Mahler measures as values of regulators

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Mahler measure of multivariate polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n}$$

$$\mathbb{T}^n = S^1 \times \cdots \times S^1$$

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1)$$
- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}, \quad k\neq 4$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathrm{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathrm{L}'(A,0)$$

$$A: y^2 = x^3-44x+112$$

An algebraic integration for Mahler measure

Deninger (1997) : General framework.

$$m(P) = m(P^*) + \frac{1}{(-2i\pi)^{n-1}} \int_\Gamma \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \dots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$\eta_n(n)(x_1, \dots, x_n)$ is a $\mathbb{R}(n-1)$-valued smooth $n-1$-form in $X(\mathbb{C})$.

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($\mathrm{L}_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

$$(K \,\text{rank}\, 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure: two-variables

Rodriguez-Villegas (1997) :

$$P(x,y) = y + x - 1 \qquad X = \{P(x,y) = 0\}$$

$$m(P) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log |y + x - 1| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

By Jensen's equality:

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{\mathrm{d}x}{x}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \eta(x,y)$$

where

$$\gamma = X \cap \{|x| = 1, |y| \geq 1\} \qquad \eta(x,y) = \log |x| \mathrm{d}i \arg y - \log |y| \mathrm{d}i \arg x$$

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Properties of $\eta(x, y)$

- $\eta(x, y) = -\eta(y, x)$
- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$
- $\mathrm{d}\eta(x, y) = \mathrm{i}\,\mathrm{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right)$

Theorem

$$\eta(x, 1 - x) = \mathrm{d}\mathrm{i}D(x)$$

Bloch–Wigner dilogarithm:

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1 - x)\log|x|$$

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad |x| < 1$$

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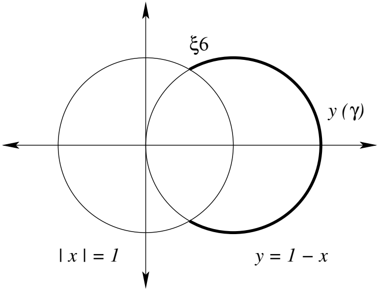
Use Stokes Theorem:

$$m(P) = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \eta(x, 1-x) = -\frac{1}{2\pi} D(\partial\gamma)$$

$x = \mathrm{e}^{2\pi \mathrm{i}\theta}$,

$$y(\gamma(\theta)) = 1 - \mathrm{e}^{2\pi \mathrm{i}\theta}, \quad \theta \in [1/6\,; 5/6]$$

$$\partial\gamma = [\bar{\xi}_6] - [\xi_6]$$



$$2\pi m(x+y+1) = D(\xi_6) - D(\bar{\xi}_6) = 2D(\xi_6) = \frac{3\sqrt{3}}{2} \mathrm{L}(\chi_{-3}, 2)$$

In general,

$P(x,y) \in \mathbb{C}[x,y]$, $X := \{P(x,y) = 0\}$

$$m(P) = m(P^*) - \frac{1}{2\pi i} \int_\gamma \eta(x,y)$$

Need

$$x \wedge y = \sum_j r_j \; z_j \wedge (1 - z_j) \qquad \text{in} \qquad \bigwedge^2 (\mathbb{C}(X)^*) \otimes \mathbb{Q}$$

Same as $\{x, y\} = 0$ in $K_2(\mathbb{C}(X)) \otimes \mathbb{Q}$.

$$\int_\gamma \eta(x,y) = \sum r_j D(z_j)|_{\partial\gamma}$$

Big picture

$$\dots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(X, \partial\gamma) \to K_2(X) \to \dots$$

$$\partial\gamma = X \cap \mathbb{T}^2$$

- $\eta(x, y)$ is exact, then $\{x, y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.
  $\rightsquigarrow$ dilogarithms, zeta function
- $\partial\gamma = \emptyset$, then $\{x, y\} \in K_2(X)$. We have $\eta(x, y)$ is not exact.
  $\rightsquigarrow$ L-series of a curve

We may get combinations of both situations.

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Example in the non-exact case

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Identities

Boyd (1997), Rodriguez-Villegas (2000)

$$7m(y^2+2xy+y-x^3-2x^2-x)=5m(y^2+4xy+y-x^3+x^2)$$

Rogers (2005)

$$m(4n^2)+m\left(\frac{4}{n^2}\right)=2m\left(2n+\frac{2}{n}\right)$$

where

$$m(k):=m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right)$$

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Idea in the Elliptic Curve case

- For $\{x, y\} \in K_2(E)$:

$$r(\{x, y\}) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \eta(x, y)$$

  γ generates $H_1(E, \mathbb{Z})^-$

- $$r(\{x, y\}) = D^E((x) \diamond (y))$$

  if $(x), (y)$ supported on $E_{tors}(\bar{\mathbb{Q}})$.

- $$\pi D^E \sim L(E, 2)$$

  is HARD.

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Properties of $\eta_n(n)(x_1, \dots, x_n)$

- Multiplicative in each variable, anti-symmetric.
  $\eta_n(n)$ is a function on $\bigwedge^n(\mathbb{C}(X)^*)_{\mathbb{Q}}$
- $\mathrm{d}\eta_n(n)(x_1, \dots, x_n) = \widehat{\mathrm{Re}_n}\left(\frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}\right)$
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Examples

$$\eta_2(2)(x, 1-x) = \mathrm{di}D(x)$$

$$\eta_3(3)(x,y,z) = \log|x| \left(\frac{1}{3}\mathrm{d}\log|y| \wedge \mathrm{d}\log|z| + \mathrm{di}\arg y \wedge \mathrm{di}\arg z\right)$$

$$+\log|y| \left(\frac{1}{3}\mathrm{d}\log|z| \wedge \mathrm{d}\log|x| + \mathrm{di}\arg z \wedge \mathrm{di}\arg x\right)$$

$$+\log|z| \left(\frac{1}{3}\mathrm{d}\log|x| \wedge \mathrm{d}\log|y| + \mathrm{di}\arg x \wedge \mathrm{di}\arg y\right)$$

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$$\eta_3(2)(x,y)$$

$$= \mathrm{i}D(x)\mathrm{di}\arg y + \frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

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$$+ \log|y| \left(\frac{1}{3} \mathrm{d}\log|z| \wedge \mathrm{d}\log|x| + \mathrm{d}i\arg z \wedge \mathrm{d}i\arg x \right)$$

$$+ \log|z| \left(\frac{1}{3} \mathrm{d}\log|x| \wedge \mathrm{d}\log|y| + \mathrm{d}i\arg x \wedge \mathrm{d}i\arg y \right)$$

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$$= iD(x)\mathrm{d}i\arg y + \frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

First variable in $\eta_n(n-1)$ behaves like the five-term relation

$$[x]+[y]+[1-xy]+\left[\frac{1-x}{1-xy}\right]+\left[\frac{1-x}{1-xy}\right]$$

Now

$$\eta_n(n-1)(x,x,x_1,\dots,x_{n-3}) = \mathrm{d}\eta_n(n-2)(x,x_1,\dots,x_{n-3})$$

First variable in $\eta_n(n-2)$ behaves like rational functional equations of $\mathcal{L}_3$.

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$$\eta_n(2)(x,x) = \mathrm{d}\eta_n(1)(x)$$

and

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Examples in three variables

- Smyth (2002):

$$\pi^2 m(1 + x + y^{-1} + (1 + x + y)z) = \frac{14}{3}\zeta(3)$$

- Condon (2003):

$$\pi^2 m\left(z - \left(\frac{1-x}{1+x}\right)(1+y)\right) = \frac{28}{5}\zeta(3)$$

- D'Andrea & L. (2003):

$$\pi^2 m\left(z(1-xy)^2 - (1-x)(1-y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}$$

New examples

Boyd & L. (2005)

$$m(x^2+1+(x+1)y+(x-1)z)=\frac{\mathrm{L}(\chi_{-4},2)}{\pi}+\frac{21}{8\pi^2}\zeta(3)$$

$$m(x^2+x+1+(x+1)y+z)=\frac{\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2)+\frac{19}{6\pi^2}\zeta(3)$$

An example in four variables

L.(2003)

$$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)=2\pi^2\mathrm{L}(\chi_{-4},2)+8\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{(-1)^{j+k+1}}{(2j+1)^3 k}$$

(2005)

$$=24\mathrm{L}(\chi_{-4},4)$$

In general, for m odd,

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{(-1)^{j+k+1}}{(2j+1)^m k}$$

$$=m\mathrm{L}(\chi_{-4},m+1)+\sum_{h=1}^{\frac{m-1}{2}}\frac{(-1)^h\pi^{2h}(2^{2h}-1)}{(2h)!}B_{2h}\mathrm{L}(\chi_{-4},m-2h+1)$$

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Exploring the n-variable world

- L. (2005)

  For
  $$z = \left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)$$
  Both $\eta_{n+1}(n+1)$ and $\eta_{n+1}(n)$ are exact.

- D'Andrea & L. (2005)

  $$X := \{\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n} = 0\} \subset \mathbb{C}^k.$$

  $\eta_k(k)$ is exact.

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Generalized Mahler measure

Gon & Oyanagi (2004)

For $f_1, \dots, f_r \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$,

$$m(f_1, \dots, f_r) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \max\{\log|f_1|, \dots, \log|f_r|\} \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n}$$

Note

$$m(f_1, f_2) = m(f_1 + z f_2)$$

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Note

$$m(f_1, f_2) = m(f_1 + z f_2)$$

Examples

The particular case when $f_j = P(x_j)$ for some $P \in \mathbb{C}[x]$.
Gon & Oyanagi (2004)

$$m(1-x_1,\ldots,1-x_n) = \sum_{j=1}^{[\frac{n}{2}]} c_{j,n} \frac{\zeta(2j+1)}{\pi^{2j}}$$

$$m(1-x_1,1-x_2) = \frac{7}{2\pi^2}\zeta(3)$$

$$m(1-x_1,1-x_2,1-x_3) = \frac{9}{2\pi^2}\zeta(3)$$

$$m(1-x_1,1-x_2,1-x_3,1-x_4) = -\frac{93}{2\pi^4}\zeta(5) + \frac{9}{\pi^2}\zeta(3)$$

Can be also computed using regulators.

$|P(x)|$ is montononous when $0 \leq \arg x \leq \pi$.

In this case, $|P(x)| = 2\left|\sin \frac{\arg x}{2}\right|$.

$$m(P(x_1), \dots, P(x_n)) = \frac{n!}{(\pi \mathrm{i})^n} \int_{0 \leq \arg x_n \leq \dots \leq \arg x_1 \leq \pi} \eta(P(x_1), x_1, \dots, x_n)$$

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$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_n}{1+x_n}\right)=\sum_{j=1}^{\left[\frac{n}{2}\right]} c'_{j,n}\frac{\zeta(2j+1)}{\pi^{2j}}$$

$$m\left(\frac{1-x_1}{1+x_1},\frac{1-x_2}{1+x_2}\right)=\frac{7}{\pi^2}\zeta(3)$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_3}{1+x_3}\right)=\frac{21}{2\pi^2}\zeta(3)$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_4}{1+x_4}\right)=-\frac{93}{\pi^4}\zeta(5)+\frac{21}{\pi^2}\zeta(3)$$

$m(1+x_1-x_1^{-1},\dots,1+x_n-x_n^{-1})=$ combination of polylogarithms.

$$m(1+x_1-x_1^{-1})=-\log(\varphi),$$

$$m(1+x_1-x_1^{-1},1+x_2-x_2^{-1})$$
$$=\frac{1}{\pi^2}\,\mathrm{Re}(\mathrm{Li}_3(\varphi^2)-\mathrm{Li}_3(-\varphi^2)+\mathrm{Li}_3(\varphi^{-2})-\mathrm{Li}_3(-\varphi^{-2}))$$

for $\varphi=\frac{-1+\sqrt{5}}{2}$.