Examples of Mahler Measures as special values of the Riemann Zeta function and L-series.

The Mahler's Measure of Polynomials – SFU and PIMS, Burnaby, BC, Canada

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1. Mahler Measure

Definition 1 *For $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic)* Mahler measure *is defined by*

$$m(P) := \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \tag{1}$$

$$= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \tag{2}$$

For more about Mahler Measures of several-variable polynomials, see [2], [5]

Jensen's formula provides a simple expression for the Mahler measure in the one-variable case. The several-variable case is more complicated. Many examples with explicit formulas have been produced. (See [4], [10], [12], [13], [14], [15], [16])

The simplest example with two variables is due to Smyth [13]:

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1) \tag{3}$$

Where

$$\mathrm{L}(\chi_{-3}, s) := \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s}$$

is the L-series in the character of conductor 3:

$$\chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 3 \\ -1 & \text{if} \quad n \equiv -1 \bmod 3 \\ 0 & \text{otherwise} \end{cases}$$

The analogous example with three variables is also due to Smyth:

$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3) \tag{4}$$

The general linear case with two variables is due to Cassaigne and Maillot, [10] : for $a, b, c \in \mathbb{C}$,

$$\pi m(a + bx + cy) = \begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{\mathrm{i}\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| & \triangle \\ \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases} \tag{5}$$

Here $\triangle$ stands for the statement that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle, and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$, and $|c|$ respectively. See Figure 1.

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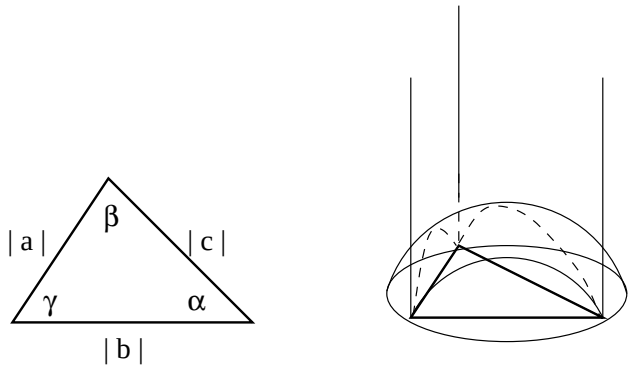

Figure 1: The main term in Caissaigne – Maillot formula is the volume of the ideal hyperbolic tetrahedron over the triangle.

D stands for the Bloch–Wigner dilogarithm (see definition later). The term with the dilogarithm can be interpreted as the volume of the ideal hyperbolic tetrahedron which has the triangle as basis and the fourth vertex is infinity (see [11], [17]).

2. Examples of higher weight

We have obtained (see [9]) examples of polynomials in several variables whose Mahler measures depend on polylogarithms, special values of the Riemann zeta function and special values of a certain L-series. See Table 1.

χ_{-4} is the real odd character of conductor 4, i.e.

$$\chi_{-4}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 4 \\ -1 & \text{if} \quad n \equiv -1 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

Let us observe that all the presented formulas share a common feature. If we assign weight 1 to any Mahler measure and to π, then all the formulas are homogeneous, meaning all the monomials have the same weight, and this weight is equal to the number of variables of the corresponding polynomial.

In addition to the formulas in Table 1, we proved

Theorem 2 *The Mahler measure of an n-variable polynomial in the first or in the third families is a homogeneous (of weight n) linear combination (with coefficients in $\mathbb{Q}[\pi]$) of special (odd) values of the Riemann zeta function.*

Analogously, the Mahler measure of an n-variable polynomial in the second family is a homogeneous (of weight n) linear combination (with coefficients in $\mathbb{Q}[\pi]$) of special (even) values of the L-series in the Dirichlet character of conductor 4.

3. Polylogarithms

We need the following definitions (see [6], [7], [8])

$\pi^2 m\left(\frac{1-x_1}{1+x_1}+\alpha\frac{1-y_1}{1+y_1}z\right)$	$7\,\zeta(3)$
$\pi^4 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}z\right)$	$62\zeta(5)+28\zeta(2)\zeta(3)$
$\pi^6 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}\frac{1-x_3}{1+x_3}+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}\frac{1-y_3}{1+y_3}z\right)$	$381\zeta(7)+372\zeta(2)\zeta(5)+336\zeta(4)\zeta(3)$
$\pi m\left((1+y)+\alpha(1-y)z\right)$	$2\mathrm{L}(\chi_{-4},2)$
$\pi^3 m\left(\frac{1-x_1}{1+x_1}(1+y)+\alpha\frac{1-y_1}{1+y_1}(1-y)z\right)$	$24\mathrm{L}(\chi_{-4},4)+6\zeta(2)\mathrm{L}(\chi_{-4},2)$
$\pi^5 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}(1+y)+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}(1-y)z\right)$	$160\mathrm{L}(\chi_{-4},6)+120\zeta(2)\mathrm{L}(\chi_{-4},4)+\frac{135}{2}\zeta(4)\mathrm{L}(\chi_{-4},2)$
$\pi^2 m\left((1+x)+\alpha(1+y)z\right)$	$\frac{7}{2}\zeta(3)$
$\pi^4 m\left(\frac{1-x_1}{1+x_1}(1+x)+\alpha\frac{1-y_1}{1+y_1}(1+y)z\right)$	$93\zeta(5)$
$\pi^6 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}(1+x)+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}(1+y)z\right)$	$\frac{15\cdot 127}{2}\zeta(7)+186\zeta(2)\zeta(5)$
$\pi^8 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}\frac{1-x_3}{1+x_3}(1+x)+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}\frac{1-y_3}{1+y_3}(1+y)z\right)$	$14\cdot 511\zeta(9)+30\cdot 127\zeta(2)\zeta(7)+48\cdot 31\zeta(4)\zeta(5)$
$\pi^3 m\left((1+w)(1+x)+\alpha(1-w)(1+y)z\right)$	$12\zeta(2)\mathrm{L}(\chi_{-4},2)+2\mathrm{i}\mathcal{L}_{3,1}(\mathrm{i},\mathrm{i})$
$\pi^5 m\left(\frac{1-x_1}{1+x_1}(1+w)(1+x)+\frac{1-y_1}{1+y_1}(1-w)(1+y)z\right)$	$144\zeta(2)\mathrm{L}(\chi_{-4},4)+90\zeta(4)\mathrm{L}(\chi_{-4},2)+16\mathrm{i}\mathcal{L}_{3,3}(\mathrm{i},\mathrm{i})+24\mathrm{i}\zeta(2)\mathcal{L}_{3,1}(\mathrm{i},\mathrm{i})$
$\pi^2 m((1+w)(1+y)+(1-w)(x-y))$	$\frac{7}{2}\zeta(3)+\frac{\pi^2}{2}\log 2$

Table 1: Here α is a nonzero complex number. The second column indicates the value of the first column for $\alpha=1$.

Definition 3 Multiple polylogarithms *are defined as the power series*

$$\mathrm{Li}_{k_1,\ldots,k_m}(x_1,\ldots,x_m) := \sum_{0<n_1<n_2<\ldots<n_m} \frac{x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}}{n_1^{k_1} n_2^{k_2} \ldots n_m^{k_m}} \tag{6}$$

which are convergent for $|x_i| < 1$. *The weight of a polylogarithm function is the number* $w = k_1 + \ldots + k_m$ *and its length is the number* m.

Definition 4 Hyperlogarithms *are defined as the iterated integrals*

$$\mathrm{I}_{k_1,\ldots,k_m}(a_1 : \ldots : a_m : a_{m+1}) :=$$

$$\int_0^{a_{m+1}} \underbrace{\frac{\mathrm{d}t}{t-a_1} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_1} \circ \underbrace{\frac{\mathrm{d}t}{t-a_2} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_2} \circ \ldots \circ \underbrace{\frac{\mathrm{d}t}{t-a_m} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_m} \tag{7}$$

where k_i *are integers,* a_i *are complex numbers, and*

$$\int_0^{b_{l+1}} \frac{\mathrm{d}t}{t-b_1} \circ \ldots \circ \frac{\mathrm{d}t}{t-b_l} = \int_{0\le t_1\le\ldots\le t_l\le b_{l+1}} \frac{\mathrm{d}t_1}{t_1-b_1} \cdots \frac{\mathrm{d}t_l}{t_l-b_l}$$

The value of the integral above only depends on the homotopy class of the path connecting 0 and a_{m+1} on $\mathbb{C} \setminus \{a_1,\ldots,a_m\}$.

It is easy to see that

$$\mathrm{I}_{k_1,\ldots,k_m}(a_1 : \ldots : a_m : a_{m+1}) = (-1)^m \mathrm{Li}_{k_1,\ldots,k_m}\left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_m}{a_{m-1}}, \frac{a_{m+1}}{a_m}\right) \tag{8}$$

$$\mathrm{Li}_{k_1,\ldots,k_m}(x_1,\ldots,x_m) = (-1)^m \mathrm{I}_{k_1,\ldots,k_m}\left(\frac{1}{x_1\ldots x_m} : \ldots : \frac{1}{x_m} : 1\right) \tag{9}$$

which gives an analytic continuation to multiple polylogarithms. For instance, with the convention about integrating over a real segment, simple polylogarithms have an analytic continuation to $\mathbb{C} \setminus [1,\infty)$.

There are modified versions of these functions which are analytic in larger sets, like the Bloch-Wigner dilogarithm,

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| \arg(1-z) \qquad z \in \mathbb{C} \setminus [1,\infty) \tag{10}$$

which can be extended as a real analytic function in $\mathbb{C} \setminus \{0,1\}$ and continuous in $\mathbb{C}$.

In the table, the numbers $\mathcal{L}_{3,n}(\mathrm{i},\mathrm{i})$ stand for certain combination of $\mathrm{Li}_{3,n}(\pm\mathrm{i},\pm\mathrm{i})$. It is possible to express these numbers as linear combinations of special values of L-series of length 2.

4. First general method for building examples

We have developed two methods that allowed us to build the examples. The first method, although more complex, can be applied to more general cases. It goes as follows:

1. Let $P_\alpha \in \mathbb{C}[x_1,\ldots,x_n]$ whose coefficients depend polynomially on a parameter $\alpha \in \mathbb{C}$. For instance, start with $P_\alpha(x) = 1 + \alpha x$, whose Mahler measure is $\log^+|\alpha|$.

2. We replace α by $\alpha\frac{1-y}{1+y}$ and obtain a polynomial $\tilde{P}_\alpha \in \mathbb{C}[x_1,\dots,x_n,y]$. In the example, $\tilde{P}_\alpha(x,y) = 1+y+\alpha(1-y)x$.

3. The Mahler measure of the second polynomial is a certain integral of the Mahler measure of the first polynomial.

$$m(\tilde{P}_\alpha) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}^1} m\left(P_{\alpha\frac{1-y}{1+y}}\right)\frac{\mathrm{d}y}{y} \quad \left(\text{ in the example, } = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}^1}\log^+\left|\alpha\frac{1-y}{1+y}\right|\frac{\mathrm{d}y}{y}\right)$$

4. If the Mahler measure depends just on the absolute value of α, we can make a change of variables $u = \left|\alpha\frac{1-y}{1+y}\right|$ (to be precise, first write $y = \mathrm{e}^{\mathrm{i}\theta}$ and then set $u = |\alpha|\tan\left(\frac{\theta}{2}\right)$). We obtain,

$$m(\tilde{P}_\alpha) = \frac{2}{\pi}\int_0^\infty m\left(P_u\right)\frac{|\alpha|\,\mathrm{d}u}{u^2+|\alpha|^2} = \frac{\mathrm{i}}{\pi}\int_0^\infty m\left(P_u\right)\left(\frac{1}{u+\mathrm{i}|\alpha|}-\frac{1}{u-\mathrm{i}|\alpha|}\right)\mathrm{d}u$$

In the example,

$$\begin{aligned} m(1+y+\alpha(1-y)x) &= \frac{\mathrm{i}}{\pi}\int_0^\infty \log^+ u\left(\frac{1}{u+\mathrm{i}|\alpha|}-\frac{1}{u-\mathrm{i}|\alpha|}\right)\mathrm{d}u \\ &= \frac{\mathrm{i}}{\pi}\int_0^1\int_s^1\frac{\mathrm{d}t}{t}\left(\frac{1}{s+\frac{\mathrm{i}}{|\alpha|}}-\frac{1}{s-\frac{\mathrm{i}}{|\alpha|}}\right)\mathrm{d}s \end{aligned}$$

$$= \frac{\mathrm{i}}{\pi}\left(\mathrm{I}_2\left(-\frac{\mathrm{i}}{|\alpha|}:1\right)-\mathrm{I}_2\left(\frac{\mathrm{i}}{|\alpha|}:1\right)\right) = -\frac{\mathrm{i}}{\pi}(\mathrm{Li}_2(\mathrm{i}|\alpha|)-\mathrm{Li}_2(-\mathrm{i}|\alpha|))$$

If we look back at the table, all the Mahler measures of polynomials that contain α have been computed by this method. The same is true for the Mahler measure of the last polynomial.

We have splitted the examples into five families. The first two families were developed starting from $1+\alpha x$, the third and fourth family start from $(1+x)+\alpha(y+z)$ (see [2], [16]). The last polynomial was obtained by integrating one particular case of Maillot's formula: $1+\alpha x+(1-\alpha)y$.

5 Second general method for building examples

The following method is very good for some specific examples and can indeed give us some information for the general n variable case.

1. Let $P(\mathbf{x},z) = p(\mathbf{x})+\alpha q(\mathbf{x})z \in \mathbb{C}[\mathbf{x},z]$ such that we know its Mahler measure as a function of α, a complex parameter.

   For instance, start with the polynomial $P(x,y,z) = (1+x)+\alpha(1+y)z$, whose Mahler measure is $\frac{2}{\pi^2}(\mathrm{Li}_3(\alpha)-\mathrm{Li}_3(-\alpha))$ for $\alpha \le 1$.

2. We will compute the Mahler measure of the polynomial

$$\tilde{P} = \left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_m}{1+x_m}\right)p(\mathbf{x})+\left(\frac{1-y_1}{1+y_1}\right)\dots\left(\frac{1-y_n}{1+y_n}\right)q(\mathbf{x})z$$

For example, consider $\tilde{P} = \frac{1-x_1}{1+x_1}(1+x) + \frac{1-y_1}{1+y_1}(1+y)z$.

It is easy to see that, for Mahler measure purposes, the general case is reduced to the following two cases: either we consider n factors in each term or we consider one term with n factors and the other with $n+1$ factors.

3. After some basic transformations (including the same change of variables as in the other method) we get a linear combination of terms of the form

$$\int_{\mathbb{T}^*} \int_0^\infty \int_0^\infty \log\max\{x|p(\mathbf{x})|, y|q(\mathbf{x})|\} \log^j x \frac{\mathrm{d}x}{x^2 \pm 1} \log^k y \frac{\mathrm{d}y}{y^2 \pm 1} \frac{\mathrm{d}\mathbf{x}}{\mathbf{x}}$$

It is crucial to use the fact that

$$\int_0^\infty \frac{x \log^k x \mathrm{d}x}{(x^2+a^2)(x^2 \pm 1)} = \frac{T_k(\log a)}{(a^2 \mp 1)} \qquad T_k[x] \in \mathbb{Q}[x] \qquad \deg T_k = k+1$$

In the example, we get

$$\pi^4 m\left(\frac{1-x_1}{1+x_1}(1+x) + \frac{1-y_1}{1+y_1}(1+y)z\right)$$

$$= \frac{1}{(2\mathrm{i})^2} \int_{\mathbb{T}^2} \int_0^\pi \int_0^\pi \log\max\left\{\left|\frac{1-\mathrm{e}^{\mathrm{i}\theta}}{1+\mathrm{e}^{\mathrm{i}\theta}}\right| |1+x|, \left|\frac{1-\mathrm{e}^{\mathrm{i}\tau}}{1+\mathrm{e}^{\mathrm{i}\tau}}\right| |1+y|\right\} \mathrm{d}\theta \mathrm{d}\tau \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= \frac{8}{(2\mathrm{i})^2} \int_{\mathbb{T}^2} \int_0^\infty \int_0^{x_1} \log\max\{x_1|1+x|, y_1|1+y|\} \frac{\mathrm{d}x_1}{x_1^2+1} \frac{\mathrm{d}y_1}{y_1^2+1} \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

4. We know that

$$\frac{1}{(2\mathrm{i})^*} \int_{\mathbb{T}^*} \log\max\{|p(\mathbf{x})|, \alpha|q(\mathbf{x})|\} \frac{\mathrm{d}\mathbf{x}}{\mathbf{x}} = \pi^* m(p(\mathbf{x}) + \alpha q(\mathbf{x})z)$$

If this formula is a multiple polylogarithm, we get a multiple polylogarithm again.

Following the example,

$$\frac{1}{(2\mathrm{i})^2} \int_{\mathbb{T}^2} \log\max\{|1+x|, \alpha|1+y|\} \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = -4\int_0^\alpha \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s}$$

Then, we set $z = \frac{y_1}{x_1}$ so we can eliminate the variable y_1. We get $x_1 \mathrm{d}z = \mathrm{d}y_1$.

$$= \frac{8}{(2\mathrm{i})^2} \int_{\mathbb{T}^2} \int_0^\infty \int_0^1 (\log x_1 + \log\max\{|1+x|, z|1+y|\}) \frac{x_1 \mathrm{d}x_1}{x_1^2+1} \frac{\mathrm{d}z}{z^2 x_1^2+1} \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= 8\int_0^\infty \int_0^1 \left(\log x_1 - 4\int_0^z \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s}\right) \frac{x_1 \mathrm{d}x_1}{x_1^2+1} \frac{\mathrm{d}z}{z^2 x_1^2+1}$$

$$= 8\pi^2 \int_0^1 \frac{\log^2 z}{2(1-z^2)} \mathrm{d}z + 8\int_0^1 \left(-4\int_0^z \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s}\right) \frac{-\log z}{1-z^2} \mathrm{d}z$$

$$= 7\pi^2\zeta(3) + 8(\mathrm{Li}_{3,2}(1,1) - \mathrm{Li}_{3,2}(-1,1) + \mathrm{Li}_{3,2}(1,-1) - \mathrm{Li}_{3,2}(-1,-1))$$

Looking back at the table, the polynomials that do not contain α have been computed exclusively in this way. The others can be computed with this method as well, except by the last polynomial.

It is possible to apply this method to obtain the more general results for any $\alpha \in \mathbb{C}$.

6. The example with $93\zeta(5)$

Back to Table 1, the first and second families were built starting from the funcion log which in a sense can be interpreted as having length 0. In that context, it is natural to expect polylogarithms of length 1 and these are easy to relate to zeta functions and L-series.

It is not clear from the methods described above how to obtain the zeta values for the third family. The basic formula has length 1 and so we expect the results to have length 2. We will describe the main tool that we used as we follow the simplification of the example above, which finally yields $93\zeta(5)$.

Very often we get the Mahler measure expressed as combinations of what is called alternating Euler sums:

$$\sum_{0<n_1<n_2<\ldots<n_m} \frac{(\pm 1)^{n_1}(\pm 1)^{n_2}\ldots(\pm 1)^{n_m}}{n_1^{k_1} n_2^{k_2}\ldots n_m^{k_m}}$$

We see that these numbers are indeed $\mathrm{Li}_{k_1,\ldots,k_m}(\pm 1,\ldots,\pm 1)$.

Recall that we have

$$\begin{aligned}\pi^4 m\left(\frac{1-x_1}{1+x_1}(1+x)+\frac{1-y_1}{1+y_1}(1+y)z\right) &= 7\pi^2\zeta(3)+8(\mathrm{Li}_{3,2}(1,1)-\mathrm{Li}_{3,2}(-1,1)) \\ &\quad +8(\mathrm{Li}_{3,2}(1,-1)-\mathrm{Li}_{3,2}(-1,-1)) \qquad (11)\end{aligned}$$

A property about alternating Euler sums, is that when they have length 2 and weight odd they can be simplified by using the formula (75) of [1]. In this particular case, it states that

$$\mathrm{Li}_{3,2}(x,y) = -\frac{1}{2}\mathrm{Li}_5(x\,y)+\mathrm{Li}_3(x)\,\mathrm{Li}_2(y)+3\mathrm{Li}_5(x)+2\mathrm{Li}_5(y)-\mathrm{Li}_2(x\,y)(\mathrm{Li}_3(x)+2\mathrm{Li}_3(y))$$

for $x, y = \pm 1$.

Taking into account that

$$\mathrm{Li}_k(1) = \zeta(k) \quad \text{and} \quad \mathrm{Li}_k(-1) = \left(\frac{1}{2^{k-1}}-1\right)\zeta(k) \qquad (12)$$

we get

$$\mathrm{Li}_{3,2}(1,1)-\mathrm{Li}_{3,2}(-1,1)+\mathrm{Li}_{3,2}(1,-1)-\mathrm{Li}_{3,2}(-1,-1) = -\frac{21}{4}\zeta(2)\zeta(3)+\frac{93}{8}\zeta(5)$$

We obtain the result by using that $\zeta(2) = \frac{\pi^2}{6}$

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