

Some aspects of Mahler Measure

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Mahler measure and Lehmer's question

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\lim_{n \rightarrow \infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

$$\omega \quad m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|$$

Kronecker's Lemma:

$$P \in \mathbb{Z}[x], P \neq 0,$$

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

Lehmer (1933)

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \\ = \log(1.176280818\dots) = 0.162357612\dots$$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

there exists $C > 0$, for all $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

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Is the polynomial above the best possible?

Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |e^{2\pi i\theta} - \alpha| d\theta = \log^+ |\alpha|$$

↳ recovers one-variable case.

Properties

- $m(P \cdot Q) = m(P) + m(Q)$
- $m(P) \geq 0$ if P has integral coefficients.
- α algebraic number, and P_α minimal polynomial over \mathbb{Q} ,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}] h(\alpha)$$

where h is the logarithmic Weil height.

- Boyd & Lawton : $P \in \mathbb{C}[x_1, \dots, x_n]$

$$\begin{aligned} \circ \quad & \lim_{k_2 \rightarrow \infty} \dots \lim_{k_n \rightarrow \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) \\ & = m(P(x_1, \dots, x_n)) \end{aligned}$$

Jensen's formula \longrightarrow simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3)$$

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \pmod{3} \\ -1 & n \equiv -1 \pmod{3} \\ 0 & n \equiv 0 \pmod{3} \end{cases}$$

∞

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Boyd & Rodriguez-Villegas (1997)

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - k\right) \stackrel{?}{=} \frac{L'(E_k, 0)}{B_k} \quad k \in \mathbb{N}$$

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\right) = 2L'(\chi_{-4}, -1)$$

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\sqrt{2}\right) = L'(A, 0)$$

$$A : y^2 = x^3 - 44x + 112$$

More examples in several variables

L (2003)

$$\pi^n m \left(1 + \left(\frac{1 - x_1}{1 + x_1} \right) \cdots \left(\frac{1 - x_n}{1 + x_n} \right) z \right)$$

= combination of $\zeta(\text{odd}) / L(\chi_{-4}, \text{even})$

$$\pi^n m \left(1 + x + \left(\frac{1 - x_1}{1 + x_1} \right) \cdots \left(\frac{1 - x_n}{1 + x_n} \right) (1 + y)z \right)$$

= combination of $\zeta(\text{odd}) / L(\chi_{-4}, \text{even})$, polylogarithms

$$\pi^n m \left(1 + \left(\frac{1 - x_1}{1 + x_1} \right) \cdots \left(\frac{1 - x_n}{1 + x_n} \right) x + \left(1 - \left(\frac{1 - x_1}{1 + x_1} \right) \cdots \left(\frac{1 - x_n}{1 + x_n} \right) \right) y \right)$$

= combination of $\zeta(\text{odd})$

Examples

$$\begin{aligned} \pi^3 m \left(1 + \left(\frac{1-x_1}{1+x_1} \right) \left(\frac{1-x_2}{1+x_2} \right) \left(\frac{1-x_3}{1+x_3} \right) z \right) \\ = 24L(\chi_{-4}, 4) + \pi^2 L(\chi_{-4}, 2) \end{aligned}$$

$$\pi^4 m \left(1 + \left(\frac{1-x_1}{1+x_1} \right) \cdots \left(\frac{1-x_4}{1+x_4} \right) z \right) = 62\zeta(5) + \frac{14}{3}\pi^2\zeta(3)$$

$$\pi^4 m \left(1 + x + \left(\frac{1-x_1}{1+x_1} \right) \left(\frac{1-x_2}{1+x_2} \right) (1+y)z \right) = 93\zeta(5)$$

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F number field)
- L-function ($L_X = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)

- Regulator map

$r : K \rightarrow$ smooth differential forms

($r = \log |\cdot|$)

Conjecture: special value of $L_X \sim_{\mathbb{Q}^*} \int_{\gamma} r(\xi)$

(E.g. Dirichlet class number formula, F real quadratic,

$\zeta'_F(0) \sim_{\mathbb{Q}^*} \log |\epsilon| \quad \epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) :

$$P(x, y) = y + x - 1 \quad C = \{P(x, y) = 0\}$$

$$m(P) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |y + x - 1| \frac{dx dy}{x y}$$

by Jensen's equality:

$$= \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{dx}{x}$$

$$= \frac{1}{2\pi i} \int_{\gamma} \log |y| \frac{dx}{x} = -\frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

where

$$\gamma = C \cap \{|x| = 1, |y| \geq 1\}$$

and

$$\eta(x, y) = \log |x| d \arg y - \log |y| d \arg x$$

- $\eta(x, y) = -\eta(y, x)$
- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$

Theorem 1

$$\eta(x, 1 - x) = dD(x)$$

Bloch–Wigner dilogarithm:

$$D(x) := \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1 - x) \log |x|$$

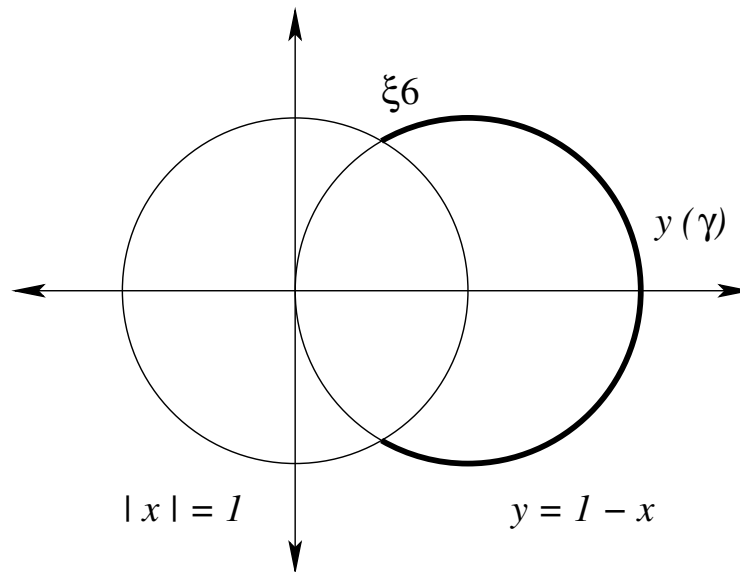
$$\operatorname{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad |x| < 1$$

Use Stokes Theorem:

$$m(P) = -\frac{1}{2\pi} D(\partial\gamma)$$

$$x = e^{2\pi i\theta} \quad y(\gamma(\theta)) = 1 - e^{2\pi i\theta}, \quad \theta \in [1/6; 5/6]$$

$$\partial\gamma = [\bar{\xi}_6] - [\xi_6]$$



$$17 \quad 2\pi m(x + y + 1) = D(\xi_6) - D(\bar{\xi}_6) = 2D(\xi_6) = \frac{3\sqrt{3}}{2}L(\chi_{-3}, 2)$$

In general, $P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

Need $\{x, y\} = 0$ in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$.

$$x \wedge y = \sum_j r_j z_j \wedge (1 - z_j)$$

in $\Lambda^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$, then

$$\int_{\gamma} \eta(x, y) = \sum r_j D(z_j)|_{\partial\gamma}$$

Big picture

$$\dots \rightarrow (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \rightarrow K_2(C, \partial\gamma) \rightarrow K_2(C) \rightarrow \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x, y)$ is exact, then $\{x, y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

\rightsquigarrow dilogarithms, zeta function

- $\partial\gamma = \emptyset$, then $\{x, y\} \in K_2(C)$. We have $\eta(x, y)$ is not exact.

\rightsquigarrow L -series of a curve

We may get combinations of both situations.

The three-variable case

$$P(x, y, z) = (1 - x) + (1 - y)z \quad S = \{P(x, y, z) = 0\}$$

$$\begin{aligned} m(P) &= m(1 - y) + \frac{1}{(2\pi i)^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1 - x}{1 - y} \right| \frac{dx dy dz}{x y z} \\ &= \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1 - x}{1 - y} \right| \frac{dx dy}{x y} = -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{dx dy}{x y} \\ &= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z) \end{aligned}$$

$$20 \quad \Gamma = S \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$\begin{aligned}
\eta(x, y, z) = & \log |x| \left(\frac{1}{3} d \log |y| d \log |z| - d \arg y d \arg z \right) \\
& + \log |y| \left(\frac{1}{3} d \log |z| d \log |x| - d \arg z d \arg x \right) \\
& + \log |z| \left(\frac{1}{3} d \log |x| d \log |y| - d \arg x d \arg y \right)
\end{aligned}$$

Theorem 2

$$\eta(x, 1 - x, y) = d \omega(x, y)$$

where

$$\omega(x, y) = -D(x) d \arg y$$

$$+ \frac{1}{3} \log |y| (\log |1 - x| d \log |x| - \log |x| d \log |1 - x|)$$

$$\eta(x, y, z) = -\eta(x, 1 - x, y) - \eta(y, 1 - y, x)$$

Maillot: if $P \in \mathbb{Q}[x, y, z]$,

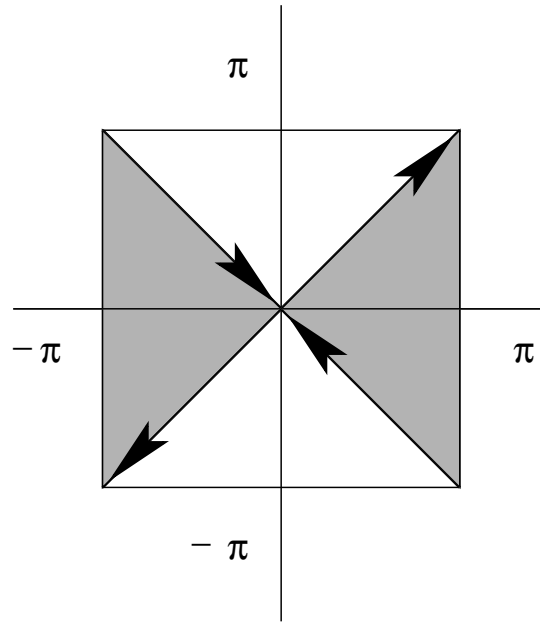
$$\partial\Gamma = \gamma = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

ω defined in

$$C = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$22 \quad \frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1 \quad C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x) + (1-y)z) = \frac{1}{4\pi^2} \int_{\gamma} \omega(x, y) + \omega(y, x)$$

Theorem 3

$$\omega(x, x) = dP_3(x)$$

$$= \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2} \zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z)$$

Need $\{x, y, z\} = 0$ in $K_3^M(\mathbb{C}(S)) \otimes \mathbb{Q}$.

$$x \wedge y \wedge z = \sum r_i x_i \wedge (1 - x_i) \wedge y_i$$

in $\wedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$, then

$$24 \quad \int_{\Gamma} \eta(x, y, z) = \sum r_i \int_{\Gamma} \eta(x_i, 1 - x_i, y_i) = \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

Let

$$R_2(x, y) = [x] + [y] + [1 - xy] + \left[\frac{1 - x}{1 - xy} \right] + \left[\frac{1 - y}{1 - xy} \right] = 0$$

in $\mathbb{Z}[\mathbb{P}_{\mathbb{C}(C)}^1]$.

F field,

$$B_2(F) := \mathbb{Z}[\mathbb{P}_F^1] / \{[0], [\infty], R_2(x, y)\}$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

25 in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.

Goncharov: zero element in $K_4^{[1]}(\mathbb{Q}(C))$.

Then

$$\int_{\gamma} \omega(x, y) = \sum r_i P_3(x_i)|_{\partial\gamma}$$

Big picture II

$$\dots \rightarrow K_4(\partial\Gamma) \rightarrow K_3(S, \partial\Gamma) \rightarrow K_3(S) \rightarrow \dots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\dots \rightarrow (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \rightarrow K_4(C, \partial\gamma) \rightarrow K_4(C) \rightarrow \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.

Examples from the world of resultants

D'Andrea & L (2003).

- $$m(\text{Res}_{\{0,m,n\}}) = m(\text{Res}_t(x + yt^m + t^n, z + wt^m + t^n))$$

$$= \frac{2}{\pi^2}(-mP_3(\varphi^n) - nP_3(-\varphi^m) + mP_3(\phi^n) + nP_3(\phi^m))$$

$$0 \leq \varphi \leq 1 \quad \text{root of } x^n + x^{n-m} - 1 = 0$$

$$1 \leq \phi \quad \text{root of } x^n - x^{n-m} - 1 = 0$$

- $$m(\text{Res}_{\{(0,0),(1,0),(0,1)\}}) = m \left(\begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix} \right)$$

$$= m((1-x)(1-y) - (1-z)(1-w)) = \frac{9\zeta(3)}{2\pi^2}$$

