalities, the first two inequalities follow.

The last inequality is an immediate consequence of the Khinchin lemma (see Appendix A.3). □

Remark that — without loss of generality — we can restrict ourselves to the investigation of call options. Indeed, making use of the put–call parity, results for call option prices can always be transformed into prices for put options.

The proofs of the results in Sections 3.1–3.3 herein are provided in the appendix.[1]

[1] For the correctness of the formulas, the exercise price K has to be non-negative, which is of course the case.

3.1. Two moments are known

If two moments of the stochastic variable S_T/S are known, the European call option price (1) satisfies the boundary conditions

$$e^{-rT} G_1(S) \leq B_{EC}(T, K, S) \leq e^{-rT} G_2(S), \tag{10}$$

with

$$G_1(S) = \begin{cases} 0 & \text{if } S \leq \dfrac{1}{\mu_1} K \\ \mu_1 S - K & \text{if } S \geq \dfrac{1}{\mu_1} K \end{cases} \tag{11}$$

and

$$G_2(S) = \begin{cases} \dfrac{1}{2}(\mu_1 S - K) + \dfrac{1}{2}\sqrt{S^2(\mu_2 - \mu_1^2) + (\mu_1 S - K)^2} & \text{if } S \leq \dfrac{2\mu_1}{\mu_2} K \\ \mu_1 S - \dfrac{\mu_1^2}{\mu_2} K & \text{if } S \geq \dfrac{2\mu_1}{\mu_2} K. \end{cases} \tag{12}$$

Note that in this last result we recognize the upper bound for the expected pay-off derived by Lo (1985).[2]

3.2. Three moments are known

Define the function $p : \mathbb{R} \to \mathbb{R} : x \mapsto p(x)$ by

$$p(x) = (\mu_2 - \mu_1^2)x^2 + (\mu_1\mu_2 - \mu_3)x + (\mu_1\mu_3 - \mu_2^2) \tag{13}$$

and denote its zeros (which always exist and which are necessarily real and positive) by v and w, with $v < w$.

Furthermore, consider the equation

$$x^3 + Ax^2 + Bx + C = 0 \tag{14}$$

with coefficients

$$\begin{cases} A = -\dfrac{2\mu_2 S + 3\mu_1 K}{2\mu_1 S} \\ B = \dfrac{2\mu_2 K}{\mu_1 S} \\ C = -\dfrac{\mu_3 K}{2\mu_1 S} \end{cases} \tag{15}$$

and denote by q the unique root of this equation in the interval $[w, +\infty[$.

If three moments of the stochastic variable S_T/S are known, the European call option price (1) satisfies the boundary conditions

$$e^{-rT} G_3(S) \leq B_{EC}(T, K, S) \leq e^{-rT} G_4(S), \tag{16}$$

with

$$G_3(S) = \begin{cases} 0 & \text{if } S \leq \dfrac{\mu_1}{\mu_2} K \\ \mu_1 S - K + \dfrac{-p(K)S^2}{\mu_3 S - \mu_2 K} & \text{if } \dfrac{\mu_1}{\mu_2} K \leq S \leq \dfrac{1}{v} K \\ \mu_1 S - K & \text{if } S \geq \dfrac{1}{v} K \end{cases} \tag{17}$$

[2] In fact it is the auxiliary function $F_2(S)$ from the proof of this result in Appendix A.1 which is identical to the result of Lo (1985).

and

$$G_4(S) = \begin{cases} \dfrac{\mu_3\mu_1 - \mu_2^2}{\mu_3 - 2q\mu_2 + q^2\mu_1} \cdot \dfrac{qS - K}{q} & \text{if } S \le \dfrac{3w - v}{2w^2} K \\ \dfrac{\mu_3\mu_1 - \mu_2^2}{\mu_3 - 2w\mu_2 + w^2\mu_1} \cdot \dfrac{wS - K}{w} & \text{if } \dfrac{3w - v}{2w^2} K \le S \le \dfrac{2}{v + w} K \\ \dfrac{1}{2}(\mu_1 S - K) + \dfrac{1}{2}\sqrt{S^2(\mu_2 - \mu_1^2) + (\mu_1 S - K)^2} & \text{if } \le \dfrac{2}{v + w} K \le S \le \dfrac{2\mu_1}{\mu_2} K \\ \mu_1 S - \dfrac{\mu_1^2}{\mu_2} K & \text{if } S \ge \dfrac{2\mu_1}{\mu_2} K. \end{cases} \tag{18}$$

3.3. Two moments and the mode are known

The mode can only be taken into account if it satisfies the condition

$$m \le \mu_1. \tag{19}$$

This is for technical reasons, which will become clear in the calculations (see Appendix A.3). Since we work with right-tailed non-negative distributions, this requirement does not impose any restrictions.

Due to the introduction of the Khinchin transform (see Appendix A.3), the results are constructed by means of two transformed moments

$$\nu_1 = 2\mu_1 - m \tag{20}$$

$$\nu_2 = 3\mu_2 - 2m\mu_1. \tag{21}$$

The natural condition $\nu_2 \ge \nu_1^2$ (see Lemma 1) implies that, besides condition (19), the mode also has to satisfy the constraints

$$\mu_1 - \sqrt{3}\sqrt{\mu_2 - \mu_1^2} \le m \le \mu_1 + \sqrt{3}\sqrt{\mu_2 - \mu_1^2}. \tag{22}$$

Consider the equation

$$x^3 + Dx^2 + Ex + F = 0 \tag{23}$$

with coefficients

$$\begin{cases} D = -3K \\ E = (4\nu_1 + 2m)SK - (2m\nu_1 + \nu_2)S^2 \\ F = 2m\nu_2 S^3 - (2m\nu_1 + \nu_2)S^2 K. \end{cases} \tag{24}$$

Case 1. $S \le \frac{1}{m} K$

If two moments and the mode of the stochastic variable S_T/S are known, the European call option price (1) satisfies the boundary conditions

$$\mathrm{e}^{-rT} G_5^a(S) \le B_{EC}(T, K, S) \le \mathrm{e}^{-rT} G_6^a(S), \tag{25}$$

with

$$G_5^a(S) = \begin{cases} 0 & \text{if } S \le \dfrac{1}{\nu_1} K \\ \mu_1 S - K + \dfrac{(K - mS)^2}{2S(\nu_1 - m)} & \text{if } S \ge \dfrac{1}{\nu_1} K \end{cases} \tag{26}$$

and

$$G_6^a(S) = \begin{cases} \dfrac{S^2(\nu_2 - \nu_1^2)(y-K)^2}{2(S^2(\nu_2 - \nu_1^2) + (y - \nu_1 S)^2)(y - mS)} & \text{if } S \le \dfrac{\nu_1(3\nu_2 - 2m\nu_1)}{\nu_2^2} K \\ \dfrac{\nu_1(\nu_2 S - \nu_1 K)^2}{2S\nu_2(\nu_2 - m\nu_1)} & \text{if } S \ge \dfrac{\nu_1(3\nu_2 - 2m\nu_1)}{\nu_2^2} K, \end{cases} \tag{27}$$

where y is the unique root in the interval $[\max(K, \frac{\nu_2}{\nu_1} S), +\infty[$ of Eq. (23).

Case 2. $S \ge \frac{1}{m} K$

If two moments and the mode of the stochastic variable S_T/S are known, the call option price (1) satisfies the boundary conditions

$$e^{-rT} G_5^b(S) \le B_{EC}(T, K, S) \le e^{-rT} G_6^b(S), \tag{28}$$

with

$$G_5^b(S) = \mu_1 S - K \tag{29}$$

and

$$G_6^b(S) = \begin{cases} \mu_1 S - K + \dfrac{S^2(\nu_2 - \nu_1^2)(z-K)^2}{2(S^2(\nu_2 - \nu_1^2) + (z - \nu_1 S)^2)(mS - z)} & \text{if } S \le \dfrac{2m\nu_1 + \nu_2}{2m\nu_2} K \\ \mu_1 S - K + \dfrac{K^2}{2mS} \dfrac{\nu_2 - \nu_1^2}{\nu_2} & \text{if } S \ge \dfrac{2m\nu_1 + \nu_2}{2m\nu_2} K, \end{cases} \tag{30}$$

where z is the unique root in the interval $[0, K]$ of Eq. (23).

4. Numerical and graphical illustrations

In the following subsections, we will compare our pricing bounds with the Black and Scholes formula, and with the bounds of Merton and Rodriguez.

Note that for a lognormally distributed price process, the moments and the mode can be calculated as[3]

- $\mu_1^* = \mathbb{E}[S_T/S] = e^{rT}$
- $\mu_2^* = \mathbb{E}[(S_T/S)^2] = e^{2rT + \sigma^2 T}$
- $\mu_3^* = \mathbb{E}[(S_T/S)^3] = e^{3rT + 3\sigma^2 T}$
- $m^* = e^{rT - \sigma^2 T}$. (31)

For the numerical examples, we will use parameter values $r = 0.1$, $\sigma = 0.2$, $T = 1$, which are the same as in Rodriguez (2003). For these choices, the moments can be calculated as

- $\mu_1^* = 1.10517$
- $\mu_2^* = 1.27125$
- $\mu_3^* = 1.52196$
- $m^* = 1.06184$. (32)

For the exercise price, we will use a value $K = 50$€.

[3] Throughout this section, we will always use the notation μ_r for general choices of the parameters, and μ_r^* if these parameters are taken from a lognormal distribution.

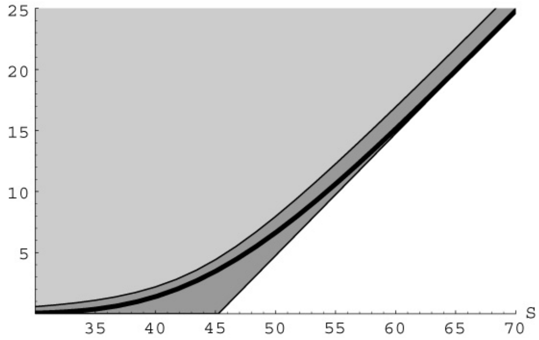

Fig. 1. Price of a European call as a function of the current asset price: Black and Scholes price versus the tolerable area if two moments are known.

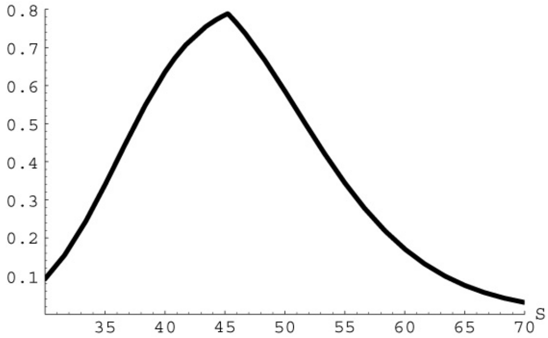

Fig. 2. Curve of the value of α (corresponding to equality of $B_{EC}^{(\alpha)}(T, K, S)$ and the Black and Scholes price) as a function of the current asset price.

4.1. Two moments are known

We start by comparing the Black and Scholes price (4) and Merton's bounds to the absolute bounds of Section 3.1, based on the knowledge of the two first moments of S_T/S. In order to make this comparison possible, for the numerical illustrations we choose the moments as if the price process were lognormally distributed, with parameters as in (32).

Fig. 1 shows the Black and Scholes price $B_{EC}^{B\&S}(T, K, S)$ (black line) together with Merton's range (pale grey) and our tolerable range $[\mathrm{e}^{-rT}G_1(S), \mathrm{e}^{-rT}G_2(S)]$ (dark grey), as a function of the current asset price S. The functions G_1 and G_2 are calculated as in Eqs. (11) and (12), with moments μ_1^* and μ_2^* as in (32). It is obvious that the Black and Scholes price completely fits in with the area that corresponds to the first two moments for the price process. For all values of S, the shape of the option price nicely corresponds to the shape of the upper bound $\mathrm{e}^{-rT}G_2(S)$. However, for options that are deep in-the-money or deep out-of-the-money, the range becomes rather narrow, and the Black and Scholes price comes close to the minimal tolerable value. This seems to indicate that the risk that the Black and Scholes formula overprices the real price for options that are deep in-the-money or deep out-of-the-money, is rather small compared to the risk of underpricing. Nevertheless, it is the risk of overpricing which is a significantly perceived phenomenon in practice and which is mentioned in the literature, e.g. in Hull (2003).

This last consideration can also be seen very clearly, if we compare the Black and Scholes price with an interpolating pricing formula based on our upper and lower bound, or

$$B_{EC}^{(\alpha)}(T, K, S) = \mathrm{e}^{-\delta T}\left[G_1(S) + \alpha(G_2(S) - G_1(S))\right], \tag{33}$$

with $0 \leq \alpha \leq 1$, and with G_1 and G_2 as defined in Eqs. (11) and (12).

Fig. 2 shows the evolution of the value of α for which the price (33) is identical to the Black and Scholes price (4), as a function of the current asset price. For options that are deep in-the-money, or deep out-of-the-money, the value of α is very low, and the Black and Scholes price is just above the minimal possible value. The maximal value of α is reached for a current asset price equal to the present value of the exercise price, or $S = \mathrm{e}^{-rT}K$.

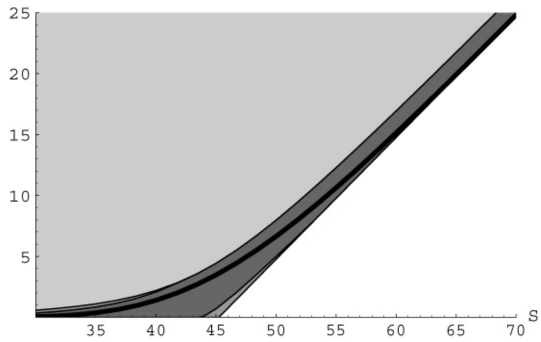

Fig. 3. Price of a European call as a function of the current asset price: Black and Scholes price versus the tolerable area if two (medium grey) and three (dark grey) moments are known.

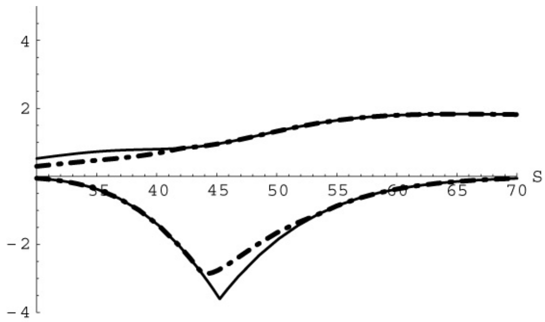

Fig. 4. Deviation of lower and upper bounds from the Black and Scholes price for a European call as a function of the current asset price if two (solid line) and three (dotted line) moments are fixed.

Note that an interpolating formula as presented in Eq. (33) could be useful for pricing the option if the range of possible values is sufficiently small (see e.g. Section 4.2). Of course, in that case we are confronted with the difficulty of choosing a plausible value for α in order to end up with an acceptable option price.

It is important to remark that for any two given moments μ_1 and μ_2, a lognormal distribution can be fitted. As a consequence, the Black and Scholes price will definitely fall in the tolerable area. However, if the third moment or the mode of the price at maturity time is known, this will not necessarily be compatible with a lognormal distribution for the price process. In that case, the Black and Scholes price can be seriously biased and the possibility even exists of a Black and Scholes price falling outside the tolerable range corresponding to the known moments. We will return to this consideration in the next two subsections.

4.2. Three moments are known

As in the previous subsection, we start by comparing the Black and Scholes price (4) to the absolute bounds of Section 3.2, but now based on the knowledge of the first three moments, where the moments are chosen as if the price process were lognormally distributed, with parameters as before, or $r = 0.1$, $\sigma = 0.2$, $T = 1$, $K = 50$€.

Fig. 3 shows the Black and Scholes price $B_{EC}^{B\&S}(T, K, S)$ (black line), Merton's range (pale grey), the tolerable area for two moments $[\mathrm{e}^{-rT}G_1(S), \mathrm{e}^{-rT}G_2(S)]$ (medium grey), and the tolerable area for three moments $[\mathrm{e}^{-rT}G_3(S), \mathrm{e}^{-rT}G_4(S)]$ (dark grey) as a function of the current asset price S. The functions G_3 and G_4 are calculated as in Eqs. (17) and (18), with moments μ_1^*, μ_2^* and μ_3^* as in (32). Again, in the case where the third moment is chosen as if the real price process follows a lognormal model, the Black and Scholes price obviously fits in with the area. In Fig. 4, we depict the deviation of each pricing bound from the Black and Scholes price. This deviation is represented by the vertical distance between the curves and the horizontal axis. The dotted line corresponds to the knowledge of three moments, the solid line that of two moments.

Remark that in the case of a third moment taken from the lognormal model, the knowledge of the third moment does not reduce fundamentally the tolerable range; the effect of an extra moment then is rather limited.

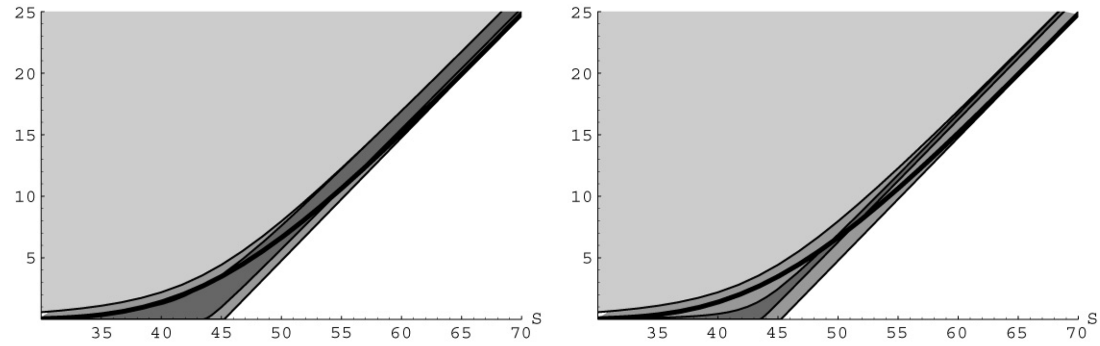

Fig. 5. Price of a European call as a function of the current asset price: Black and Scholes price versus the tolerable area if the third moment differs from the lognormal value.

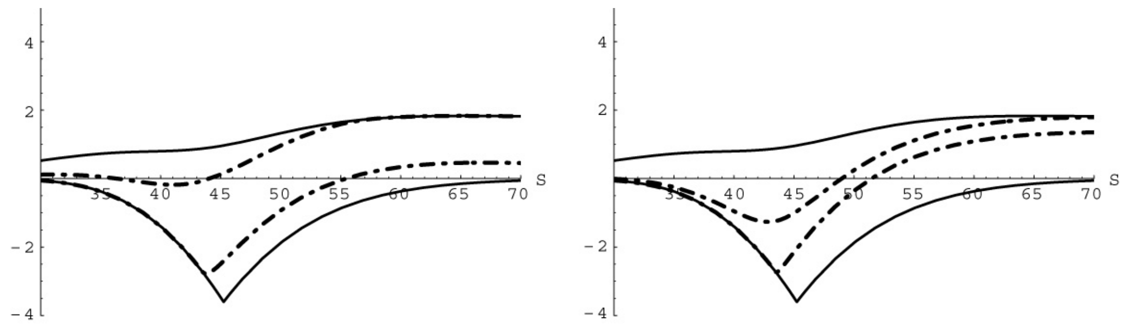

Fig. 6. Deviation of lower and upper bounds from the Black and Scholes price for a European call as a function of the current asset price if two (solid line) and three (dotted line) moments are fixed, with a third moment lower than the lognormal third moment.

The previous figures and remarks were made under the assumption that the lognormal model is correct: indeed, we compared the Black and Scholes price with the absolute bounds calculated under the hypothesis that the first three moments for the real price process coincide with the first three moments of the lognormal model. However, if the lognormal model is invalid, the pictures change completely, especially when the "real" third moment is smaller than μ_3^*, i.e. when the price process exhibits a left tail that is much heavier than the right tail.

Fig. 5 shows the Black and Scholes price and the tolerable area (for two and three moments) for two choices of the third moment, different from the third moment μ_3^* of (32).[4] In the left graph, where $\mu_3 = 1.49$, the Black and Scholes price is situated at the boundaries of the tolerable area: for options that are deep in-the-money, the Black and Scholes price is just too low, while for options that are deep out-of-the-money, the Black and Scholes price is just too high. The right graph shows the situation for $\mu_3 = 1.47$: the assumption of a lognormal distribution as in the Black and Scholes pricing formula now is completely incompatible with the assumption about the skewness, and the Black and Scholes price falls outside the tolerable range for almost all values of S. In Fig. 6, as before we depict the deviation of each pricing bound from the Black and Scholes price for the same choices of the third moment (the dotted line corresponds to the knowledge of three moments, the solid line that of two moments). If the third moment of the real price process is equal to $\mu_3 = 1.49$ as in the left graph, the Black and Scholes price is not compatible with reality for a current asset price between 36.39€ and 44.19€ or above 55.42€. If the third moment of the real price process is equal to $\mu_3 = 1.47$ as in the right graph, the Black and Scholes price is not compatible with reality for a current asset price between 29.61€ and 48.96€ or above 51.46€.

In Table 1 we compare our absolute bounds for option prices with the results of the Black and Scholes formula and with the bounds in Rodriguez (2003, p. 158), for different choices of the third moment. In this respect, it is important to recall that for the bounds of Rodriguez, just as is the case for the Black and Scholes prices, the asset price

[4] Note that the minimal possible value for the third moment follows from the formula $\mu_3 \geq \frac{\mu_2^2}{\mu_1}$ (see Lemma 1); for the current parameter choices, this minimal value is 1.46228.

Table 1
Results for the prices of a European call option for different choices of the third moment, with $\mu_1^* = 1.10517$ and $\mu_2^* = 1.27125$ as in (32), corresponding to a lognormal distribution with $r = 0.1$ and $\sigma = 0.2$

μ_3 (skewness)	S	New lower bound	New upper bound	Black and Scholes price	Lower bound Rodriguez (2003, p. 158)	Upper bound Rodriguez (2003, p. 158)
1.47	30	0	**0.0480**	0.0538	0	0.1806
	40	0	**0.4432**	1.3950	**0.7965**	2.7767
(−4.05479)	50	**6.2785**	**6.8827**	6.6348	6.1724	9.8149
	60	**16.2215**	**16.6795**	15.1292	14.9845	19.1693
	70	**26.1678**	**26.6088**	24.8157	24.7867	29.0754
1.49	30	0	**0.1686**	0.0538	0	0.1806
	40	0	**1.2302**	1.3950	**0.7965**	2.7767
(−2.25766)	50	5.7045	**7.5998**	6.6348	**6.1724**	9.8149
	60	**15.4681**	**16.9279**	15.1292	14.9845	19.1693
	70	**25.2685**	**26.6351**	24.8157	24.7867	29.0754
1.51	30	0	0.2840	0.0538	0	**0.1806**
	40	0	**1.7926**	1.3950	**0.7965**	2.7767
(−0.460529)	50	5.2266	**7.9618**	6.6348	**6.1724**	9.8149
	60	14.7816	**16.9279**	15.1292	**14.9845**	19.1693
	70	24.7581	**26.6351**	24.8157	**24.7867**	29.0754
1.53	30	0	0.3949	0.0538	0	**0.1806**
	40	0	**2.1950**	1.3950	**0.7965**	2.7767
(+1.3366)	50	4.8226	**7.9618**	6.6348	**6.1724**	9.8149
	60	14.7581	**16.9279**	15.1292	**14.9845**	19.1693
	70	24.7581	**26.6351**	24.8157	**24.7867**	29.0754

Note that for the lognormal distribution based on these parameters and with these two moments, the third moment can be calculated as $\mu_3^* = 1.52196$, as in (32), with skewness equal to 0.6136. The remaining parameters are $T = 1$ and $K = 50€$ as before.

is assumed to follow a lognormal distribution. This also means that the third moment is not used for the computation of the values in these columns, which explains the recurring results in the last three columns. Black and Scholes prices that are incompatible with the absolute bounds, are underlined. These values occur when the skewness[5] is negative and in absolute value larger than the skewness for the lognormal distribution: this means that the left tail of the real distribution is heavier than is the case for the lognormal model. For each choice of the parameters considered in the table, the most accurate upper and lower bound are put in bold type. Note that our upper bound is more accurate in almost all cases; for the lower bound, our method gives tighter results for higher values of the current asset price in the case of a negative and rather high skewness. The parameter values are the same as before, or $r = 0.1$, $\sigma = 0.2$, $T = 1$, $K = 50€$.

4.3. Two moments and the mode are known

We now compare the Black and Scholes price (4) to the absolute bounds of Section 3.3, based on the knowledge of the first two moments and the mode, where the moments and the mode are chosen as if the price process were lognormally distributed, with parameters as before, or $r = 0.1$, $\sigma = 0.2$, $T = 1$, $K = 50€$.

Fig. 7 shows the Black and Scholes price $B_{EC}^{B\&S}(T, K, S)$ (black line), Merton's range (pale grey), the tolerable area for two moments $[\mathrm{e}^{-rT} G_1(S), \mathrm{e}^{-rT} G_2(S)]$ (medium grey), and the tolerable area for two moments and the mode $[\mathrm{e}^{-rT} G_5(S), \mathrm{e}^{-rT} G_6(S)]$ (dark grey) as a function of the current asset price S, for parameter choices as before. The functions G_5 and G_6 are calculated as in Eqs. (26), (27), (29) and (30), with moments μ_1^*, μ_2^* and mode m^* as in (32).

[5] The skewness in the table is calculated by means of the Fisher formula: skewness $= \frac{\mathrm{E}[(X-\mu)^3]}{\mathrm{E}[(X-\mu)^2]^{3/2}}$ or with our notation for the non-central moments: skewness $= \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}}$.

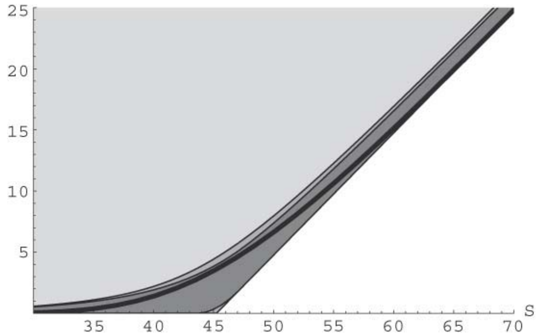

Fig. 7. Price of a European call as a function of the current asset price: Black and Scholes price versus the tolerable area if two moments (medium grey) and the mode (dark grey) are known.

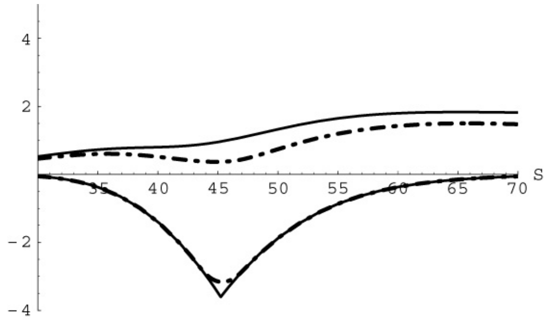

Fig. 8. Deviation of lower and upper bounds from the Black and Scholes price for a European call as a function of the current asset price if two moments (solid line) and the mode (dotted line) are fixed.

One can see that – if the first two moments and the mode of the lognormal model correspond to the values of the "real" price process– the Black and Scholes price seems to be rather high in the neighbourhood of the accrued value or for options that are almost at-the-money, and rather low for options that are deep in-the-money or deep out-of-the-money. In Fig. 8, we depict the deviation of each pricing bound from the Black and Scholes price (the dotted line corresponds to the knowledge of two moments and the mode, the solid line that of two moments).

Fig. 9 shows the Black and Scholes price and the tolerable range (for two moments with and without the given mode) for a choice of the mode of 0.87, which is different from the mode m^* of (32).[6] As was the case in Section 4.2, the tolerable range becomes rather narrow, and the Black and Scholes price is situated near the boundaries of the range. In Fig. 10, again we depict the deviation of each pricing bound from the Black and Scholes price, for the same choice for the mode (the dotted line corresponds to the knowledge of two moments and the mode, the solid line that of two moments). Note that the deviation is much smaller than for the case where the mode of the lognormal model is equal to the real mode of the price process.

5. Extension to arithmetic Asian call options

Due to the fact that our method only requires the knowledge of the first two or three moments of the quantity under investigation, it can be used for a broad range of option types. As an example, we illustrate our results for arithmetic Asian options, for which the pricing is far from evident.

[6] Note that the maximal possible value for the mode follows from the condition $m \leq \mu_1$ (see Section 3); for the present parameter choices, this maximal value is 1.10517.

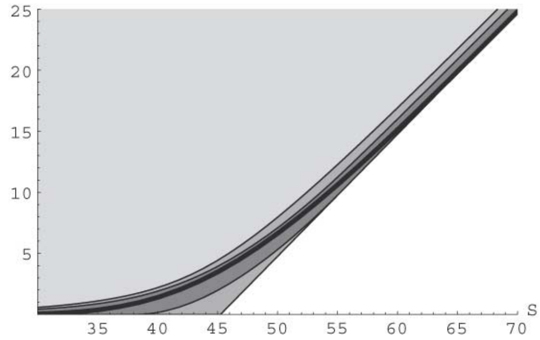

Fig. 9. Price of a European call as a function of the current asset price: Black and Scholes price versus the tolerable area if the mode differs from the lognormal value.

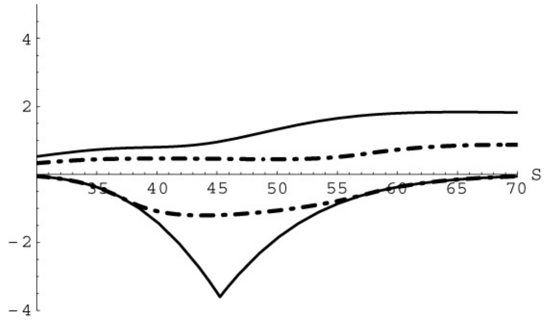

Fig. 10. Deviation of lower and upper bounds from the Black and Scholes price for a European call as a function of the current asset price if two moments (solid line) and the mode (dashed line) are fixed, with a mode lower than the lognormal mode.

The price of an arithmetic Asian call option (European style) that matures at time T with exercise price K and with n averaging dates, is given by

$$B_{AC}(T, S, K, n) = \mathrm{e}^{-rT}\ \mathbb{E}^{Q}\left[\left(\frac{1}{n}\sum_{i=0}^{n-1} S_{T-i} - K\right)_{+}\right], \tag{34}$$

where as before r denotes the risk-free interest rate, Q is the unique equivalent probability measure, and the stochastic process $\{S_t, t \geq 0\}$, starting in $S_0 = S$, describes the price process of the underlying risky asset.

Due to the appearance of a sum of dependent variables, an explicit expression for the price of such a call is even not known in a Black and Scholes setting, for the simple reason that a sum of (dependent) lognormal variables is no longer lognormal. Although a new lognormal distribution can be a good approximation for the distribution of such a sum (see e.g. Levy (1992)) and although this approximation has been used for pricing purposes (see e.g. Levy (1992) and Jacques (1996)), it is shown in Vyncke et al. (2004) that for options that are out-of-the-money, this approach causes structural underpricing. An approximation by means of inverse Gaussian distributions (see Jacques (1996) and Milevsky and Posner (1998)) shows exactly the same kinds of problems, especially for options that are deep in-the-money or out-of-the-money.

Applying the methodology described in Section 2.2, we can calculate bounds for arithmetic Asian options, when estimates are known for the first two or three moments. As before, we compare our results with results obtained in a Black and Scholes setting. The graphs show the bounds as a function of the initial stock price S for an exercise price $K = 100$€, time to maturity T equal to 120 days, 30 averaging dates, and parameters $r = \ln(1.09)/365$ and $\sigma = 0.2/\sqrt{365}$. These values are the same as in Jacques (1996) and Vyncke et al. (2004).

We suggest calculating the bounds of the option price, starting directly from the moments for the sum $A_T = \frac{1}{n}\sum_{i=0}^{n-1} S_{T-i}$. If the price process is modelled by a lognormal process, the moments for this sum A_T can be calculated

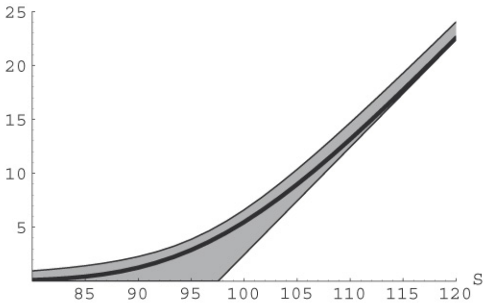

Fig. 11. Price of an Asian call as a function of the current asset price: estimated Black and Scholes price versus the tolerable area if two moments are known.

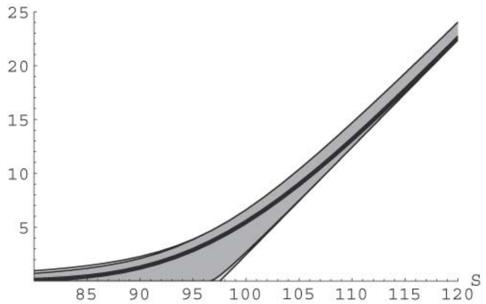

Fig. 12. Price of an Asian call as a function of the current asset price: estimated Black and Scholes price versus the tolerable area if two (medium grey) and three (dark grey) moments are known.

as

$$\begin{aligned}
&\bullet\quad \mu_1^{**} = \mathbb{E}[A_T/S] = \frac{1}{n}\sum_{i=0}^{n-1} \mathrm{e}^{r(T-i)}\\
&\bullet\quad \mu_2^{**} = \mathbb{E}[(A_T/S)^2] = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \mathrm{e}^{r(T-i+T-j)}\mathrm{e}^{\sigma^2\min(T-i,T-j)}\\
&\bullet\quad \mu_3^{**} = \mathbb{E}[(A_T/S)^3] = \frac{1}{n^3}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \mathrm{e}^{r(T-i+T-j+T-k)}\cdot \mathrm{e}^{2\sigma^2\min(T-i,T-j,T-k)}\mathrm{e}^{\sigma^2\mathrm{mid}(T-i,T-j,T-k)},
\end{aligned} \tag{35}$$

where $\mathrm{mid}(a,b,c) = a+b+c-\min(a,b,c)-\max(a,b,c)$.

For the choices of the parameters as suggested, they are equal to

$$\begin{aligned}
&\bullet\quad \mu_1^{**} = 1.02522\\
&\bullet\quad \mu_2^{**} = 1.06273\\
&\bullet\quad \mu_3^{**} = 1.11381.
\end{aligned} \tag{36}$$

The following graphs compare the tolerable range for the exact price of the arithmetic Asian call option with moments as given above, with estimates of the option price if it is assumed that the underlying price process is lognormally distributed. Following the idea of Levy (see Levy (1992)), an estimate of the option price can be derived by fitting a new lognormal distribution to the arithmetic averaging process A_T. For the parameters (r, σ, T and n) as before, this new lognormal distribution has parameters $\breve{r} = \ln(1.07872)/365$ and $\breve{\sigma} = 0.183054/\sqrt{365}$.

Fig. 11 refers to the knowledge of two moments of the sum process A_T, while in Fig. 12 the relation between the estimated Black and Scholes price and the tolerable area is shown in the case where three moments are known.

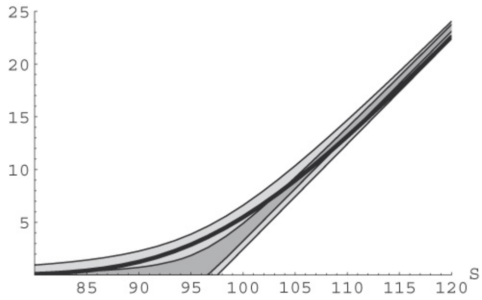

Fig. 13. Price of an Asian call as a function of the current asset price: estimated Black and Scholes price versus the tolerable area if the third moment differs from the lognormal moment.

Both figures were produced under the assumption that the lognormal model is correct, and we compared the estimated Black and Scholes price with the absolute bounds, calculated with the first three moments of the fitted lognormal model. However, if the lognormal model is invalid, again the pictures can be very different.

Indeed, Fig. 13 shows the situation when the third moment differs from the third moment μ_3^{**} of (36). In the graph, we drew up the area for $\mu_3 = 1.105$; it is clear that now the Black and Scholes price is not acceptable for all values of S. Even for options that are almost at-the-money, the estimated Black and Scholes price is not correct.

6. Conclusion

In this paper we showed how close bounds on option prices can be derived when limited information about the underlying price process is available. The method only uses values for successive moments of the price of the underlying asset at maturity time, and not the complete distribution. We showed that in many cases the Black and Scholes option pricing formula performs very well, but that there exist situations (and not only very particular ones) where this famous formula yields prices which are not compatible with specific characteristics of the underlying asset. After an application of our approach to European call options, we showed how to extend the method e.g. to arithmetic Asian call options.

Acknowledgement

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Appendix A. Proofs of the results

Call option prices are determined by means of the calculation of expected values of the form $\mathbb{E}\left[(S_T - K)_+\right]$. As mentioned in Section 2.2, we want to construct close bounds for such expected values if information is available for the moments of the variable under investigation. For G, we will use one-, two- or three-point distributions belonging to the set $\mathcal{B}$, while for the polynomial $P(x)$ we will choose that polynomial that matches $(x - K)_+$ at the mass points of G such that (7) is satisfied.

Following the method, this variable here is S_T. However, in order to be able to present the results as a function of the (moving) current asset price S, which is much more interesting than a result as a function of the (fixed) exercise price, we use not the moments of S_T but those of S_T/S. Since the results in this paper are based on a method as introduced and developed in the earlier papers Jansen et al. (1986) and Heijnen (1989), for each of the proofs, we will first derive the bounds as a function of the exercise price, if two or more moments for the price S_T are given. Afterwards, we will transform the results into bounds as a function of the current asset price, making use of successive moments of the ratio S_T/S. Adding the actualization factor e^{-rT} completes the proofs.

Since we work with two types of moments, two different notations will be needed. When talking about the ratio S_T/S, which is the case in the main part of the paper, we use the notation μ_k for the moments and m for the mode, or

$$\mu_k = \mathbb{E}\left[(S_T/S)^k\right] \tag{A.1}$$

as mentioned in Section 3. When working with the original price process, which is the case here in the appendix for the construction of the proofs, we will use the notation $\tilde{\mu}_k$ for the moments and $\tilde{m}$ for the mode, or

$$\tilde{\mu}_k = \mathbb{E}\left[(S_T)^k\right]. \tag{A.2}$$

The same principle is used for the transformed moments in the case where the mode is given, or

$$\nu_1 = 2\mu_1 - m, \qquad \nu_2 = 3\mu_2 - 2m\mu_1 \tag{A.3}$$

and

$$\tilde{\nu}_1 = 2\tilde{\mu}_1 - \tilde{m}, \qquad \tilde{\nu}_2 = 3\tilde{\mu}_2 - 2\tilde{m}\tilde{\mu}_1. \tag{A.4}$$

A.1. Proof for Section 3.1

1. We start by proving that, if two moments $\tilde{\mu}_1$ and $\tilde{\mu}_2$ of the stochastic variable S_T are known, the expected value $\mathbb{E}\left[(S_T - K)_+\right]$ satisfies the boundary conditions

$$F_1(K) \le \mathbb{E}\left[(S_T - K)_+\right] \le F_2(K), \tag{A.5}$$

with

$$F_1(K) = \begin{cases} \tilde{\mu}_1 - K & \text{if } K \le \tilde{\mu}_1 \\ 0 & \text{if } K \ge \tilde{\mu}_1 \end{cases} \tag{A.6}$$

and

$$F_2(K) = \begin{cases} (\tilde{\mu}_1 - K) + \dfrac{\tilde{\mu}_2 - \tilde{\mu}_1^2}{\tilde{\mu}_2}\, K & \text{if } K \le \dfrac{\tilde{\mu}_2}{2\tilde{\mu}_1} \\[2ex] \dfrac{1}{2}(\tilde{\mu}_1 - K) + \dfrac{\sqrt{(\tilde{\mu}_2 - \tilde{\mu}_1^2) + (\tilde{\mu}_1 - K)^2}}{2} & \text{if } K \ge \dfrac{\tilde{\mu}_2}{2\tilde{\mu}_1}. \end{cases} \tag{A.7}$$

For the upper bounds, the proof can be found in Heijnen (1989). One just has to replace 'd' by 'K' in the formula of the paper, and then taking the limit for b going to infinity (range $[0, +\infty[$ instead of $[0, b]$) yields (A.7).

The proof for the lower bounds (A.6) is obvious since in the case where $K \le \tilde{\mu}_1$, $P(x) = x - K$ in (7), and in the case where $K \ge \tilde{\mu}_1$, the X-axis coincides with (the limiting case for b going to infinity of) $P(x)$.

2. Making use of the relations $\tilde{\mu}_1 = S \cdot \mu_1$ and $\tilde{\mu}_2 = S^2 \cdot \mu_2$, the result can easily be transformed into the bounds of (11) and (12).

A.2. Proof for Section 3.2

1. We start by proving that, if three moments $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$ of the stochastic variable S_T are known, the expected value $\mathbb{E}\left[(S_T - K)_+\right]$ satisfies the boundary conditions

$$F_3(K) \le \mathbb{E}\left[(S_T - K)_+\right] \le F_4(K), \tag{A.8}$$

with

$$F_3(K) = \begin{cases} \tilde{\mu}_1 - K & \text{if } K \le c \\ (\tilde{\mu}_1 - K) + \dfrac{-\tilde{p}(K)}{\tilde{\mu}_3 - K\tilde{\mu}_2} & \text{if } c \le K \le \dfrac{\tilde{\mu}_2}{\tilde{\mu}_1} \\ 0 & \text{if } K \ge \dfrac{\tilde{\mu}_2}{\tilde{\mu}_1} \end{cases} \tag{A.9}$$

and

$$F_4(K) = \begin{cases} (\tilde{\mu}_1 - K) + \dfrac{\tilde{\mu}_2 - \tilde{\mu}_1^2}{\tilde{\mu}_2} K & \text{if } K \leq \dfrac{\tilde{\mu}_2}{2\tilde{\mu}_1} \\ (\tilde{\mu}_1 - K) + \dfrac{\sqrt{(\tilde{\mu}_2 - \tilde{\mu}_1^2) + (\tilde{\mu}_1 - K)^2} - (\tilde{\mu}_1 - K)}{2} & \text{if } \dfrac{\tilde{\mu}_2}{2\tilde{\mu}_1} \leq K \leq \dfrac{c + c'}{2} \\ \dfrac{\tilde{\mu}_3 \tilde{\mu}_1 - \tilde{\mu}_2^2}{\tilde{\mu}_3 - 2c'\tilde{\mu}_2 + c'^2 \tilde{\mu}_1} \cdot \dfrac{c' - K}{c'} & \text{if } \dfrac{c + c'}{2} \leq K \leq \dfrac{2c'^2}{3c' - c} \\ \dfrac{\tilde{\mu}_3 \tilde{\mu}_1 - \tilde{\mu}_2^2}{\tilde{\mu}_3 - 2s\tilde{\mu}_2 + s^2 \tilde{\mu}_1} \cdot \dfrac{s - K}{s} & \text{if } K \geq \dfrac{2c'^2}{3c'^{-c}} \end{cases} \tag{A.10}$$

where c and c' are the zeros (which always exist and which are necessarily real and positive) of

$$\tilde{p}(x) = (\tilde{\mu}_2 - \tilde{\mu}_1^2)x^2 + (\tilde{\mu}_1\tilde{\mu}_2 - \tilde{\mu}_3)x + (\tilde{\mu}_1\tilde{\mu}_3 - \tilde{\mu}_2^2) \tag{A.11}$$

with $c < c'$, and where s is the unique root in the interval $[c', +\infty[$ of the equation

$$2\tilde{\mu}_1\, x^3 - (2\tilde{\mu}_2 + 3K\tilde{\mu}_1)\, x^2 + 4K\tilde{\mu}_2\, x - K\tilde{\mu}_3 = 0. \tag{A.12}$$

For the upper bounds, the proof can be found in Jansen et al. (1986). One just has to replace 't' by 'K' and 'a' by '0' in the formula of the paper, and then taking the limit for b going to infinity yields (A.10).

For the lower bounds, Lemma 2 of Jansen et al. (1986) can be used to generate two-point and three-point distributions with the given moments $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$. In fact the two-point distribution with mass $(\tilde{\mu}_1 - c')/(c - c')$ in c and mass $(\tilde{\mu}_1 - c)/(c' - c)$ in c' is the unique two-point distribution on $[0, +\infty[$ with moments $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$ (c and c' the zeros of (A.11)). Therefore, one has to conclude that in the case of $c \geq K$, the best lower bound is $\tilde{\mu}_1 - K$ (take $P(x) = x - K$ in (7)), and in the case of $c' \leq K$, the best lower bound is 0 (take $P(x) = 0$ in (7)).

More calculations only have to be done for $c < K < c'$. We define two sub-cases: $c < K \leq 0' < c'$ and $c < 0' < K < c'$ (with $0' = \tilde{\mu}_2/\tilde{\mu}_1$).

(i) $c < K \leq 0' < c'$

Using the definitions of Jansen et al. (1986), we look for the point s for which $u(0, s) = K$.[7] Because $c < K \leq 0'$, we are sure that $s = (\tilde{\mu}_3 - K\tilde{\mu}_2)/(\tilde{\mu}_2 - K\tilde{\mu}_1) > c'$. The polynomial $P(x)$ that is needed in (7) equals:

$$P(x) = \frac{x}{s^2}\left[s(s - K) + K(x - s) - (x - s)^2\right]. \tag{A.13}$$

This polynomial passes through the points $(0, 0)$, $(K, 0)$ and $(s, s - K)$, is tangent to $(x - K)_+$ in s and remains lower than or equal to $(x - K)_+$ on $[0, +\infty]$. Therefore the best lower bound in this case is $q_s(s - k)$, with q_s the mass in s as defined in Lemma 2 of Jansen et al. (1986). After some calculations one easily finds the lower bound of (17).

(ii) $c < 0' < K < c'$

Now we first have to introduce an upper bound b in the range of the distribution functions. Later on, we will take the limit for b going to infinity.

Using the definitions of Jansen et al. (1986) we look for the point r for which $u(r, b) = K$. Because $0' < K < c'$, we are sure that $r = (\tilde{\mu}_3 - (b + K)\tilde{\mu}_2 + bK\tilde{\mu}_1)/(\tilde{\mu}_2 - (b + K)\tilde{\mu}_1 + bK) < c$. The polynomial $P(x)$ that is used in (7) equals:

$$P(x) = \frac{(x - r)^2(x - K)}{(b - r)^2}. \tag{A.14}$$

This polynomial passes through the points $(r, 0)$, $(K, 0)$ and $(b, b - K)$, is tangent to $(x - K)_+$ in r and remains lower than or equal to $(x - K)_+$ on $[0, b]$. Therefore the best lower bound in this case is $q_b(b - K)$, with q_b the mass in b as defined in Lemma 2 of Jansen et al. (1986). After some calculations and after taking the limit for b going to infinity, this best lower bound becomes zero, which completes the proof of (A.9).

[7] In Jansen et al. (1986) the function $u(r, s)$ is used to generate three-point distributions with the given moments $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$. If $r < c$ and $s > c'$ then there exists such a distribution with spectrum $\{r, u, s\}$, and the masses q_r, q_u and q_s can be expressed analytically.

2. Making use of the relations $\tilde{\mu}_1 = S \cdot \mu_1$, $\tilde{\mu}_2 = S^2 \cdot \mu_2$ and $\tilde{\mu}_3 = S^3 \cdot \mu_3$, the results of (A.9) and (A.10) can be transformed into the bounds of (17) and (18). The relation between the zeros v and w of $p(x)$ of (13) and c and c' of $\tilde{p}(x)$ of (A.11) is that $v = S \cdot c$ and $w = S \cdot c'$. The same reasoning holds for the root q of Eq. (14) and the root s of Eq. (A.12): $q = S \cdot s$.

A.3. Proof for Section 3.3

1. If besides the first two moments, also the mode of the underlying variable is known, use can be made of a Khinchin transform, as explained in the following lemma.

Lemma A.1. *If a non-negative variable X has mode m and moments μ_1 and μ_2, then there exists a non-negative random variable Y with moments $\nu_1 = 2\mu_1 - m$ and $\nu_2 = 3\mu_2 - 2m\mu_1$, such that for any function $f : \mathbb{R} \to \mathbb{R}$ the following equality holds:*

$$\mathbb{E}[f(X)] = \mathbb{E}[g(Y)]. \tag{A.15}$$

The "Khinchin transform" $g : \mathbb{R} \to \mathbb{R}$ is defined as

$$g(y) = \frac{1}{y-m} \int_0^{y-m} f(t+m)\mathrm{d}t. \tag{A.16}$$

For a proof, see Feller (1971).

As a consequence, if besides the first two moments also the mode is known, the problem (6) can be transformed to find

$$\sup_{F \in \mathcal{B}^*} \int_0^{+\infty} g(x)\mathrm{d}F(x) \quad \text{and} \quad \inf_{F \in \mathcal{B}^*} \int_0^{+\infty} g(x)\mathrm{d}F(x), \tag{A.17}$$

where $\mathcal{B}^*$ is the class of all distribution functions with domain $\mathbb{R}^+$ and with first two moments ν_1 and ν_2, and where

$$g(y) = \frac{1}{y-m} \int_0^{y-m} (t+m-K)_+ \mathrm{d}t. \tag{A.18}$$

2. We then start by proving that, if two moments $\tilde{\mu}_1$ and $\tilde{\mu}_2$ and the mode $\tilde{m}$ of the stochastic variable S_T are known, the expected value $\mathbb{E}\left[(S_T - K)_+\right]$ satisfies the boundary conditions

$$F_5(K) \le \mathbb{E}\left[(S_T - K)_+\right] \le F_6(K), \tag{A.19}$$

with $F_5(K)$ and $F_6(K)$ specified as follows:

- if $K \le \tilde{m}$

$$F_5(K) = \tilde{\mu}_1 - K \tag{A.20}$$

and

$$F_6(K) = \begin{cases} (\tilde{\mu}_1 - K) + \dfrac{K^2}{2\tilde{m}} \dfrac{\tilde{\nu}_2 - \tilde{\nu}_1^2}{\tilde{\nu}_2} & \text{if } K \le \dfrac{2\tilde{m}\tilde{\nu}_2}{2\tilde{m}\tilde{\nu}_1 + \tilde{\nu}_2} \\ (\tilde{\mu}_1 - K) + \dfrac{(\tilde{\nu}_2 - \tilde{\nu}_1^2)(z-K)^2}{2((\tilde{\nu}_2 - \tilde{\nu}_1^2) + (z - \tilde{\nu}_1)^2)(\tilde{m} - z)} & \text{if } K \ge \dfrac{2\tilde{m}\tilde{\nu}_2}{2\tilde{m}\tilde{\nu}_1 + \tilde{\nu}_2}, \end{cases} \tag{A.21}$$

where z is the unique root in the interval $[0, K]$ of the equation

$$x^3 + Ax^2 + Bx + C = 0 \tag{A.22}$$

with coefficients

$$\begin{aligned} A &= -3K \\ B &= (4\tilde{\nu}_1 + 2\tilde{m})K - (2\tilde{m}\tilde{\nu}_1 + \tilde{\nu}_2) \\ C &= 2\tilde{m}\tilde{\nu}_2 - (2\tilde{m}\tilde{\nu}_1 + \tilde{\nu}_2)K. \end{aligned} \tag{A.23}$$

• if $K \geq \tilde{m}$

$$F_5(K) = \begin{cases} (\tilde{\mu}_1 - K) + \dfrac{(K-\tilde{m})^2}{2(\tilde{\nu}_1 - \tilde{m})} & \text{if } K \leq \tilde{\nu}_1 \\ 0 & \text{if } K \geq \tilde{\nu}_1 \end{cases} \tag{A.24}$$

and

$$F_6(K) = \begin{cases} \dfrac{\tilde{\nu}_1(\tilde{\nu}_2 - K\tilde{\nu}_1)^2}{2\tilde{\nu}_2(\tilde{\nu}_2 - \tilde{m}\tilde{\nu}_1)} & \text{if } K \leq \dfrac{\tilde{\nu}_2^2}{\tilde{\nu}_1(3\tilde{\nu}_2 - 2\tilde{m}\tilde{\nu}_1)} \\ \dfrac{(\tilde{\nu}_2 - \tilde{\nu}_1^2)(y-K)^2}{2((\tilde{\nu}_2 - \tilde{\nu}_1^2) + (y-\tilde{\nu}_1)^2)(y-\tilde{m})} & \text{if } K \geq \dfrac{\tilde{\nu}_2^2}{\tilde{\nu}_1(3\tilde{\nu}_2 - 2\tilde{m}\tilde{\nu}_1)}, \end{cases} \tag{A.25}$$

where y is the unique root in the interval $[\max(K, \frac{\tilde{\nu}_2}{\tilde{\nu}_1}), +\infty[$ of Eq. (A.22).

Proof of the results for $K \leq \tilde{m}$

In (A.17) we substitute the Khinchin transform of $(x-K)_+$, i.e.

$$g(x) = \begin{cases} \dfrac{(\tilde{m}-K)^2}{2(\tilde{m}-x)} & \text{if } 0 \leq x < K \ (\leq \tilde{m}) \\ \dfrac{x+\tilde{m}-2K}{2} & \text{if } x \geq K. \end{cases} \tag{A.26}$$

Because of (19), one always has $K < 0^*$ ($K \leq \tilde{m} \leq \tilde{\nu}_1 < 0^*$), so $g(0^*) = (0^* + \tilde{m} - 2K)/2$ and $g'(0^*) = 1/2$.[8]

For the upper bounds, the reasoning is completely analogous to Section 5.2 of Heijnen (1989). We first look for polynomials of degree 2 for which the equality in (7) holds for a two-point distribution with mass points 0 and 0^*. Such a polynomial P has to pass through the points $(0, g(0))$ and $(0^*, g(0^*))$, has to be tangent to (A.26) in 0^* and $P'(0) \geq g'(0)$ must hold. Using (2.1) and (2.2) of Heijnen (1989), this is equivalent to the condition

$$\frac{1}{2}\left(\frac{1}{2} + \frac{(\tilde{m}-K)^2}{2\tilde{m}^2}\right) < \frac{1}{0^*}\left(\frac{0^* + \tilde{m} - 2K}{2} - \frac{(\tilde{m}-K)^2}{2\tilde{m}}\right). \tag{A.27}$$

After some calculations, (A.27) can be transformed into the first condition of (A.21). The best upper bound is then $q_0 g(0) + q_{0^*} g(0^*)$, which turns out to be the upper bound (A.21)(a).

Then we look for polynomials of degree 2 for which the equality in (7) holds for other two-point distributions with mass points r and r^* where $r \in [0, K]$. Note that firstly because $K < \tilde{\nu}_1$ and because the range of the distributions is $[0, +\infty[$, we know that r^* exists for any $r \in [0, K]$, and secondly because $K < 0^*$ it is always true that $r^* > K$. Such a polynomial P has to pass through the points $(r, g(r))$ and $(r^*, g(r^*))$ and has to be tangent to (A.26) in both r and r^*. Using (2.1) and (2.2) of Heijnen (1989), the existence of such a P becomes, after some calculations, equivalent to the existence of a unique root in $[0, K]$ of Eq. (A.22). The left-hand side of (A.22) is a polynomial of degree 3 (let us call it Q) with point of inflexion K. The condition in (A.21)(b) guarantees that $Q(0) \leq 0$. Also $Q(K) > 0$ for $K < \tilde{m}$. So the condition in (A.21)(b) is sufficient to guarantee a unique root of Q in $[0, K]$. The best upper bound is then $q_r g(r) + q_{r^*} g(r^*)$ which, after some calculations, is transformed into the upper bound (A.21)(b).

For the best lower bounds, the reasoning is the following. Because of $K \leq \tilde{m}$ and (19), we have $K \leq \tilde{\mu}_1 \leq \tilde{\nu}_1$, so K^* exists and $K^* > K$ (except for the limiting case $K = \tilde{m} = \tilde{\mu}_1 = \tilde{\nu}_1$ where the best lower bound is obviously zero). Taking $P(x) = (x + \tilde{m} - 2K)/2$ in (7) with the two-point distribution with spectrum $\{K, K^*\}$ immediately gives $\tilde{\mu}_1 - K$ as best lower bound.

[8] In Heijnen (1989), r^* is defined as the mass point that corresponds to r having a two-point distribution with spectrum $\{r, r^*\}$ with the given moments $\tilde{\nu}_1$ and $\tilde{\nu}_2$. If $r < b^*$ or $r > 0^*$, such a distribution exists and the masses q_r and q_{r^*} can be expressed analytically. In particular $(r^*)^* = r$, $0^* = \tilde{\nu}_2/\tilde{\nu}_1$ and $\lim_{b \to +\infty} b^* = \tilde{\nu}_1$.

Proof of the results for $K \geq \tilde{m}$

In (A.17) we substitute the Khinchin transform of $(x - K)_+$, i.e.

$$g(x) = \frac{(x - K)_+^2}{2(x - \tilde{m})}. \tag{A.28}$$

For the upper bounds, the proof can be found in Heijnen (1989); one just has to replace 'd' by 'K' and '0^*' by '$\tilde{\nu}_2/\tilde{\nu}_1$' in the formulae of the paper. Taking the limit for b going to infinity completes the proof, except for one consideration. In Heijnen (1989) the existence of a unique solution of (A.22) in $[\max(K; \tilde{\nu}_2/\tilde{\nu}_1); +\infty[$ is guaranteed by a condition that is slightly stronger than (19). So we still have to verify that the conditions $K \geq \tilde{m}$, (19) and the condition in (A.25)(b) together are strong enough to guarantee the same. The reasoning is the following. The left-hand side of (A.22) is a polynomial of degree 3, let us call it again Q. Q has its point of inflexion in K and $Q(K) < 0$ for $K \geq \tilde{m}$ while $Q(+\infty) > 0$, so (A.22) has a unique real solution in $[K, +\infty[$. Because of conditions (A.25)(b) and (19) also $Q(\tilde{\nu}_2/\tilde{\nu}_1) < 0$, which implies at least one real solution of (A.22) in $[\tilde{\nu}_2/\tilde{\nu}_1; +\infty[$. The two statements on K and $\tilde{\nu}_2/\tilde{\nu}_1$ together guarantee a unique solution of (A.22) in $[\max(K; \tilde{\nu}_2/\tilde{\nu}_1); +\infty[$.

For the lower bounds we first have to introduce an upper bound in b in the range of the distribution functions. Later on we will take the limit for b going to infinity.

Using Lemma 3.1 of Heijnen (1989) one learns that $b^* \overset{<}{\to} \nu_1$ if $b \to +\infty$. So in the case where $K < \tilde{\nu}_1$ we can choose b large enough to have $K < b^* < \tilde{\nu}_1$. Using formula (2.1) of Heijnen (1989) generates a polynomial P of degree 2 through the points $(b^*, g(b^*))$ and $(b, g(b))$ which is tangent to (A.28) in b^*. This polynomial is useful for (7) if $P(0) \leq 0$. Some more calculations show that $\lim_{b\to+\infty} P(0) < 0$ if $\tilde{m} \leq K < \tilde{\nu}_1$, which is the case here. So we can choose b large enough to guarantee that $P(0) < 0$. Then the best lower bound is $q_{b^*} g(b^*) + q_b g(b)$ with q_{b^*} and q_b the masses in b^* and b as defined in Lemma 3.1 of Heijnen (1989). Taking the limit of this lower bound for b going to infinity gives $(\tilde{\nu}_1 - K)^2/2(\tilde{\nu}_1 - \tilde{m})$ as best lower bound for this case, which can easily be transformed into the expression (A.24)(a). The best lower bound in the case where $K \geq \tilde{\nu}_1$ is obviously zero, since the best lower bound in the case where $K < \tilde{\nu}_1$ tends to zero for K going to $\tilde{\nu}_1$.

3. Making use of the relations $\tilde{\mu}_1 = S \cdot \mu_1$, $\tilde{\mu}_2 = S^2 \cdot \mu_2$ and $\tilde{m} = S \cdot m$, the results of (A.20), (A.21) and (A.24), (A.25) can be transformed into the bounds of (29), (30) and (26), (27). Note that the Eqs. (23) and (A.22) are equivalent.

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