The Minimal entropy martingale measure (MEMM) for a Markov-modulated Lévy model

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Abstract

This paper deals with the characterization problem of the minimal entropy martingale measure (MEMM) for a Markov-modulated exponential Lévy model. This model is characterized by the presence of a factor modulating the risky asset price movements between different regimes, which allow to stress the strong dependence of financial assets price with structural change in the environment. Our main result are obtained from the key idea that working conditionally on the modulator-factor process, the problem is essentially reduced to studying processes with independent increments. Our work generalizes some previous works in the literature which have treated either the exponential Lévy case or the exponential-additive case.

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1 Introduction

Markov-modulated (or Regime-switching) models were originally introduced in order to model the macroeconomic events which influence asset prices (See Hamilton 1989). These cycles are modeled by an underlying Markov chain that drives the asset prices through structurally different market scenarios (due to trades or to arrival of significant new information). In the context of derivative pricing these models lead to incomplete markets and therefore, there are in general infinitely many equivalent martingale measures (EMM) or equivalently there is no unique preference-independent price for options.

In recent years there has been many papers in the area of characterization of martingales measures in incomplete markets. The mainstream of this research concerned with the projection-based methods in which one looks at a martingale measure which is the "closest" to the physical or real world probability measure relative to a criteria. For example, Föllmer and Sondermann (1986), Föllmer and Schweizer (1991), Schweizer (1991,1996) use quadratic or L^2-distance; Chan (1999), Miyahara (1999) and Frittelli (2000) use Kullback-Leibler distance (or relative entropy) which leads to the so-called *minimal entropy equivalent martingale measure*.

Many arguments play in favor of a MEMM. Firstly, due to the fact that it comes from the minimization of relative entropy with respect to the real probability measure it retains every

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information we know about the randomness structure underlying the market thus it is consistent with the efficient market hypothesis. Secondly, the well-known duality relationship [See Fritelli (2000) and references therein] between minimization of the relative entropy and maximization of exponential utility makes the minimal entropy martingale measure economically meaningful. Thirdly, if the minimal entropy martingale exists, it is always equivalent to the objective probability measure unlike some other martingale measures such as the minimal variance martingale measure which may not be equivalent to the objective probability measure.

Many authors have studied this equivalent martingale measure in different contexts. For example, Chan (1999) has studied the problem of pricing contingent claims in a Lévy model and provided a solution based on the MEMM. Frittelli (2000) has looked at the problem of existence and uniqueness of the MEMM in a general incomplete market model and provided its economical interpretation in terms of exponential utility functions. Miyahara (1999), Fujiwara and Miyahara (2003) have obtained some results for the MEMM in the geometric Lévy models. Recently, Fujiwara (2009) has extended these results to the case of an exponential additive process.

In this paper, we extend the result of Fujiwara (2009) to a general Markov-modulated Lévy model whose the main feature is the presence of a modulator factor which changes the characteristic of the dynamics of risky asset under different regimes. Our main contribution is to give an expression, when it exists, for the minimal entropy equivalent measure for this class of models. The paper is organized as follows. In section 2, we describe the model set-up and give some preliminaries tools. In particular, we recall some facts on the theory of Markov additive processes which are the mathematical structure behind our model. The main problem is discussed in section 3 where we mimic the approach of Fujiwara (2009). The section 4 contains an example which illustrates the feasability of the results obtained and the last section 5 finally concludes.

2 Model description and preliminaries

The general model considered here can be viewed as an extension of the exponential-Lévy model described in Cont and Tankov (2003, p.283) where a factor of modulation is introduced to allow for more flexibility, especially for the time-inhomogeneity property.

2.1 The model set-up

We consider a financial market with two primary securities, namely a money market account B and a stock S which are traded continuously over the time horizon $\mathcal{T} := [0, T]$, where $T \in (0, \infty)$ represents the maturity time for investment. To formalize this market, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the real-world probability. Further, we will add to this set-up a filtration which specifies the flow of informations available for the investors.

Let $X := \{X_t : t \in \mathcal{T}\}$ denote an irreducible homogeneous continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite state space $\mathbb{S} = \{s_1, s_2, ..., s_M\} \subset \mathbb{R}^M$ and characterized by a rate (or intensity) matrix $\mathbb{A} := \{a_{ij} : 1 \leq i, j \leq M\}$. The entry of the a_{ij} matrix represent the rates at which the process X jumps to state i to state j, that is , the transition rates. Following Elliott (1993), we can identify $\mathbb{S}$ with the basis set of the linear space $\mathbb{R}^M$. From now on, we set $s_i = \mathbf{e}_i := (0, 0, ..., \underbrace{1}_{i-th}, ..., 0)$.

Let r_t denotes the instantaneous interest rate of the money market account B at time t. We suppose that $r_t := r(t, X_t) = \langle \underline{r}, X_t \rangle$ where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^M$ and

$\underline{r} = (r_1, r_2, ..., r_M) \in {\mathbb{R}_+}^M$. The price dynamics of B is given by:

$$B_t = B_0 \exp\Big(\int_0^t r_s ds\Big),\ B_0 = 1; \quad t \in \mathcal{T}. \tag{2.1}$$

Let μ_t and σ_t denote the appreciation rate and the volatility of the stock S at time t, we suppose respectively that:

$$\mu_t = \langle \underline{\mu}, X_t \rangle, \quad \sigma_t = \langle \underline{\sigma}, X_t \rangle$$

where $\underline{\mu} = (\mu_1, \mu_2, ..., \mu_M) \in \mathbb{R}^M$ and $\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_2) \in {\mathbb{R}_+}^M$.

The stock price process S is described by the following exponential Markov modulated Lévy process:

$$S_t = S_0 \exp(Y_t), \quad S_0 > 0 \tag{2.2}$$

with

$$Y_t = \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\Big) ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}\backslash\{0\}} z \widetilde{N}^X(ds, dz) - \int_0^t \int_{\mathbb{R}\backslash\{0\}} (e^z - 1 - z) \rho^X(dz) ds \tag{2.3}$$

and where $N^X(dt, dz)$ denotes the differential form of a Markov-modulated random measure on $\mathcal{T} \times \mathbb{R}\backslash\{0\}$. We recall from Elliott and Osakwe (2006) and Elliott and Royal (2006) that a Markov-modulated random measure on $\mathcal{T} \times \mathbb{R}\backslash\{0\}$ is a family $\{N^X(dt, dz; \omega) : \omega \in \Omega\}$ of non-negative measures on the measurable space $(\mathcal{T} \times \mathbb{R}\backslash\{0\}, \mathcal{B}(\mathcal{T}) \otimes \mathcal{B}(\mathbb{R}\backslash\{0\}))$, which satisfy $N^X(\{0\}, \mathbb{R}\backslash\{0\}; \omega) = 0$ and has the following compensator, or dual predictable projection:

$$\rho^X(dz)dt := \sum_{i=1}^M \langle X_{t^-}, \mathbf{e}_i \rangle \rho_i(dz) dt \tag{2.4}$$

where $\rho_i(dz)$ is the density for the jump size when the Markov chain X is in state $\mathbf{e}_i$ and verifying $\int_{\mathbb{R}\backslash\{0\}} (|z|^2 \wedge 1)\ \rho_i(dz) < \infty$. The compensated random measure associated to $N^X(dt, dz)$ is defined by

$$\widetilde{N}^X(dt, dz) := N^X(dt, dz) - \rho^X(dz)dt \tag{2.5}$$

$W := (W_t)_{t \in \mathcal{T}}$ denote the standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ which is supposed to be independent of X and N^X. However, we will assume that a switch of X from state $\mathbf{e}_j$ to state $\mathbf{e}_k$ and a jump of Y do not happen simultaneously, a.s.

We notice that the expression (2.2) admit an another representation by using *Doléans-Dade (or stochastic) exponential* $\mathcal{E}(\cdot)$. Indeed,

$$S_t = S_0 \mathcal{E}(\widehat{Y}_t) \tag{2.6}$$

with the process $\{\widehat{Y}_t : t \in \mathcal{T}\}$ defined by

$$\widehat{Y}_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}\backslash\{0\}} (e^z - 1) \widetilde{N}^X(ds, dz) \tag{2.7}$$

Sometimes, we will use the following process $(\widetilde{R}_t)$ introduced in Fujiwara and Miyahara (2003)

$$\widetilde{R}_t := \int_0^t \frac{1}{\widetilde{S}_{s^-}} d\widetilde{S}_s \tag{2.8}$$

where

$$\widetilde{S}_t := \frac{S_t}{B_t} = e^{-\int_0^t r_s ds} S_t, \tag{2.9}$$

From (2.2) and (2.10), it is easy to see that

$$\widetilde{R}_t = \widehat{Y}_t - \int_0^t r_s ds. \tag{2.10}$$

2.2 Some preliminaries

In this section, we review some notions related to the *Markov additive processes* (MAP) which are the mathematical object behind Markov-modulated Lévy models. In particular, we define this object and give some of its fundamental properties which will be useful in the following.

2.2.1 Markov additive Processes

The mathematical theory of Markov additive processes can be traced back from the works of Ezhov and Skorokhod (1969a) and (1969b), Çinlar (1972a) and (1972b) and Grigelionis (1978). We will recall some useful results of this theory. In all this section, $\mathcal{T} := [0, T]$ with $T > \infty$.

Definition 2.1

A stochastic process $\{(J_t, Z_t) : t \in \mathcal{T}\}$ adapted to a right-continuous filtration $\mathbf{F} := \{\mathcal{F}_t : t \in \mathcal{T}\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the component $Z = \{Z_t : t \in \mathcal{T}\}$ takes its values in $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ and is continuous from the right with limits at the left whereas the component $J = \{J_t : t \in \mathcal{T}\}$ takes its values in a measure space $(\mathbf{D}, \mathcal{D})$, will be called a ***Markov additive process*** *with respect to $\mathbf{F}$ if for any Borel set $A \in \mathcal{B}(\mathbb{R}^m)$, $B \in \mathcal{D}$ and $0 \leq s < t \leq T$ we have almost surely (a.s)*

$$\mathbb{P}\{Z_t - Z_s \in A, J_t \in B | \mathcal{F}_s\} = \mathbb{P}\{Z_t - Z_s \in A, J_t \in B | J_s\} \tag{2.11}$$

or, more generally, for any bounded random variable η that is measurable with respect to the σ-algebra $\sigma\Big\{(Z_t - Z_s, J_t) : t \in [s, T]\Big\}$ and $s \in \mathcal{T}$ we have almost surely

$$\mathbb{E}^{\mathbb{P}}\Big[\eta\Big|\mathcal{F}_s\Big] = \mathbb{E}^{\mathbb{P}}\Big[\eta\Big|J_s\Big] \tag{2.12}$$

For the following, we introduce for $0 \leq s \leq t, with\ s, t \in \mathcal{T}$

- $\mathcal{F}^J_{s,t} := \sigma(J_u; u \in \mathcal{T}, s \leq u \leq t)$;
- $\mathcal{F}^J_t := \mathcal{F}^J_{0,t}$;
- $\mathcal{F}^Z_t := \sigma(Z_u; u \in \mathcal{T}, 0 \leq u \leq t)$;
- $\mathcal{G}_t := \mathcal{F}^J_t \vee \mathcal{F}^Z_t$;

- $\overline{\mathcal{G}}_t = \mathcal{F}_T^J \vee \mathcal{F}_t^Z$.

Remark 2.1 *The two filtrations* $\mathbf{G} := \{\mathcal{G}_t : t \in \mathcal{T}\}$ *and* $\overline{\mathbf{G}} := \{\overline{\mathcal{G}}_t : t \in \mathcal{T}\}$ *are of particular importance for our model. The filtration* $\overline{\mathbf{G}}$ *could be seen as the information set given the whole history of the Markov chain* X.

We now state a fundamental property of Markov additive process $\{(J_t, Z_t) : t \in \mathcal{T}\}$.

Theorem 2.1 (Grigelionis, 1978) *The component* Z *is a process with conditional independent increments with respect to the* σ*-algebra* $\mathcal{F}_T^J$*, i.e., for any* $A \in \mathcal{B}(\mathbb{R}^m)$ *and* $0 \leq s \leq t$*, with* $s, t \in \mathcal{T}$ *we have almost surely*

$$\mathbb{P}\{Z_t - Z_s \in A | \overline{\mathcal{G}}_s\} = \mathbb{P}\{Z_t - Z_s \in A | \mathcal{F}_T^J\} \tag{2.13}$$

or equivalently, for $0 \leq s \leq t$ *with* $s, t \in \mathcal{T}$ *and* $y \in \mathbb{R}^m$ *we have almost surely*

$$\mathbb{E}^{\mathbb{P}}\Big[\exp i(y, Z_t - Z_s)\Big|\overline{\mathcal{G}}_s\Big] = \mathbb{E}^{\mathbb{P}}\Big[\exp i(y, Z_t - Z_s)\Big|\mathcal{F}_T^J\Big] \tag{2.14}$$

Proof. The proof of this theorem is obtained by using lemma 1 in Grigelionis (1978). ■

Remark 2.2 *This last property highlights a feature of the MAP which will be very useful in the sequel. Indeed, if we work on a suitable stochastic basis* $(\Omega, \mathcal{F}, \mathbb{P})$ *which allows the existence of some regular version of the conditional probability* $\widehat{\mathbb{P}}(\omega, .)$ *for an* $\omega \in \Omega$ *then one can switch for the new probability space* $(\Omega, \mathcal{F}, \widehat{\mathbb{P}}(\omega, .))$ *under which the component* Z *is an additive process (i.e., has independent increments).*

From now on, we shall suppose this setting thus we set Ω to be the canonical space $\mathbb{D}_T(\mathbb{R}^m)$ of càdlàg functions

$$\beta : t \in [0, T] \to \beta(t) \in \mathbb{R}^m$$

endowed with the Skorohod topology such that $\mathcal{D}(\mathbb{R}^m)$ be its Borel σ-field. The filtration taken on that space is the family $\mathbf{D}(\mathbb{R}^m) = \{\mathcal{D}_t(\mathbb{R}^m) = \bigcap_{u>t} \mathcal{D}_u^0(\mathbb{R}^m)\}$, where $\mathcal{D}_u^0(\mathbb{R}^m)$ denotes the σ-field generated by all maps : $\beta \rightsquigarrow \beta(s)$ for $s \leq t$.
Let $(J, Z) = \{(J_t, Z_t) : t \in \mathcal{T}\}$ be a Markov additive process defined on $(\Omega, \mathcal{D}(\mathbb{R}^m), \mathbf{D}(\mathbb{R}^m), \mathbb{P})$ with Z denoting its additive part.

Lemma 2.1 (Çinlar, 1972a)
There exists a regular version $\widehat{\mathbb{P}}$ *of the conditional probability* $\mathbb{P}\{\cdot|\mathcal{F}_T^J\}$ *on the* σ*-algebra* $\overline{\mathcal{G}_t}$ *i.e.,, for* $\mathbb{P}-$*almost all* $\omega \in \Omega$ $\widehat{\mathbb{P}}(\omega, .)$ *is a probability measure on* $\overline{\mathcal{G}}_T$ *and* $\widehat{\mathbb{P}}(\omega, A)$ *is a version of* $\mathbb{P}\{A|\mathcal{F}_T^J\}$ *for every* $A \in \overline{\mathcal{G}}_T$.

Proof. See Çinlar (1972a). ■

The following proposition, formulated by Grigelionis (1978) gives a characterization criterion of a Markov additive process:

Theorem 2.2 (Grigelionis, 1978)
A stochastic process $\{(J_t, Z_t) : t \in \mathcal{T}\}$ *adapted to a complete right-continuous filtration* $\mathbf{F} := \{\mathcal{F}_t : t \in \mathcal{T}\}$ *is a Markov additive process (MAP) if and only if the following conditions are satisfied:*

1. *The component Z has an expansion $Z_t = Z_t^{'} + Z_t^{''}$ where $Z^{'}$ is continuous from the right with limits at the left and such that for all $t \in \mathcal{T}$, $Z_t^{'}$ is $\mathcal{F}_T^J$-measurable and $Z^{''}$ is an $(\overline{\mathbf{G}}, \mathbb{P})$-semimartingale whose the spot-characteristics triplet (A_t, ν_t, γ_t) are $\mathcal{F}_T^J$-measurable;*

2. *The component J has the Markov property with respect to the filtration $\mathbf{G} = \{\mathcal{G}_t : t \in \mathcal{T}\}$ i.e.,, for any $0 \leq s < t \leq T$ and $B \in \mathcal{D}$ we have almost surely*

$$\mathbb{P}\{J_t \in B | \mathcal{G}_s\} = \mathbb{P}\{J_t \in B | J_s\} \tag{2.15}$$

With all the ingredients above we will state a result allowing us, when we have a MAP, to decompose the density process associated to a change of measure into some useful components related to this MAP.

Theorem 2.3
Let $\mathbb{Q}$ be an arbitrary probability measure such that $\mathbb{Q} \overset{loc}{\ll} \mathbb{P}$ i.e., $\mathbb{Q}|_{\overline{\mathcal{G}}_t} \ll \mathbb{P}|_{\overline{\mathcal{G}}_t}$ for each $t \in \mathcal{T}$. We denote by $\Lambda^{J,Z}$ the Radon-Nikodym density process defined by

$$\Lambda_t^{J,Z} = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_t} \tag{2.16}$$

Then the following holds $\mathbb{P}$-almost surely:
$\Lambda^{J,Z}$ admits the unique(up to $\mathbb{P}$-null sets) decomposition

$$\Lambda_t^{J,Z} = \Lambda_t^Z .\ \Lambda_T^J \quad for \quad t \in \mathcal{T} \tag{2.17}$$

where

- $\Lambda^Z := \left\{ \frac{d\widehat{\mathbb{Q}}(\omega,.)}{d\widehat{\mathbb{P}}(\omega,.)}\Big|_{\overline{\mathcal{G}}_t} ; t \in \mathcal{T} \right\}$ *is the Radon-Nikodym density process of a regular version $\widehat{\mathbb{Q}}(\omega,.)$ of $\mathbb{Q}(\cdot|\mathcal{F}_T^J)$ with respect to a regular version $\widehat{\mathbb{P}}(\omega,.)$ of $\mathbb{P}(\cdot|\mathcal{F}_T^J)$;*

- $\Lambda_T^J = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_0} \left(= \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T^J} \right)$ *is the Radon-Nikodym derivative characterizing a change of measure under which the Markov chain is modified.*

Proof.
1)Let $\mathbb{Q}$ be an arbitrary probability locally absolutely continuous with respect to $\mathbb{P}$. Let $\Lambda_t^{J,Z}$ be a version of the Radon-Nikodym derivative of $\mathbb{Q}|_{\overline{\mathcal{G}}_t}$ w.r.t $\mathbb{P}|_{\overline{\mathcal{G}}_t}$. By lemma (2.1) we have that for $\mathbb{P}-$almost all $\omega \in \Omega$, there exist a regular version of $\mathbb{P}(\cdot|\mathcal{F}_T^J)$ and $\mathbb{Q}(\cdot|\mathcal{F}_T^J)$ which we will denote respectively by $\widehat{\mathbb{P}}(\omega,.)$ and $\widehat{\mathbb{Q}}(\omega,.)$.
Since $\mathbb{Q} \overset{loc}{\ll} \mathbb{P}$, we have that for $\mathbb{P}-$almost all $\omega \in \Omega$, $\widehat{\mathbb{Q}}(\omega,.)$ is also locally absolutely continuous with respect to $\widehat{\mathbb{P}}(\omega,.)$.
Indeed, for each $A \in \overline{\mathcal{G}}_t$ satisfying $\widehat{\mathbb{P}}(\omega, A) = 0$ we have

$$\begin{aligned}
\widehat{\mathbb{Q}}(\omega, A) &:= \mathbb{E}^{\mathbb{Q}}[1_A|\mathcal{F}_T^J](\omega) \\
&= \mathbb{E}^{\mathbb{P}}[\Lambda_t^{J,Z}.1_A|\mathcal{F}_T^J](\omega).\frac{1}{\mathbb{E}^{\mathbb{P}}[\Lambda_t^{J,Z}|\mathcal{F}_T^J](\omega)} \quad \text{(by Bayes' rule)} \\
&= \mathbb{E}^{\widehat{\mathbb{P}}(\omega,.)}\left[\Lambda_t^{J,Z}.1_A\right]\frac{1}{\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^{J,Z}\Big|\mathcal{F}_T^J\right](\omega)} \quad \text{(by definition of } \mathbb{E}^{\widehat{\mathbb{P}}(\omega,.)}[\cdot]) \\
&= \frac{1}{\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^{J,Z}\Big|\mathcal{F}_T^J\right](\omega)} \int_\Omega \Lambda_t^{J,Z}(\omega^{'})1_A(\omega^{'})\widehat{\mathbb{P}}(\omega, d\omega^{'}) \\
&= 0.
\end{aligned} \tag{2.18}$$

Since $\widehat{\mathbb{Q}}(\omega,.)|_{\overline{\mathcal{G}}_t}$ is absolutely continuous with respect to $\widehat{\mathbb{P}}(\omega,.)|_{\overline{\mathcal{G}}_t}$, by theorem III-3.4 of Jacod and Shiryaev (2003) there exists for $\mathbb{P}-$almost all $\omega \in \Omega$, a density process $\Lambda_Z(\omega)$ such that

$$\Lambda_t^Z = \frac{d\widehat{\mathbb{Q}}(\omega,.)|_{\overline{\mathcal{G}}_t}}{d\widehat{\mathbb{P}}(\omega,.)|_{\overline{\mathcal{G}}_t}}, \ \forall 0 \leq t \leq T. \tag{2.19}$$

By definition, Λ^Z is a $\overline{\mathbf{G}}$-martingale for $\widehat{\mathbb{P}}(\omega,.)$(also for $\mathbb{P}$).

For each $A \in \overline{\mathcal{G}_t}$ we have for $\mathbb{P}-$almost all $\omega \in \Omega$

$$\begin{aligned} \widehat{\mathbb{Q}}(\omega, A) &= \mathbb{E}^{\widehat{\mathbb{P}}(\omega,.)}[\Lambda_t^Z.1_A] \\ &= \mathbb{E}^{\mathbb{P}}[\Lambda_t^Z.1_A|\mathcal{F}_T^J](\omega) \end{aligned} \tag{2.20}$$

otherwise,

$$\begin{aligned} \widehat{\mathbb{Q}}(\omega, A) &= \mathbb{E}^{\mathbb{Q}}\Big[1_A\Big|\mathcal{F}_T^J\Big](\omega) \\ &= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{J,Z}.1_A\Big|\mathcal{F}_T^J\Big](\omega)\frac{1}{\mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{J,Z}\Big|\mathcal{F}_T^J\Big](\omega)} \quad \text{(by Bayes' rule).} \end{aligned} \tag{2.21}$$

By setting

$$\Lambda_T^J = \mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{J,Z}\Big|\mathcal{F}_T^J\Big] \tag{2.22}$$

and taking the above two expressions of $\widehat{\mathbb{Q}}(\omega, A)$, we obtain that either

$$\mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^Z\Lambda_T^J.1_A\Big|\mathcal{F}_T^J\Big](\omega) = \mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{J,Z}.1_A\Big|\mathcal{F}_T^J\Big](\omega) \tag{2.23}$$

or

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\Lambda_t^Z\Lambda_T^J - \Lambda_t^{J,Z}\Big).1_A\Big|\mathcal{F}_T^J\Big](\omega) = 0 \tag{2.24}$$

and by taking expectation in the both sides, we have for $\mathbb{P}$ a.s

$$\Lambda_t^{J,Z} = \Lambda_t^Z\Lambda_T^J \quad \forall t \in [0,T]. \tag{2.25}$$

Now we have to show that Λ_T^J is the Radon-Nikodym derivative ($\mathcal{F}_T^J$-measurable) characterizing a change of measure under which the Markov chain is modified.
Indeed, by (2.22)

$$\begin{aligned} \Lambda_T^J &:= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{J,Z}\Big|\mathcal{F}_T^J\Big] = \mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{J,Z}\Big|\overline{\mathcal{G}}_0\Big] \\ &= \Lambda_0^{J,Z} \qquad (2.26) \\ &= \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_0} = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T^J} \qquad (2.27) \end{aligned}$$

By setting

$$\Lambda_t^J := \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^J\Big|\mathcal{F}_t^J\Big], \quad \forall t \in [0,T] \tag{2.28}$$

we define a $(\mathcal{F}^J, \mathbb{P})$ martingale process such that $\mathbb{E}^{\mathbb{P}}[\Lambda_t^J] = 1$ so Λ^J is a density process under which the Markov chain would be eventually modified. ■

Remark 2.3 *The process (X,Y) defined by model (2.1)-(2.2) in section 2.1 is a Markov additive process. In particular the log-return process Y is a conditional additive process given the σ-algebra $\mathcal{F}_T^X$; hence, from (2.3) we have this canonical decomposition for Y_t associated with the truncation function $h(x) := x\mathbb{I}_{\{|x|\leq 1\}}$:*

$$Y_t = \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\Big)ds - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\rho^X(dz)ds + \int_0^t \sigma_s dW_s$$
$$+ \int_0^t \int_{\mathbb{R}\setminus\{0\}} h(z)\widetilde{N}^X(ds,dz) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} (z - h(z))N^X(ds,dz) \quad (2.29)$$

Hence its spot-characteristic triplet (A_t, ν_t, γ_t) is given by

$$A_t := \int_0^t \sigma_s^2 ds \quad (2.30)$$

$$\nu_t(dy) := \int_0^t \rho^X(dy)ds \quad (2.31)$$

and

$$\gamma_t := \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\Big)ds - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - h(z))\rho^X(dz)ds \quad (2.32)$$

3 Minimal entropy equivalent martingale measure for Markov-modulated Lévy process

In this section, we address the main aim of our study: existence and characterization of the minimal entropy equivalent martingale measure for the Markov-modulated Lévy model. It is known that the financial market described by equations (2.2) and (2.1) is incomplete, so neither existence nor uniqueness of equivalent martingale measures are assured. Nevertheless, we show in this section that under mild conditions there is a MEMM and its uniqueness results from the strict convexity of the relative entropy operator.

3.1 Preliminaries

Let $\mathcal{G}$ be a sub σ-algebra of $\overline{\mathcal{G}}_T$ and $\mathcal{P}$ the set of probability measure on $(\Omega, \mathcal{G})$.

Definition 3.1 *For $\mathbb{Q} \in \mathcal{P}(\Omega, \mathcal{G})$, the **relative entropy** of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as:*

$$\mathbb{H}_{\mathcal{G}}(\mathbb{Q},\mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{Q}}\left[\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}}\right)\right] & \text{if} \quad \mathbb{Q} << \mathbb{P} \quad \text{on} \quad \mathcal{G} \\ +\infty & \text{otherwise} \end{cases} \quad (3.1)$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}}$ denotes the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{G}$.
Also, it verifies the following

- $\mathbb{H}_{\mathcal{G}}(\mathbb{Q},\mathbb{P}) \geq 0$ *and* $\mathbb{H}_{\mathcal{G}}(\mathbb{Q},\mathbb{P}) = 0$ *if and only if* $\mathbb{Q} = \mathbb{P}$*;*
- *The functional* $\mathbb{Q} \longmapsto \mathbb{H}_{\mathcal{G}}(\mathbb{Q},\mathbb{P})$ *is strictly convex.*

Now we recall a lemma (See Fujiwara and Miyahara (2003)) which is useful for the following.

Lemma 3.1 *Let $\mathcal{K}$ be a sub σ-field of a σ-algebra $\overline{\mathcal{G}}_T$ and $\mathbb{Q}$ an element of $\mathcal{P}(\Omega, \mathcal{G})$ which is absolutely continuous with respect to $\mathbb{P}$. Then we have the following equivalence $(a) \sim (b) \sim (c)$*

- *(a) $\mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}) \geq 0$;*
- *(b)Let $\mathbb{P}_1 \in \mathcal{P}(\Omega, \mathcal{G})$ be equivalent to $\mathbb{P}$ on $\mathcal{G}$ and $\log\left(\frac{d\mathbb{P}_1}{d\mathbb{P}}\Big|_{\mathcal{G}}\right)$ be integrable with respect to $\mathbb{Q}$, then*

$$\mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P}) \geq \int_{\Omega} \log\left(\frac{d\mathbb{P}_1}{d\mathbb{P}}\Big|_{\mathcal{G}}\right) d\mathbb{Q}$$

- *(c)If $\mathcal{K} \subset \mathcal{G}$, then*

$$\mathbb{H}_{\mathcal{K}}(\mathbb{Q}, \mathbb{P}) \leq \mathbb{H}_{\mathcal{G}}(\mathbb{Q}, \mathbb{P})$$

In particular, from statement (c) the relative entropy $\mathbb{H}_{\mathcal{K}}(\mathbb{Q}, \mathbb{P})$ is nondecreasing on the elements of a filtration.

Now we define several spaces that will be useful in the sequel.

- $\mathcal{M}^a(\widetilde{S}) := \left\{\mathbb{Q} \overset{loc}{\ll} \mathbb{P} : (\widetilde{S}_t) \text{ is a } (\overline{\mathbf{G}}, \mathbb{Q})\text{-local martingale}\right\}$;
- $\mathcal{M}^e(\widetilde{S}) := \left\{\mathbb{Q} \overset{loc}{\sim} \mathbb{P} : (\widetilde{S}_t) \text{ is a } (\overline{\mathbf{G}}, \mathbb{Q})\text{- martingale}\right\}$;

Elements of $\mathcal{M}^e(\widetilde{S})$ are called *equivalent martingale measures (EMMs)* for $\widetilde{S}$.

Definition 3.2 *The* minimal entropy martingale measure *is a probability measure $\mathbb{Q}^\star \in \mathcal{M}^e(\widetilde{S})$ such that*

$$\mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}^\star, \mathbb{P}) = \min_{\mathbb{Q} \in \mathcal{M}^a(\widetilde{S})} \mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}, \mathbb{P}) \tag{3.2}$$

If the MEMM exists, by definition it is unique. Moreover[See Fritelli (2001), Theorem 2.2] under the assumption

$$\inf_{\mathbb{Q} \in \mathcal{M}^e(\widetilde{S})} \mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}, \mathbb{P}) < \infty \tag{3.3}$$

it is equivalent to $\mathbb{P}$.

3.2 Main results

We set the following condition:

Condition (H)
There exists a $\mathcal{F}_t^X$-adapted process with càglàd path $(\theta_t^X)_{\{t \in \mathcal{T}\}}$ that satisfies the following
(i)

$$\int_0^T \int_{\{z>1\}} e^z e^{\theta_s^X (e^z - 1)} \rho^X(dz) ds < \infty \quad \mathbb{P}\ (a.s) \tag{3.4}$$

(ii) for almost all $t \in \mathcal{T}$,

$$\mu_t + \theta_t^X \sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{\theta_s^X (e^z - 1)} - 1\right)(e^z - 1)\rho^X(dz) ds = r_t \quad \mathbb{P}\ (a.s) \tag{3.5}$$

Definition 3.3 *Let $\{\theta_t^X : t \in \mathcal{T}\}$ be a process satisfying the condition (H). We define a probability measure $\mathbb{Q}^{\theta^X}$ on $(\Omega, \overline{\mathcal{G}}_T)$ such that*

$$\forall A \in \overline{\mathcal{G}}_t, \quad \mathbb{Q}^{\theta^X}(A) := \mathbb{E}^{\mathbb{P}}[L_t.\mathbb{I}_A] \tag{3.6}$$

where $L_t = \Lambda_t^{\theta^X}.\ U_T$ with

$$\Lambda_t^{\theta^X} := \frac{\exp\Big(\int_0^t \theta_s^X d\widetilde{R}_s\Big)}{\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^t \theta_s^X d\widetilde{R}_s\Big)\Big|\mathcal{F}_T^X\Big]} \tag{3.7}$$

and U_T be a $\mathcal{F}_T^X$-measurable positive function such that $\mathbb{E}^{\mathbb{P}}[U_T] = 1$.

Remark 3.1 *The probability measure $\mathbb{Q}^{\theta^X}$ defined above is absolutely continuous with respect to $\mathbb{P}$ and by theorem (2.3) it admits the decomposition*

$$\frac{d\mathbb{Q}^{\theta^X}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_t} = \frac{d\widehat{\mathbb{Q}}^{\theta^X}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_t}.\ U_T \tag{3.8}$$

where

$$\frac{d\widehat{\mathbb{Q}}^{\theta^X}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_t} := \frac{\exp\Big(\int_0^t \theta_s^X d\widetilde{R}_s\Big)}{\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^t \theta_s^X d\widetilde{R}_s\Big)\Big|\mathcal{F}_T^X\Big]} \tag{3.9}$$

Proposition 3.1
The probability measure $\mathbb{Q}^{\theta^X}$ defined above is an element of $\mathcal{M}^e(\widetilde{S})$.

Proof. We only need to show that $(\widetilde{S}_t)$ is a $(\overline{\mathbf{G}}, \mathbb{Q})$- martingale.
So, using Bayes' rule we have $\forall\ 0 \le s \le t \le T$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\widetilde{S}_t\Big|\overline{\mathcal{G}}_s\Big] &= \frac{\mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{\theta^X} U_T \widetilde{S}_t\Big|\overline{\mathcal{G}}_s\Big]}{\mathbb{E}^{\mathbb{P}}\Big[\Lambda_t^{\theta^X} U_T\Big|\overline{\mathcal{G}}_s\Big]}, \\
&= \mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}} e^{-\int_0^t r_u du} S_s e^{\int_s^t dY_u}\Big|\overline{\mathcal{G}}_s\right] \\
&= \widetilde{S}_s \mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}} e^{-\int_s^t r_u du} e^{\int_s^t dY_u}\Big|\overline{\mathcal{G}}_s\right].
\end{aligned} \tag{3.10}$$

The proposition is proved if under the conditions (3.4)-(3.5)

$$\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}} e^{-\int_s^t r_u du} e^{\int_s^t dY_u}\Big|\overline{\mathcal{G}}_s\right] = 1, \quad \forall 0 \le s \le t \le T \tag{3.11}$$

we write from proposition VIII-22 in Cont and Tankov (2005)

$$Y_t = \widehat{Y}_t - \frac{1}{2}\int_0^t \sigma_s^2 ds - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(e^z - 1 - z\Big)\widetilde{N}^X(ds, dz) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(e^z - 1 - z\Big)\rho^X(dz)ds \tag{3.12}$$

Then,

$$\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}} e^{-\int_s^t r_u du} e^{\int_s^t dY_u} =$$

$$\underbrace{\exp\left\{\widehat{K}_s(\theta^X) - \widehat{K}_t(\theta^X) - \int_s^t r_u du - \frac{1}{2}\int_s^t \sigma_u^2 du - \int_s^t \int_{\mathbb{R}\setminus\{0\}} \left(e^{\theta_u^X(e^z-1)} - 1 - \theta_u^X(e^z-1)\right)\rho^X(dz)du\right\}}_{I}$$

$$\times \underbrace{\exp\left\{\int_s^t (\theta_u^X+1)\mu_u du + \int_s^t (\theta_u^X+1)\sigma_u dW_u + \int_s^t \int_{\mathbb{R}\setminus\{0\}} \left(z + \theta_u^X(e^z-1)\right)\widetilde{N}^X(du;dz)\right\}}_{II} \tag{3.13}$$

where [See Appendix 2]

$$\widehat{K}_t(\theta^X) = \int_0^t \left[\theta_s^X \mu_s + \frac{1}{2}(\theta_s^X)^2\sigma_s^2\right]ds + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left(e^{\theta_s^X(e^z-1)} - 1 - \theta_s^X(e^z-1)\right)\rho^X(dz)ds, \quad \forall\, t \in [0,T] \tag{3.14}$$

The term I in (3.13) is $\overline{\mathcal{G}}_s$-measurable, hence

$$\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}} e^{-\int_s^t r_u du} e^{\int_s^t dY_u}\middle|\overline{\mathcal{G}}_s\right] = \mathbb{E}^{\mathbb{P}}\left[\exp\left\{\int_s^t (\theta_u^X+1)\mu_u du + \int_s^t (\theta_u^X+1)\sigma_u dW_u \right.\right.$$
$$\left.\left. + \int_s^t \int_{\mathbb{R}\setminus\{0\}} \left(z + \theta_u^X(e^z-1)\right)\widetilde{N}^X(du;dz)\right\}\middle|\overline{\mathcal{G}}_s\right]$$
$$\times \exp\left\{\widehat{K}_s(\theta^X) - \widehat{K}_t(\theta^X) - \int_s^t r_u du - \frac{1}{2}\int_0^t \sigma_u^2 du - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left(e^{\theta_u^X(e^z-1)} - 1 - \theta_u^X(e^z-1)\right)\rho^X(dz)du\right\} \tag{3.15}$$

After some algebra and using theorem (2.1) we have

$$\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda_t^{\theta^X}}{\Lambda_s^{\theta^X}} e^{-\int_s^t r_u du} e^{\int_s^t dY_u}\middle|\overline{\mathcal{G}}_s\right] = \exp\left\{\int_s^t (\mu_u - r_u + \theta_u^X\sigma_u^2)du \right.$$
$$\left. + \int_s^t \int_{\mathbb{R}\setminus\{0\}} \left(e^{\theta_u^X(e^z-1)} - 1\right)(e^z-1)\rho^X(dz)du\right\} \tag{3.16}$$

Hence, using the condition (3.5) we have the result. ■

Remark 3.2 *Condition (3.4) is useful to assure integrability of some expressions used in the proof. In the literature, (3.5) is known as* Martingale condition *because the probability measure* $\mathbb{Q}^{\theta^X}$ *defined from the process* θ^X *is a martingale measure for the discounted price process of risky asset.*

For pricing purposes, we need to know the dynamics of price process of the risky asset under the martingale probability measure $\mathbb{Q}^{\theta^X}$. The following proposition states a result in this direction

Proposition 3.2
Under risk-neutral probability measure $\mathbb{Q}^{\theta^X}$,

1. *the log-return process Y is expressed as*

$$Y_t = \widehat{\gamma}_t + \widehat{M}_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} h(z)\underline{\widetilde{N}}^X(ds,dz) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} (z-h(z))N^X(ds,dz) \quad \textit{for} \quad t \in \mathcal{T} \tag{3.17}$$

where

- *$(\widehat{\gamma}_t)$ is a $\mathcal{F}_T^X$-measurable continuous process defined by*

$$\widehat{\gamma}_t := \int_0^t \Big(r_s - \frac{1}{2}\sigma_s^2\Big)ds - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - h(z))\underline{\rho}^{\theta^X}(dz)ds \tag{3.18}$$

- *$(\widehat{M}_t)$ is a continuous Gaussian process with independent increments under $\widehat{\mathbb{Q}}^{\theta^X}(\omega,.)$, defined by*

$$\widehat{M}_t := \int_0^t \sigma_s dW_s^{\theta^X} \tag{3.19}$$

with W^{θ^X} denoting the standard Brownian motion under $\mathbb{Q}^{\theta^X}$;

- *$\underline{\widetilde{N}}^X$ is the compensated measure of N^X under $\mathbb{Q}^{\theta^X}$ defined by*

$$\underline{\widetilde{N}}^X(ds,dz) = N^X(ds,dz) - \underline{\rho}^{\theta^X}(dz)ds \tag{3.20}$$

with $\underline{\rho}^{\theta^X}(dz) = e^{\theta^X(e^z-1)}\rho^X(dz)$.

Also, $\widehat{Y}$ admits the following expansion

$$\widehat{Y}_t = \int_0^t r_s ds + \int_0^t \sigma_s dW_s^{\theta} + \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1)\underline{\widetilde{N}}^X(ds,dz) \quad \textit{for} \quad t \in \mathcal{T}. \tag{3.21}$$

2. *Let $U_t := \frac{d\mathbb{Q}^{\theta^X}}{d\mathbb{P}}\Big|_{\mathcal{F}_t^X}$ for all $0 \le t \le T$. If (U_t) is a multiplicative functional (See Asmussen (2003)) and $U_0 = 1$ then X has the Markov property with respect to the filtration $\mathbf{G} = \{\mathcal{G}_t, \quad 0 \le t \le T\}$.*

Proof.
1) For $\mathbb{P}$ a.s. $\omega \in \Omega$, $\Lambda_t^{\theta^X} = \frac{d\widehat{\mathbb{Q}}^{\theta^X}(\omega,.)}{d\widehat{\mathbb{P}}(\omega,.)}|_{\overline{\mathcal{G}}_t} = e^{D_t^{\theta^X}}$ where $D_t^{\theta^X} = \int_0^t \theta_s^X d\widehat{Y}_s - \log \widehat{K}_t(\theta)$ i.e.,

$$D_t^{\theta^X} = \int_0^t \sigma_s \theta_s^X dW_s - \frac{1}{2}\int_0^t \sigma_s^2 (\theta_s^X)^2 ds + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \theta_s^X (e^z - 1)\widetilde{N}^X(ds,dz) \tag{3.22}$$

$$- \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(e^{\theta_s^X(e^z-1)} - 1 - \theta_s^X(e^z - 1)\Big)\rho^{X_s}(dz)ds. \tag{3.23}$$

By Itô's formula,

$$\Lambda_t^{\theta^X} = e^{D_t^{\theta^X}} = 1 + \int_0^t \Lambda_{s^-}^{\theta^X} \theta_s^X \sigma_s dW_s - \int_0^t \int_{\mathbb{R}\backslash\{0\}} \Lambda_{s^-}^{\theta^X} \Big(e^{\theta_s^X (e^z - 1)} - 1 \Big) \widetilde{N}^X(ds, dz) \tag{3.24}$$

Thus, by setting $M_t := \int_0^t \sigma_s dW_s$ which is a local martingale under $\mathbb{P}$ the quadratic covariation

$$\begin{aligned} [M; \Lambda^{\theta^X}]_t &= \int_0^t \Lambda_{s^-}^{\theta^X} \theta_s^X \sigma_s^2 ds \\ &= \int_0^t \Lambda_s^{\theta^X} \theta_s^X \sigma_s^2 ds \end{aligned} \tag{3.25}$$

so from theorem III-39 of Protter (2003) we have that the process $\widehat{M}$ defined by

$$\widehat{M}_t = M_t - \int_0^t \frac{d[M; \Lambda^{\theta^X}]_s}{\Lambda_s^{\theta^X}} = \int_0^t \sigma_s dW_s - \int_0^t \theta_s^X \sigma_s^2 ds \tag{3.26}$$

is a continuous local martingale under $\widehat{\mathbb{Q}}^{\theta^X}(\omega, .)$ and we could write $\widehat{M}_t = \int_0^t \sigma_s dW_s^{\theta^X}$ where $W_t^{\theta^X} = W_t - \int_0^t \theta_s^X \sigma_s ds$.
Also, from the theorem 2.3 we have the decomposition

$$\frac{d\mathbb{Q}^{\theta^X}}{d\mathbb{P}}|_{\overline{\mathcal{G}}_t} = \Lambda_t^{\theta^X} U_T$$

and

$$[W; \Lambda^{\theta^X} U_T]_t = \int_0^t \Lambda_s^{\theta^X} U_T \theta_s^X \sigma_s ds.$$

Hence, by theorem III-39 of Protter (2003) $W_t^{\theta^X} = W_t - \int_0^t \frac{d[W, \Lambda^{\theta^X} U]_s}{\Lambda_t^{\theta^X} U_T}$ is a locally martingale under $\mathbb{Q}^{\theta^X}$.
Also

$$[W^{\theta^X}, W^{\theta^X}]_t = [W, W]_t = t$$

and by the Lévy characterization theorem, W^{θ^X} is the standard Brownian motion under $\mathbb{Q}^{\theta^X}$ and consequently $\widehat{M}_t = \int_0^t \sigma_s dW_s^{\theta^X}$ is a continuous Gaussian process with independent increments under $\widehat{\mathbb{Q}}^{\theta^X}(\omega, .)$.

Now, we will construct the compensated measure of N^X with respect to the family $\overline{\mathbf{G}} = \{\overline{\mathcal{G}}_t, \quad 0 \leq t \leq T\}$ and the probability measure $\mathbb{Q}^{\theta^X}$.
Let $\underline{\widetilde{N}}^X$ be the process defined for any $A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$ (See Fujiwara (2009)) by

$$\begin{aligned} \underline{\widetilde{N}}_t^X(A) &:= N^X\Big((0, t], A\Big) - \int_0^t \int_{\mathbb{R}\backslash\{0\}} \mathbb{I}_A(z) \rho^X(dz) ds \\ &= \int_0^t \int_{\mathbb{R}\backslash\{0\}} \mathbb{I}_A(z) \widetilde{N}^X(ds, dz) \end{aligned} \tag{3.27}$$

we have $\Big[\underline{\widetilde{N}}^X(A), \Lambda^{\theta^X} U_T\Big]_t = \int_0^t \int_{\mathbb{R}\backslash\{0\}} \mathbb{I}_A(z) \Lambda_{s^-}^{\theta^X} U_T \Big(e^{\theta_s (e^z - 1)} - 1 \Big) N^X(ds, dz)$.
Since $\rho^X(dz)dt$ is the predictable projection of the Poisson measure $N^X(dt, dz)$,

$$< \underline{\widetilde{N}}^X(A); \Lambda^\theta U_T >_t = \int_0^t \int_{\mathbb{R}\backslash\{0\}} \mathbb{I}_A(z) \Lambda_{s^-}^\theta U_T \Big(e^{\theta_s (e^z - 1)} - 1 \Big) \rho^{X_s}(dz) ds \tag{3.28}$$

So, by theorem III-3.11 of Jacod and Shiryaev (2003), the new process $\underline{\widetilde{N}}$ defined by

$$\begin{aligned}\underline{\widetilde{N}}_t^X(A) &= \underline{\widetilde{N}}_t^X - \int_0^t \frac{d < \underline{\widetilde{N}}^X(A); \Lambda^\theta U_T >_s}{\Lambda^\theta_{s^-} U_T} \\ &= N^X\Big((0,t], A\Big) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \mathbb{I}_A(z) e^{\theta_s(e^z-1)} \rho^X(dz) ds\end{aligned} \tag{3.29}$$

is also a locally martingale under $\mathbb{Q}^{\theta^X}$.
$\mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[[\underline{\widetilde{N}}, \underline{\widetilde{N}}]_t\Big] = \mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\int_0^t \int_{\mathbb{R}\setminus\{0\}} \mathbb{I}_A(z) e^{\theta_s(e^z-1)} \rho^X(dz) ds\Big] < \infty$ by condition (3.4).
Then by corollary II-3 of Protter (2003), $\underline{\widetilde{N}}^X$ is a martingale under $\mathbb{Q}^{\theta^X}$ with respect to family $\overline{\mathbf{G}} = \{\overline{\mathcal{G}}_t, \quad 0 \le t \le T\}$, hence it is the compensated measure of N^X under $\mathbb{Q}^{\theta^X}$ i.e.,

$$\underline{\widetilde{N}}^X(ds, dz) = N^X(ds, dz) - \underline{\rho}^{\theta^X}(dz) ds \tag{3.30}$$

where $\underline{\rho}^{\theta^X}(dz) := e^{\theta_s(e^z-1)} \rho^X(dz)$.

From decomposition 2.3 and using the results above and condition (3.5), we obtain this expression of Y under $\mathbb{Q}^{\theta^X}$:

$$Y_t = \widehat{\gamma}_t + \widehat{M}_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} h(z) \underline{\widetilde{N}}(ds, dz) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} (z - h(z)) N^X(ds, dz) \quad \text{for} \quad \forall t \in \mathcal{T} \tag{3.31}$$

where $\widehat{\gamma}$ is a continuous $\mathcal{F}_T^X$-measurable process defined by

$$\widehat{\gamma}_t := \int_0^t \Big(r_s - \frac{1}{2}\sigma_s^2\Big) ds - \int_0^t \int_{\{0<|z|\le 1\}} (e^z - 1 - z) \underline{\rho}^{\theta^X}(dz) ds - \int_0^t \int_{\{|z|>1\}} (e^z - 1) \underline{\rho}^{\theta^X}(dz) ds \tag{3.32}$$

The expression of $\widehat{Y}$ is obtained straightforwardly as we do for Y.

2) Let $(\mathbb{P}_x)_{x\in\mathbb{S}}$ a family of probabilities defined on $(\Omega, \overline{\mathcal{G}}_T)$ and associated to the Markov semi-group $(P_t(x, .))$ which characterizes X under $\mathbb{P}$. We define another family of probabilities $(\mathbb{Q}_x^{\theta^X})_{x\in\mathbb{S}}$ on $(\Omega, \overline{\mathcal{G}}_T)$ such that

$$\mathbb{Q}_x^{\theta^X} = \mathbb{Q}^{\theta^X}[A | X_0 = x], \quad A \in \overline{\mathcal{G}}_T. \tag{3.33}$$

$\mathbb{Q}_x^{\theta^X}$ is locally absolutly continuous with respect to $\mathbb{P}_x$ and

$$\frac{d\mathbb{Q}_x^{\theta^X}}{d\mathbb{P}_x}\Big|_{\mathcal{F}_t^X} = U_t, \quad t \in \mathcal{T}. \tag{3.34}$$

The Markov property for X under $(\mathbb{P}, \mathbf{G})$ could be written as

$$\mathbb{E}^{\mathbb{P}_x}\Big[Z_t \circ \tau_s \Big| \mathcal{G}_s\Big] = \mathbb{E}^{\mathbb{P}_{X_s}}\Big[Z_t\Big], \quad \forall s \le t \le t + s \le T \tag{3.35}$$

for any $\mathcal{F}_t^X$-measurable Z_t and where τ_t is the shift operator (see Asmussen (2003)).
We want to prove that X is also Markov under $(\mathbb{Q}^{\theta^X}, \mathbf{G})$.
For any $\mathcal{F}_t^X$-measurable Z_t and $x \in \mathbb{S}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_x^{\theta^X}}\Big[Z_t \circ \tau_s \Big| \mathcal{G}_s\Big] &= \frac{\mathbb{E}^{\mathbb{P}_x}\Big[U_{t+s} Z_t \circ \tau_s \Big| \mathcal{G}_s\Big]}{\mathbb{E}^{\mathbb{P}_x}\Big[U_{t+s} \Big| \mathcal{G}_s\Big]} \quad (\text{ by Bayes' rule}) \\
&= \frac{\mathbb{E}^{\mathbb{P}_x}\Big[U_s.(U_t \circ \tau_s).(Z_t \circ \tau_s) \Big| \mathcal{G}_s\Big]}{\mathbb{E}^{\mathbb{P}_x}\Big[U_s.(U_t \circ \tau_s) \Big| \mathcal{G}_s\Big]} \\
&= \frac{\mathbb{E}^{\mathbb{P}_x}\Big[(U_t.Z_t) \circ \tau_s \Big| \mathcal{G}_s\Big]}{\mathbb{E}^{\mathbb{P}_x}\Big[U_t \circ \tau_s \Big| \mathcal{G}_s\Big]} \\
&= \frac{\mathbb{E}^{\mathbb{P}_{X_s}}\Big[U_t.Z_t\Big]}{\mathbb{E}^{\mathbb{P}_{X_s}}\Big[U_t\Big]} \quad (\text{by } 3.35) \\
&= \mathbb{E}^{\mathbb{P}_{X_s}}\Big[U_t.Z_t\Big] \\
&= \mathbb{E}^{\mathbb{Q}_{X_s}^{\theta^X}}\Big[Z_t\Big] \qquad (3.36)
\end{aligned}$$

where we have used the fact that U is a multiplicative functional which verifies $\mathbb{E}^{\mathbb{P}_x}[U_t] = 1$, $\forall x \in \mathbb{S}$. ■

Remark 3.3 *Theorem 2.2 and proposition 3.2 imply that (X, Y) is a Markov additive process under the risk-neutral measure $\mathbb{Q}^{\theta^X}$.*

Proposition 3.3
Under condition (H) *we have that for all $\mathbb{Q} \in \mathcal{M}^a(\widetilde{S})$, if $\widehat{\mathbb{Q}}(\omega, .)$ and $\widehat{\mathbb{P}}(\omega, .)$ denote respectively some regular versions of conditional probability $\mathbb{Q}(.|\mathcal{F}_T^X)$ and $\mathbb{P}(.|\mathcal{F}_T^X)$ Then*

$$\mathbb{H}_{\mathcal{G}_T}(\widehat{\mathbb{Q}}, \widehat{\mathbb{P}}) \geq \mathbb{H}_{\mathcal{G}_T}(\widehat{\mathbb{Q}}^{\theta^X}, \widehat{\mathbb{P}}) \quad \mathbb{P}\ (a.s) \qquad (3.37)$$

where $\widehat{\mathbb{Q}}^{\theta^X}$ is defined by (3.9).

Proof.
Let $\mathbb{Q} \in \mathcal{M}^a(\widetilde{S})$. By theorem (2.3), $\mathbb{Q}$ admits the decomposition

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_t} = \Lambda_t^Y \Lambda_T^X, \quad \text{for all} \quad 0 \leq t \leq T \qquad (3.38)$$

where we set $\Lambda_t^Y := \frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_t}$ with $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$ denoting the regular versions of $\mathbb{Q}(.|\mathcal{F}_T^X)$ and $\mathbb{P}(.|\mathcal{F}_T^X)$ respectively. Otherwise, we have from Theorem III-33 of Protter (2003) that $\{\widetilde{R}_t : 0 \leq t \leq T\}$ and $\{\int_0^t \theta_s^X d\widetilde{R}_s : 0 < t \leq T\}$ are $(\overline{\mathbf{G}})-$local martingale under $\mathbb{Q}$ since $\{1/\widetilde{S}_t^- : 0 \leq t \leq T\}$ and $\{\theta_t^X : t \in \mathcal{T}\}$ are càglàd. Therefore, there exists a sequence $(T_n)_{n\geq 0}$ of stopping times such that $T_n \nearrow T$ as $n \longrightarrow +\infty$ and $\{\int_0^{t\wedge T_n} \theta_s^X d\widetilde{R}_s : 0 \leq t \leq T\}$ is an $(\overline{\mathbf{G}})$-martingale under $\mathbb{Q}$.

Now, by using lemma 3.1 we have $\mathbb{P}$ $a.s$

$$\begin{aligned}
\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}|\widehat{\mathbb{P}}) &\geq \mathbb{H}_{\overline{\mathcal{G}}_{T_n}}(\widehat{\mathbb{Q}}|\widehat{\mathbb{P}}) \text{ (since } \overline{\mathcal{G}}_{T_n} \subset \overline{\mathcal{G}}_T) \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\log\Big(\frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_{T_n}}\Big)\Big|\mathcal{F}_T^X\Big] \\
&\geq \mathbb{E}^{\mathbb{Q}}\Big[\log\Big(\frac{d\widehat{\mathbb{Q}}^{\theta^X}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_{T_n}}\Big)\Big|\mathcal{F}_T^X\Big] \qquad (3.39)
\end{aligned}$$

where

$$\frac{d\widehat{\mathbb{Q}}^{\theta^X}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_{T_M}} := \frac{\exp\Big(\int_0^{T_n}\theta_s^X d\widetilde{R}_s\Big)}{\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^{T_n}\theta_s^X d\widetilde{R}_s\Big)\Big|\mathcal{F}_T^X\Big]} \quad (\mathbb{P}\ a.s) \qquad (3.40)$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\log\left(\frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_{T_n}}\right)\Bigg|\mathcal{F}_T^X\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{T_n}\theta_s^X d\widetilde{R}_s - \widehat{K}_{T_n}(\theta) + \int_0^{T_n}\theta_s^X r_s ds\Bigg|\mathcal{F}_T^X\right] \qquad (3.41)$$

but

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\int_0^{T_n}\theta_s^X d\widetilde{R}_s\Bigg|\mathcal{F}_T^X\right] &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^{T_n}\theta_s^X d\widetilde{R}_s\Bigg|\mathcal{G}_0\right] \\
&= 0 \quad (\mathbb{P}\ a.s)
\end{aligned}$$

since $\Big\{\int_0^{t\wedge T_n}\theta_s^X d\widetilde{R}_s : 0\leq t\leq T\Big\}$ is a $\overline{\mathbf{G}}$-martingale under $\mathbb{Q}$. Hence because $\{r_t : 0\leq t\leq T\}$ and $\{\widehat{K}_t(\theta) : 0\leq t\leq T\}$ are $\mathcal{F}_T^X-$measurable, we then have $\mathbb{P}\ a.s$

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\log\Big(\frac{d\widehat{\mathbb{Q}}^{\theta}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_{T_n}}\Big)\right] = \int_0^{T_n}\theta_s^X r_s ds - \widehat{K}_{T_n}(\theta) \quad \forall n\in\mathbb{N} \qquad (3.42)$$

On the other side, we have by Dominated Convergence Theorem:

$$\lim_{n\to+\infty}\Big\{\int_0^{T_n}\theta_s^X r_s ds - \widehat{K}_{T_n}(\theta)\Big\} = \int_0^{T}\theta_s^X r_s ds - \widehat{K}_{T}(\theta) \quad (\mathbb{P}\ a.s) \qquad (3.43)$$

Thus from (3.39), for all $\mathbb{Q}\in\mathcal{M}^a(\widetilde{S})$

$$\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}|\widehat{\mathbb{P}}) \geq \int_0^{T}\theta_s^X r_s ds - \widehat{K}_{T}(\theta) \quad (\mathbb{P}\ a.s) \qquad (3.44)$$

But, we have that $(\int_0^t\theta_s^X d\widetilde{R}_s)_{0\leq t\leq T}$ is a $\overline{\mathbf{G}}$-martingale under $\mathbb{Q}^{\theta^X}$ [See Appendix 6.3] and therefore

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{\theta^X}}\left[\int_0^{T}\theta_s^X d\widetilde{R}_s\Bigg|\mathcal{F}_T^X\right] &= \mathbb{E}^{\mathbb{Q}^{\theta}}\left[\int_0^{T}\theta_s^X d\widetilde{R}_s\Bigg|\overline{\mathcal{G}}_0\right] \\
&= 0. \quad (\mathbb{P}\ a.s) \qquad (3.45)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}^{\theta^X}, \widehat{\mathbb{P}}) &:= \mathbb{E}^{\mathbb{Q}^{\theta^X}}\left[\int_0^T \theta_s^X d\widetilde{R}_s + \int_0^T \theta_s^X r_s ds - \widehat{K}_T(\theta)\middle|\mathcal{F}_T^X\right] \\
&= \mathbb{E}^{\mathbb{Q}^{\theta^X}}\left[\int_0^T \theta_s^X r_s ds\middle|\mathcal{F}_T^X\right] - \mathbb{E}^{\mathbb{Q}^{\theta}}\left[\widehat{K}_T(\theta)\middle|\mathcal{F}_T^X\right] \\
&= \int_0^T \theta_s^X r_s ds - \widehat{K}_T(\theta) \quad (\mathbb{P}\ a.s)
\end{aligned}$$

Thus

$$\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}, \widehat{\mathbb{P}}) \geq \mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}^{\theta^X}, \widehat{\mathbb{P}}) \quad (\mathbb{P}\ a.s) \tag{3.46}$$

for all $\mathbb{Q} \in \mathcal{M}^a(\widetilde{S})$ and we can conclude that

$$\begin{aligned}
\mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}, \widehat{\mathbb{P}}) &\geq \mathbb{H}_{\overline{\mathcal{G}}_T}(\widehat{\mathbb{Q}}^{\theta^X}, \widehat{\mathbb{P}}) \\
&= \int_0^T \theta_s^X r_s ds - \widehat{K}_T(\theta) \quad (\mathbb{P}\ a.s)
\end{aligned} \tag{3.47}$$

This ends the proof of proposition 3.3. ■

In the previous proposition, we showed the optimality under certain conditions of one component of the decomposition of $\mathbb{Q}^{\theta^X}$. Specifically, this component is optimal among all "projections" on the σ-algebra $\mathcal{F}_T^X$ of probability measures $\mathbb{Q} \in \mathcal{M}^a(\widetilde{S})$. So, to obtain a global result we need to add some conditions on the other component. This is the subject of next section.

3.3 An optimization problem

Theorem (2.3) tells us that the density process related to a probability measure $\mathbb{Q} \in \mathcal{M}^a(\widetilde{S})$ can be decomposed as a product of a density process associated to the conditional probabilities given $\mathcal{F}_T^X$ and a Radon-Nikodym derivative associated to the change of measure of the Markov chain X. This decomposition is crucial because it permits to reduce with proposition (3.3) the problem of finding a MEMM for a Markov-modulated Lévy model to an optimization problem involving only the characteristics of the Markov chain component X.
In particular, we need to know how the Markov chain changes through a change of measure. So, we begin by giving a procedure to characterize this change of measure, namely by using a Girsanov-type theorem. In this setting we set the optimization problem and give the conditions for the existence of a solution.

3.3.1 Girsanov's change of measure for the continuous-time Markov chain

The Markov chain X is completely characterized under $\mathbb{P}$ by $(\Pi_0, \mathbb{A})$, where Π_0 is the initial probability distribution and $\mathbb{A} := (a_{ij})_{1\leq i,j\leq N}$ is the intensity matrix. We recall that an intensity matrix (for X) is a square matrix $\mathbb{M} := (m_{ij})_{1\leq i,j\leq N}$ with elements m_{ij} which satisfy the following conditions

1. $\forall i, j = 1, 2, ..., N$ with $i \neq j, \quad m_{ij} > 0$;
2. $\sum_{i=1}^M m_{ij} = 0$, so $m_{ii} < 0$.

In the sequel, we suppose without loss of generality that the change of measure modifies only the intensity matrix. Now, we introduce some notations as in Dufour and Elliott (1999).

- Let $\mathbf{a} := \Big(a_{11}, a_{22}, ..., a_{NN}\Big)' \in \mathbb{R}^M$ and $\mathbb{A}_0 := \mathbb{A} - \mathbf{Diag}(\mathbf{a})$, where $\mathbf{Diag}(\mathbf{a})$ is a diagonal matrix with the elements given by the vector $\mathbf{a}$;
- Let $\mathcal{C} := \Big\{(c_{ij})_{1\leq i,j\leq N}$ where c_{ij} verifies conditions (1)-(2) above$\Big\}$ the set of all possible intensity matrix for X;
- For $\mathbb{A}$, $\mathbb{C} \in \mathcal{C}$ we define $\mathbb{D} := \mathbb{C}/\mathbb{A}$ the matrix defined by $\mathbb{D} := (c_{ij}/a_{ij})_{1\leq i,j\leq N}$;
- $\mathbf{1} := (1, 1, ..., 1)' \in \mathbb{R}^M$ and $\mathbb{I}$ the identity matrix.

We have assumed that X is irreducible so $a_{ij} \neq 0$ and thus $\mathbb{D}$ is well defined. Now, consider a vector of counting process $\mathbf{N} := \{\mathbf{N}_t : t \in \mathcal{T}\}$ with

$$\mathbf{N}_t = \int_0^t \Big(\mathbb{I} - \mathbf{Diag}(X_{s^-})\Big) dX_s \tag{3.48}$$

Its component $N_t(i)$ counts the number of times the chain X jumps to state s_i during the time interval $[0, t]$ for $i = 1, 2, ..., N$.

Dufour and Elliott (1999) showed that the process defined by

$$\widetilde{\mathbf{N}}_t := \mathbf{N}(t) - \int_0^t \mathbb{A}_0 X_{u^-} du, \quad t \in \mathcal{T} \tag{3.49}$$

is a $(\{\mathcal{F}_t^X\}_{t\in\mathcal{T}}, \mathbb{P})-$martingale. Hence, we can state

Proposition 3.4 *The process $\Lambda^C := \{\Lambda_t^C : t \in \mathcal{T}\}$ defined by*

$$\Lambda_t^C = 1 + \int_0^t \Lambda_{u^-}^C [\mathbb{D}_0 X_{u^-} - \mathbf{1}]' d\widetilde{\mathbf{N}}_u \tag{3.50}$$

is a $(\{\mathcal{F}_t^X\}_{t\in\mathcal{T}}, \mathbb{P})-$martingale under some regularity conditions on the coefficients of $\mathbb{D}$. Also, by setting

$$\frac{d\mathbb{Q}^C}{d\mathbb{P}}\Big|_{\mathcal{F}_t^X} := \Lambda^C(t) \tag{3.51}$$

we define an absolutely continuous probability measure $\mathbb{Q}^C$ w.r.t $\mathbb{P}$ under which X is a Markov chain characterized by the intensity matrix $\mathbb{C}$.

For the proof, we refer to Dufour and Elliott (1999).

An explicit expression of this likelihood ratio is given by the following proposition

Proposition 3.5

$$\frac{d\mathbb{Q}^C}{d\mathbb{P}}\Big|_{\mathcal{F}_t^X} := \exp\Big\{ - \int_0^t [\mathbb{D}_0 X_{s^-} - \mathbf{1}]' \mathbb{A}_0 X_{s^-} ds\Big\} \prod_{0<s\leq t} \Big(1 + [\mathbb{D}_0 X_{s^-} - \mathbf{1}]' \Delta N_s\Big) \tag{3.52}$$

Furthermore,

$$\frac{d\mathbb{Q}^C}{d\mathbb{P}}\Big|_{\mathcal{F}_t^X} := \prod_{1\le i\neq j\le M} \exp\left\{ -\int_0^t (\widehat{\lambda}_s^{ij} - \lambda_s^{ij})ds + \int_0^t \log\Big(\frac{\widehat{\lambda}_s^{ij}}{\lambda_s^{ij}}\Big) d\mathcal{N}_s^{ij} \right\} \tag{3.53}$$

with for $i \neq j$,

- $\mathcal{N}_t^{ij} := \sum_{0<s\le t} \langle X_{s^-}, \mathbf{e}_i\rangle\langle X_s, \mathbf{e}_j\rangle$ *the number of times where X jumps from $\mathbf{e}_i$ to $\mathbf{e}_j$ during the time interval $[0,t]$ ($d\mathcal{N}_t^{ij}$ denotes its differential notation);*
- $\lambda_t^{ij} := a_{ij}\langle X_{t^-}, \mathbf{e}_i\rangle\langle X_t, \mathbf{e}_j\rangle$;
- $\widehat{\lambda}_t^{ij} := c_{ij}\langle X_{t^-}, \mathbf{e}_i\rangle\langle X_t, \mathbf{e}_j\rangle$

Proof. See Appendix. ■

Now, by setting for $1 \le i \neq j \le M$ and $s \in [0,T]$

$$\alpha_s^{ij} := \log\Big(\frac{\widehat{\lambda}_s^{ij}}{\lambda_s^{ij}}\Big) \tag{3.54}$$

we have

$$\begin{aligned} \Lambda_T^C &= \prod_{1\le i\neq j\le M} \exp\left\{ -\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}} - 1)ds + \int_0^T \alpha_s^{ij} d\mathcal{N}_s^{ij} \right\} \\ &= \exp\left\{ \sum_{1\le i\neq j\le M} \left(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}} - 1)ds + \int_0^T \alpha_s^{ij} d\mathcal{N}_s^{ij} \right) \right\} \end{aligned} \tag{3.55}$$

and in the sequel we will denote Λ_T^C by Λ_T^α.

For a given processes $\theta^X := \{\theta_t^X : t \in [0,T]\}$ satisfying condition (H) and $\alpha := \{\alpha_t^{ij} : 1 \le i \neq j \le M; t \in [0,T]\}$ we introduce the functional $F(\alpha)$ defined by

$$\begin{aligned} F(\alpha) := \mathbb{E}^{\mathbb{P}}\Bigg[&\left\{ \sum_{1\le i\neq j\le M} \Big(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}} - 1 - \alpha_s^{ij})ds \Big) + \int_0^T \theta_s^X r_s ds - \widehat{K}_T(\theta^X) \right\} \\ &\times \exp\left\{ \sum_{1\le i\neq j\le M} \left(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}} - 1)ds + \int_0^T \alpha_s^{ij} d\mathcal{N}_s^{ij} \right) \right\} \Bigg] \end{aligned} \tag{3.56}$$

Let $\mathbb{Q}^{(\alpha,\ \theta^X)}$ a probability measure absolutely continuous with respect to $\mathbb{P}$ on $\overline{\mathcal{G}}_T$ and characterized by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{(\alpha,\ \theta^X)}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_T} := \Lambda_T^{\theta^X} . \Lambda_T^\alpha \tag{3.57}$$

Lemma 3.2 *Under the probability measure $\mathbb{Q}^{(\alpha,\ \theta^X)}$ defined above, the functional $F(\alpha)$ admit a simple expression*

$$F(\alpha) = \sum_{1\le i\neq j\le M} \alpha_{ij}\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\mathcal{N}_T^{ij}\Big]$$
$$+\sum_{j=1}^{M}\left[\frac{1}{2}\theta_j^2\sigma_j^2 + \int_{\mathbb{R}\setminus\{0\}}\Big(e^{-\theta_j(e^z-1)} - 1 + \theta_j(e^z-1)\Big)\underline{\rho}_j^{\theta}(dz) - \sum_{\substack{i=1\\ i\neq j}}^{M} a_{ij}(e^{\alpha_{ij}}-1)\right]\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\tau_j\Big] \tag{3.58}$$

where we have set

- *τ_i the occupation time of the Markov chain X in the state $\mathbf{e}_i$ during the time interval $[0,T]$;*
- *$\alpha_t^{ij} := \alpha_{ij}\langle X_{t^-},\mathbf{e}_i\rangle\langle X_t,\mathbf{e}_j\rangle$;*
- *$\theta_t^X = \sum_{j=1}^M \theta_j\langle X_{t^-},\mathbf{e}_i\rangle$*
- *$\underline{\rho}_j^{\theta}(dz) := e^{\theta_j(e^z-1)}\rho_j(dz)$;*

Proof. See Appendix (6.4). ■

Theorem 3.1 *If it exists a matrix $\widetilde{\alpha} := \{\widetilde{\alpha}_{ij} : 1\le i\neq j\le M\}$ solution of the problem*

$$\min_{\alpha} F(\alpha) \tag{3.59}$$

Then the minimal entropy equivalent martingale measure for the model (2.1)-(2.2) exists and is defined by

$$\frac{d\mathbb{Q}^{(\widetilde{\alpha},\ \theta^X)}}{d\mathbb{P}}\Big|_{\mathbb{Q}^{(\widetilde{\alpha},\ \theta^X)}} := \Lambda_T^{\theta^X}\,.\,\Lambda_T^{\widetilde{\alpha}} \tag{3.60}$$

Proof.
Let $\mathbb{Q}$ be an arbitrary probability of $\mathcal{M}^a(\widetilde{S})$. By proposition 2.3 we can always write that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_T} = \Lambda_T^Y\Lambda_T^{\alpha} \tag{3.61}$$

where $\Lambda_T^{\alpha} = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T^X}$ and $\Lambda_T^Y = \frac{d\widehat{\mathbb{Q}}}{d\widehat{\mathbb{P}}}\Big|_{\overline{\mathcal{G}}_T}$. By an easy calculation we obtain

$$\begin{aligned}
\mathbb{H}_{\mathcal{G}_T}(\mathbb{Q}|\mathbb{P}) &= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^{\alpha}\log(\Lambda_T^{\alpha})\mathbb{E}^{\mathbb{P}}[\Lambda_T^Y|\mathcal{F}_T^X] + \Lambda_T^{\alpha}\mathbb{E}^{\mathbb{P}}[\Lambda_T^Y\log(\Lambda_T^Y)|\mathcal{F}_T^X]\Big]\\
&= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^{\alpha}\log(\Lambda_T^{\alpha}) + \Lambda_T^{\alpha}\mathbb{E}^{\mathbb{P}}[\Lambda_T^Y\log(\Lambda_T^Y)|\mathcal{F}_T^X]\Big] \quad \text{since} \quad \mathbb{E}^{\mathbb{P}}[\Lambda_T^Y|\mathcal{F}_T^X] = 1\\
&= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^{\alpha}\log(\Lambda_T^{\alpha}) + \Lambda_T^{\alpha}\mathbb{H}_{\mathcal{G}_T}(\widehat{\mathbb{Q}},\widehat{\mathbb{P}})\Big]\\
&\ge \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^{\alpha}\log(\Lambda_T^{\alpha}) + \Lambda_T^{\alpha}\mathbb{H}_{\mathcal{G}_T}(\widehat{\mathbb{Q}}^{\theta^X},\widehat{\mathbb{P}})\Big] \qquad \text{(by proposition 3.3)}\\
&= \mathbb{E}^{\mathbb{P}}\Bigg[\Bigg\{\sum_{1\le i\neq j\le M}\Big(-\int_0^T\lambda_s^{ij}(e^{\alpha_s^{ij}}-1-\alpha_s^{ij})ds\Big) + \int_0^T\theta_s^X r_s ds - \widehat{K}_T(\theta^X)\Bigg\}\\
&\times \exp\Bigg\{\sum_{1\le i\neq j\le M}\Bigg(-\int_0^T\lambda_s^{ij}(e^{\alpha_s^{ij}}-1)ds + \int_0^T\alpha_s^{ij}d\mathcal{N}_s^{ij}\Bigg)\Bigg\}\Bigg]\\
&= F(\alpha).
\end{aligned}$$

If we admit the existence of $\widetilde{\alpha}$, then

$$\begin{aligned} F(\alpha) &\geq F(\widetilde{\alpha}) \\ &= \mathbb{E}^{\mathbb{Q}^{(\widetilde{\alpha},\ \theta^X)}}\Big[\log(\Lambda_T^{\widetilde{\alpha}}) + \log(\Lambda_T^{\theta^X})\Big] \\ &= \mathbb{H}_{\overline{\mathcal{G}}_T}(\mathbb{Q}^{(\widetilde{\alpha},\ \theta^X)}|\mathbb{P}) \end{aligned}$$

■

Remark 3.4 *The above result tells us that once the Esscher parameter θ^X is founded, the search of the MEMM consist to minimize the funtional $F(\alpha)$ over the space of matrix $\alpha := \{\alpha_{ij} : 1 \leq i \neq j \leq M\}$. To have an explicit result, we need the explicit expression of $\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\tau_j\Big]$ and $\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\mathcal{N}_T^{ij}\Big]$ which we do not have since T is finite. However, by considering the asymtotic case $(T \to \infty)$ and with the hypothesis that X admit a stationary distribution Π under $\mathbb{Q}^{(\alpha,\ \theta^X)}$ the problem may be solved using Lagrange multipliers.*

4 Example

We worked out an example to show the feasibility of the theory developed before. We take the most simple example of Markov-modulated Lévy model namely the regime-switching Black-Scholes model. This model is recovered from the general setting describe in section 2.1 by imposing $N(.,.) = 0$ in equation (2.2). For ease of exposition, we take $M = 2$ (two-regime case).

In this case, the condition (H) yields a unique process $\theta := \{\frac{r_t - \mu_t}{\sigma_t^2} : t \in \mathcal{T}\}$ and thus we have to minimize the functional

$$\begin{aligned} f(\alpha) = &\sum_{1\leq i\neq j\leq 2} \alpha_{ij}\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\mathcal{N}_T^{ij}\Big] \\ &+\sum_{j=1}^{M}\left[\frac{1}{2}(\frac{r_j-\mu_j}{\sigma_j^2})^2\sigma_j^2+\int_{\mathbb{R}\setminus\{0\}}\Big(e^{-\theta_j(e^z-1)}-1+\theta_j(e^z-1)\Big)\underline{\rho}_j^{\theta}(dz)-\sum_{\substack{i=1\\ i\neq j}}^{M} a_{ij}(e^{\alpha_{ij}}-1)\right]\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\tau_j\Big] \end{aligned} \tag{4.1}$$

If we suppose that X admit a stationary distribution $\Pi = (\pi_1, \pi_2)$ under $\mathbb{Q}^{(\alpha,\ \theta^X)}$, then by the ergodic theorem we have these approximation when $T \to \infty$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\tau_j\Big] &\approx \pi_j T, \quad j = 1, 2 \\ \mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Big[\mathcal{N}_T^{ij}\Big] &\approx a_{ij}e^{\alpha_{ij}}\pi_j T, \quad 1 \leq i \neq j \leq 2. \end{aligned} \tag{4.2}$$

And the optimization problem (3.59) is equivalent to this one

$$\min_{(\Pi,\ \alpha)}\left\{\sum_{1\leq i\neq j\leq 2}\alpha_{ij}e^{\alpha_{ij}}a_{ij}\pi_j T + \sum_{j=1}^{2}\left[\frac{1}{2}(r_j-\mu_j)^2 - \sum_{\substack{i=1\\ i\neq j}}^{2} a_{ij}(e^{\alpha_{ij}}-1)\right]\pi_j T\right\} \tag{4.3}$$

under the constraints

$$\begin{aligned} \pi_1 + \pi_2 &= 1 \\ \pi_2 a_{21} e^{\alpha_{21}} - \pi_1 a_{12} e^{\alpha_{12}} &= 0 \end{aligned} \tag{4.4}$$

This problem can be easily solve by using Lagrangian techniques for optimization. Indeed, we can prove that the solution $(\Pi,\ \alpha)$ is such that:

$$\begin{aligned} \pi_1 &= \frac{a_{12} e^{\alpha_{12}}}{a_{12} e^{\alpha_{12}} + a_{21} e^{\alpha_{21}}} \\ \pi_2 &= \frac{a_{21} e^{\alpha_{21}}}{a_{12} e^{\alpha_{12}} + a_{21} e^{\alpha_{21}}} \end{aligned} \tag{4.5}$$

with $(\alpha_{12},\ \alpha_{21})$ solution of the system of equations

$$a_{21}^2 e^{2\alpha_{21}} \alpha_{12} + a_{12}^2 e^{2\alpha_{12}} \alpha_{21} = 0 \tag{4.6}$$

$$\begin{aligned} a_{21}\alpha_{21} e^{\alpha_{21}} (a_{12} e^{\alpha_{12}} + a_{21} e^{\alpha_{21}}) = \Big[&(1-\alpha_{21}) a_{21} e^{\alpha_{21}} - (1-\alpha_{12}) a_{12} e^{\alpha_{12}} \\ &+ a_{12} - a_{21} - \frac{1}{2}(r_1 - \mu_1)^2 + \frac{1}{2}(r_2 - \mu_2)^2 \Big] a_{12} e^{\alpha_{12}} \end{aligned} \tag{4.7}$$

which can be solved numerically.

Remark 4.1 *There are few results in the literature for the problem of determination of an MEMM for regime-switching models in continuous time. The first one is that by Elliott* et al. *(2005) which in the setting of a Markov-modulated Black-Scholes model showed that the equivalent martingale measure defined by a regime switching Esscher transform is the minimal entropy martingale measure (MEMM). Our model encompasses their setting and our result is more precise. Indeed, the MEMM obtained in our setting minimizes the inconditional relative entropy not only the conditional relative entropy like in their paper.*

5 Conclusion

In this paper, we showed how the minimal entropy martingale measure can be determined for a general Markov-modulated Lévy model. Our technique follows those developed by Miyahara and Fujiwara (2003) but we use the particular properties of the Markov additive processes to overcome some difficulties.

6 Appendix

6.1 Appendix 1

We give a proof for the proposition (3.5).
Proof. For the first part, we just have to prove that

$$\Lambda_t^C = \exp\Big\{ -\int_0^t [\mathbb{D}_0 X_s - \mathbf{1}]' \mathbb{A}_0 X_s ds \Big\} \prod_{0<s\leq t} \Big(1 + [\mathbb{D}_0 X_s - \mathbf{1}]' \Delta N_s\Big) \tag{6.1}$$

but it's a simple application of the Doléans-Dade exponential formula (See, Dufour and Elliott (1999)). For the second part, we have to show that

$$\Lambda_t^C = \prod_{1\leq i\neq j\leq M} \exp\left\{ -\int_0^t (\widehat{\lambda}_s^{ij} - \lambda_s^{ij}) ds + \int_0^t \log\Big(\frac{\widehat{\lambda}_s^{ij}}{\lambda_s^{ij}}\Big) d\mathcal{N}_s^{ij} \right\} \tag{6.2}$$

we remark that for $k = 1, 2, ..., N$

$$\mathbb{D}_0 X_{t^-} = \Big(\frac{c_{1k}}{a_{1k}} \langle X_{t^-}, \mathbf{e}_k\rangle, ..., \underbrace{0}_{k-th}, ..., \frac{c_{Nk}}{a_{Nk}} \langle X_{t^-}, s_k\rangle\Big)'$$

and

$$\mathbb{A}_0 X_{t^-} = \Big(a_{1k} \langle X_{t^-}, \mathbf{e}_k\rangle, ... \underbrace{0}_{k-th}, ..., a_{Nk} \langle X_{t^-}, \mathbf{e}_k\rangle\Big)'$$

hence,

$$[\mathbb{D}_0 X_{s^-} - \mathbf{1}]' \mathbb{A}_0 X_{s^-} = \sum_{i=1}^M \sum_{j=1, i\neq j}^M \langle X_s, \mathbf{e}_i\rangle \Big(c_{ij} \langle X_{s^-}, \mathbf{e}_j\rangle - a_{ij} \langle X_{s^-}, \mathbf{e}_j\rangle\Big) \tag{6.3}$$

so,

$$[\mathbb{D}_0 X_{s^-} - \mathbf{1}]' \mathbb{A}_0 X_{s^-} = \sum_{i=1}^M \sum_{j=1, i\neq j}^M \Big(\widehat{\lambda}_s^{ij} - \lambda_s^{ij}\Big) \tag{6.4}$$

Also, by noting that $\Delta N_s := N_s - N_{s^-}$, we have that $\Delta N_s = (0, ..., \underbrace{1}_{j-th}, 0, ..., 0)'$ when

$$\langle X_{s^-}, \mathbf{e}_i\rangle \langle X_s, \mathbf{e}_j\rangle = 1, \quad \text{for } i \neq j;\ i, j = 1, ..., N \text{ and } 0 < s \leq t \leq T.$$

Thus, we obtain

$$\begin{aligned} \prod_{0<s\leq t} \Big(1 + [\mathbb{D}_0 X_{s^-} - \mathbf{1}]' \Delta N_s\Big) &= \sum_{i=1}^M \sum_{j=1, i\neq j}^M \int_0^t \Big(\frac{c_{ij} \langle X_s, \mathbf{e}_i\rangle \langle X_{s^-}, \mathbf{e}_j\rangle}{a_{ij} \langle X_s, \mathbf{e}_i\rangle \langle X_{s^-}, \mathbf{e}_j\rangle}\Big) d\mathcal{N}_s^{ij} \\ &= \sum_{i=1}^M \sum_{j=1, i\neq j}^M \int_0^t \Big(\frac{\widehat{\lambda}_s^{ij}}{\lambda_s^{ij}}\Big) d\mathcal{N}_s^{ij} \end{aligned} \tag{6.5}$$

and hence,

$$\prod_{0<s\leq t} \Big(1 + [\mathbb{D}_0 X_{s^-} - \mathbf{1}]' \Delta N_s\Big) = \exp\Big\{ \int_0^t \log\Big(\frac{\widehat{\lambda}_s^{ij}}{\lambda_s^{ij}}\Big) d\mathcal{N}_s^{ij} \Big\} \tag{6.6}$$

■

6.2 Appendix 2

We have to show that $\forall\ t \in [0,T]$

$$\begin{aligned}\widehat{K}_t(\theta^X) &:= \mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^t \theta_s^X d\widehat{Y}_s\Big)\Big|\mathcal{F}_T^X\Big] \\ &= \int_0^t \Big[\theta_s^X\mu_s + \frac{1}{2}(\theta_s^X)^2\sigma_s^2\Big]ds + \int_0^t\int_{\mathbb{R}\setminus\{0\}}\Big(e^{\theta_s^X(e^z-1)} - 1 - \theta_s^X(e^z-1))\Big)\rho^X(dz)ds\end{aligned} \tag{6.7}$$

Proof. Formally, we can write by using Itô's Formula:

$$\begin{aligned}\exp\Big(\int_0^t \theta_s^X d\widehat{Y}_s\Big) &= 1 + \int_0^t \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)\Big[\theta_s^X\mu_s + \int_{\mathbb{R}\setminus\{0\}}\theta_s^X(e^z-1)\widetilde{N}^X(ds;dy)\Big]ds \\ &+ \int_0^t \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)\theta_s^X\sigma_s dW_s + \frac{1}{2}\int_0^t \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)(\theta_s^X)^2\sigma_s^2 ds \\ &+ \int_0^t\int_{\mathbb{R}\setminus\{0\}} \exp\Big(\int_0^s \theta_{s^-}^X d\widehat{Y}_s\Big)\Big(e^{\theta_{s^-}^X(e^z-1)} - 1 - \theta_{s^-}^X(e^z-1)\Big)N^X(ds,dz)\end{aligned} \tag{6.8}$$

or,

$$\begin{aligned}\exp\Big(\int_0^t \theta_s^X d\widehat{Y}_s\Big) &= 1 + \int_0^t \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)\theta_s^X\sigma_s dW_s \\ &+ \int_0^t\int_{\mathbb{R}\setminus\{0\}} \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)\Big(e^{\theta_s^X(e^z-1)} - 1\Big)\widetilde{N}^X(ds;dy) \\ &+ \int_0^t \exp\Big(\int_0^s \theta_s^X d\widehat{Y}_s\Big)\Big[\theta_s^X\mu_s + \frac{1}{2}(\theta_s^X)^2\sigma_s^2 + \int_{\mathbb{R}\setminus\{0\}}\Big(e^{\theta_s^X y} - 1 - \theta_s^X(e^z-1)\Big)\rho^X(dz)\Big]ds\end{aligned} \tag{6.9}$$

Conditioning both sides of (6.9) on $\mathcal{F}_T^X$,

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^t \theta_s^X d\widehat{Y}_s\Big)\Big|\mathcal{F}_T^X\Big] = \\ &\qquad 1 + \int_0^t \Big[\theta_u^X\mu_u + \frac{1}{2}(\theta_u^X)^2\sigma_u^2\Big]\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^u \theta_s^X d\widehat{Y}_s\Big)\Big|\mathcal{F}_T^X\Big]du \\ &+ \int_0^t\int_{\mathbb{R}\setminus\{0\}}\Big(e^{\theta_u^X(e^z-1)} - 1 - \theta_u^X(e^z-1)\Big)\rho^X(dz)\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^u \theta_s^X d\widehat{Y}_s\Big)\Big|\mathcal{F}_T^X\Big]du\end{aligned} \tag{6.10}$$

Where we have used the fact that W and $\widetilde{N}^X(.;.)$ are martingales.

Then, by solving (6.10) we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^t \theta_s^X d\widehat{Y}_s\Big)\Big|\mathcal{F}_T^X\Big] = \exp\Bigg[&\int_0^t \Big(\theta_s^X\mu_s + \frac{1}{2}(\theta_s^X)^2\sigma_s^2\Big)ds \\ &+ \int_0^t\int_{\mathbb{R}\setminus\{0\}}\Big(e^{\theta_s^X(e^z-1)} - 1 - \theta_s^X(e^z-1)\Big)\rho^X(dz)ds\Bigg]\end{aligned} \tag{6.11}$$

■

So we deduce an explicit expression for

$$\Lambda_t^{\theta^X} = \frac{\exp\Big(\int_0^t \theta_s^X d\widetilde{R}_s\Big)}{\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^t \theta_s^X d\widetilde{R}_s\Big)\Big|\mathcal{F}_T^X\Big]},\ t\in[0,T] \tag{6.12}$$

as

$$\Lambda_t^{\theta^X} = \exp\Big[\int_0^t \theta_s^X \sigma_s dW_s - \frac{1}{2}\int_0^t (\theta_s^X)^2\sigma_s^2 ds + \int_0^t \theta_s^X (e^z-1)\widetilde{N}^X(ds,dz) \\ - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(e^{\theta_s^X(e^z-1)} - 1 - \theta_s^X(e^z-1)\Big)\rho^X(dz)ds\Big] \tag{6.13}$$

6.3 Appendix 3

Indeed, it follows from (2.10) and (3.21)

$$\int_0^t \theta_s^X d\widetilde{R}_s = \int_0^t \theta_s^X \sigma_s dW_s^\theta + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \theta_s^X (e^z-1)\widetilde{N}^X(ds,dz)$$

and,
$a)$ - $\Big\{\int_0^T \theta_s^X \sigma_s dW_s^\theta : 0\le t\le T\Big\}$ is a continuous local martingale with his predictable quadratic variation which verifies

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^\theta}\Big[\Big\langle \int_0^T \theta_s^X \sigma_s dW_s^\theta\Big\rangle_T\Big] &= \mathbb{E}^{\mathbb{Q}^\theta}\Big[\Big(\int_0^T \theta_s^X \sigma_s dW_s^{\theta^X}\Big)^2\Big] \\
&= \mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\Big(\int_0^T \theta_s^X \sigma_s dW_s^\theta\Big)^2\Big|\mathcal{F}_T^X\Big]\Big] \\
&= \mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\int_0^T (\theta_s^X)^2\sigma_s^2 ds\Big] \\
&< \infty
\end{aligned} \tag{6.14}$$

so by corollary II-3 of Protter (2003) $\Big\{\int_0^T \theta_s^X \sigma_s dW_s^{\theta^X} : 0\le t\le T\Big\}$ is a $\overline{\mathbf{G}}$-continuous martingale under $\mathbb{Q}^{\theta^X}$.

$b)$- Also, condition (3.4) implies that $\forall t\in\mathcal{T}$,

$$\mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\int_0^t \int_{\mathbb{R}\setminus\{0\}} \theta_s^X (e^z-1)\rho^X(dz)ds\Big] \le \mathbb{E}^{\mathbb{Q}^{\theta^X}}\Big[\int_0^T \int_{\mathbb{R}\setminus\{0\}} e^z e^{\theta_s^X(e^z-1)}\rho^X(dz)ds\Big] < \infty \tag{6.15}$$

hence by corollary II-3 of Protter (2003),
$\Big\{\int_0^t \int_{\mathbb{R}\setminus\{0\}} \theta_s^X (e^z-1)\widetilde{N}^X(ds,dz)) : 0\le t\le T\Big\}$ is a $\overline{\mathbf{G}}$-martingale under $\mathbb{Q}^{\theta^X}$.

6.4 Appendix 4

By definition,

$$F(\alpha) := \mathbb{E}^{\mathbb{P}}\Bigg[\Bigg\{\sum_{1\le i\neq j\le M}\Big(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}}-1-\alpha_s^{ij})ds\Big)+\int_0^T \theta_s^X r_s ds - \widehat{K}_T(\theta^X)\Bigg\}$$
$$\times \exp\Bigg\{\sum_{1\le i\neq j\le M}\Bigg(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}}-1)ds+\int_0^T \alpha_s^{ij}d\mathcal{N}_s^{ij}\Bigg)\Bigg\}\Bigg] \quad (6.16)$$

so, by taking into account the expressions of $\Lambda_T^{\theta^X}$, Λ_T^{α} and $\widehat{(}K)(\theta^X)$

$$\begin{aligned} F(\alpha) &= \mathbb{E}^{\mathbb{P}}\Bigg[\mathbb{E}^{\mathbb{P}}\Bigg[\Lambda_T^{\alpha}\Lambda_T^{\theta^X}\log\Big(\Lambda_T^{\alpha}\Lambda_T^{\theta^X}\Big)\Bigg|\mathcal{F}_T^X\Bigg]\Bigg] \\ &= \mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Bigg[\log\Big(\Lambda_T^{\alpha}\Lambda_T^{\theta^X}\Big)\Bigg] \\ &= \mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Bigg[\log\Lambda_T^{\alpha}\Bigg]+\mathbb{E}^{\mathbb{Q}^{(\alpha,\ \theta^X)}}\Bigg[\log\Lambda_T^{\theta^X}\Bigg] \end{aligned} \quad (6.17)$$

also, by using the expressions from (3.2) and (3.58) we have

$$\begin{aligned} \log\Lambda_T^{\alpha} &= \sum_{1\le i\neq j\le M}\Bigg(-\int_0^T \lambda_s^{ij}(e^{\alpha_s^{ij}}-1)ds+\int_0^T \alpha_s^{ij}d\mathcal{N}_s^{ij}\Bigg) \\ &= \sum_{1\le i\neq j\le M}\Bigg(-a_{ij}(e^{\alpha_{ij}}-1)\tau_j+\alpha_{ij}\mathcal{N}_T^{ij}\Bigg) \end{aligned} \quad (6.18)$$

$$\log\Lambda_T^{\theta^X} = \int_0^T \theta_s^X\sigma_s dW_s^{\theta^X}+\frac{1}{2}\int_0^T(\theta_s^X\sigma_s)^2ds+\int_0^T\int_{\mathbb{R}\setminus\{0\}}\theta_{s-}^X(e^z-1)\underline{\widetilde{N}}^X(ds,dz)$$
$$+\int_0^T\int_{\mathbb{R}\setminus\{0\}}\Big(e^{-\theta_s^X(e^z-1)}-1+\theta_s^X(e^z-1)\Big)\underline{\rho}^X(dz)ds \quad (6.19)$$

Thus, since $\{W_t^{\theta^X} : t\in\mathcal{T}\}$ and $\{\underline{\widetilde{N}}^X(t,\cdot) : t\in\mathcal{T}\}$ are local-martingales under $\mathbb{Q}^{(\alpha,\ \theta^X)}$ we obtain the expression of $F(\alpha)$ given by lem 3.2.

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