Introduction to Mahler Measure

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1. Definition of Mahler Measure and Lehmer's question

Looking for large primes, Pierce [18] proposed the following in 1918:
Consider $P \in \mathbb{Z}[x]$ monic, and write

$$P(x) = \prod_i (x - \alpha_i)$$

then, we look at

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

Since the α_i are integers and by applying Galois theory, it is easy to see that $\Delta_n \in \mathbb{Z}$. Note that if $P = x - 2$, we get the sequence $\Delta_n = 2^n - 1$. The idea is to look for primes among the factors of Δ_n. The prime divisors of such integers must satify some congruence conditions that are quite restrictive, hence they are easier to factorize than a randomly given number. Moreover, one can show that $\Delta_m | \Delta_n$ if $m|n$. Then we may look at the numbers

$$\frac{\Delta_p}{\Delta_1} \qquad p \text{ prime}$$

In order to minimize the number of trial divisions, the sequence Δ_n should grow slowly. Lehmer [15] studied $\frac{\Delta_{n+1}}{\Delta_n}$, observed that

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

and suggested the following definition:

Definition 1 *Given $P \in \mathbb{C}[x]$, such that*

$$P(x) = a \prod_i (x - \alpha_i)$$

define the Mahler measure [2] of P as

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\} \tag{1}$$

The logarithmic Mahler measure is defined as

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i| \tag{2}$$

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[2] The name Mahler came later after the person who successfully extended this definition to the several-variable case.

When does $M(P)=1$ for $P\in\mathbb{Z}[x]$? We have

Lemma 2 *(Kronecker) Let $P=\prod_i(x-\alpha_i)\in\mathbb{Z}[x]$, if $|\alpha_i|\leq 1$, then the α_i are zero or roots of the unity.*

By Kronecker's Lemma, $P\in\mathbb{Z}[x]$, $P\neq 0$, then $M(P)=1$ if and only if P is the product of powers of x and cyclotomic polynomials. This statement characterizes integral polynomials whose Mahler measure is 1.

Lehmer found the example

$$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)=\log(1.176280818\ldots)=0.162357612\ldots$$

and asked the following (Lehmer's question, 1933):

Is there a constant $C>1$ such that for every polynomial $P\in\mathbb{Z}[x]$ with $M(P)>1$, then $M(P)\geq C$?

Lehmer's question remains open nowadays. His 10-degree polynomial remains the best possible result.

We will list some important results in the direction of solving Lehmer's question.

Theorem 3 *(Smyth [21], 1971) If $P\in\mathbb{Z}[x]$ is monic, irreducible, $P\neq\pm P^*$ (nonreciprocal), then*

$$M(P)\geq M(x^3-x-1)=\theta=1.324717\ldots \tag{3}$$

Corollary 4 *If $P\in\mathbb{Z}[x]$ is monic, irreducible, and of odd degree, then*

$$M(P)\geq\theta$$

Theorem 5 *(Dobrowolski [7], 1979) If $P\in\mathbb{Z}[x]$ is monic, irreducible and noncyclotomic of degree d, then*

$$M(P)\geq 1+c\left(\frac{\log\log d}{\log d}\right)^3 \tag{4}$$

where c is an absolute positive constant.

Theorem 6 *(Schinzel) If $P\in\mathbb{Z}[x]$ is monic of degree d having all real roots and satisfies $P(1)P(-1)\neq 0$ and $|P(0)|=1$, then*

$$M(P)\geq\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{d}{2}} \tag{5}$$

Theorem 7 *(Borwein, Hare, Mossinghoff [1], 2002) If $P\in\mathbb{Z}[x]$ is monic, nonreciprocal with odd coefficients, then*

$$M(P)\geq M(x^2-x-1)=\frac{1+\sqrt{5}}{2} \tag{6}$$

Theorem 8 *(Bombieri, Vaaler [3], 1983) Let $P\in\mathbb{Z}[x]$ with $M(P)<2$, then P divides a polynomial $Q\in\mathbb{Z}[x]$ whose coefficients belong to $\{-1,0,1\}$.*

2. Mahler Measure in several variables

Definition 9 *For $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the logarithmic Mahler measure is defined by*

$$m(P) := \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \tag{7}$$

$$= \frac{1}{(2\pi\mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \tag{8}$$

It is possible to prove that this integral is not singular and that $m(P)$ always exists.

Because of Jensen's formula [3]:

$$\int_0^1 \log |\mathrm{e}^{2\pi\mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha| \tag{9}$$

we recover the formula for the one-variable case.

Let us mention some elementary properties.

Proposition 10 *For $P, Q \in \mathbb{C}[x_1, \dots, x_n]$*

$$m(P \cdot Q) = m(P) + m(Q) \tag{10}$$

Proposition 11 *Let $P \in \mathbb{C}[x_1, \dots, x_n]$ such that $a_{m_1, \dots, m_n}$ is the coefficient of $x_1^{m_1} \dots x_n^{m_n}$ and P has degree d_i in x_i. Then*

$$|\, a_{m_1, \dots, m_n}| \leq \binom{d_1}{m_1} \dots \binom{d_n}{m_n} M(P) \tag{11}$$

$$M(P) \leq L(P) \leq 2^{d_1 + \dots + d_n} M(P) \tag{12}$$

where $L(P)$ is the length of the polynomial, the sum of the absolute values of the coefficients.

It is also true that $m(P) \geq 0$ if P has integral coefficients.

Let us also mention the following amazing result:

Theorem 12 *(Boyd [2],1981, Lawton [14], 1983) For $P \in \mathbb{C}[x_1, \dots, x_n]$*

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, \dots, x_n)) \tag{13}$$

Because of the above theorem, Lehmer's question in the several-variable case reduces to the one-variable case.

The formula for the one-variable case tells us some information about the nature of the values that Mahler measure can reach. For instance, the Mahler measure of a polynomial in one variable with integer coefficients must be an algebraic number.

It is natural, then, to wonder what happens with the several-variable case. Is there any simple formula, besides the integral? Unfortunately, this case is much more complicated and we only have some paticular examples. On the other hand, the values are very interesting.

3. Polylogarithms and L-functions

Before going into the several variable case, let us recall the definitions of polylogarithms and L-functions. More about polylogarithms can be found in Goncharov [10], [11], [12]

[3] $\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

Definition 13 *Polylogarithms are defined as the power series*

$$\mathrm{Li}_k(x) := \sum_{0<n} \frac{x^n}{n^k} \tag{14}$$

which are convergent for $|x| < 1$ *and* $k \geq 2$*. The number* k *is called the weight of the polylogarithm.*

We see that for $x = 1$, we recover the Riemann zeta function in k.

Polylogarithms have analytic continuation to $\mathbb{C} \setminus [1, \infty)$ through the integral

$$\mathrm{Li}_k(x) := -\int_{0 \leq t_1 \leq \ldots \leq t_k \leq 1} \frac{\mathrm{d}t_1}{t_1 - \frac{1}{x}} \frac{\mathrm{d}t_2}{t_2} \cdots \frac{\mathrm{d}t_k}{t_k} \tag{15}$$

There are modified versions of these functions which are analytic in larger sets, like the Bloch-Wigner dilogarithm,

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| \arg(1-z) \qquad z \in \mathbb{C} \setminus [1, \infty) \tag{16}$$

which can be extended as a real analytic function in $\mathbb{C} \setminus \{0, 1\}$ and continuous in $\mathbb{C}$. For more information about this function see Zagier [26].

Definition 14 *The L-series in the character* χ *is defined to be the function*

$$\mathrm{L}(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

We are going to use the real characters

$$\chi_{-f}(n) := \left(\frac{-f}{n}\right)$$

where the symbol in the right is Kronecker's extension to Jacobi's symbol. In particular,

$$\begin{aligned}
\chi_{-3}(n) &= \left(\frac{n}{3}\right) \\
\chi_{-4}(n) &= \begin{cases} \left(\frac{-1}{n}\right) & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases} \\
\chi_{-8}(n) &= \begin{cases} \left(\frac{-2}{n}\right) & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}
\end{aligned}$$

4. Examples for two and three variables

- The simplest example with two variables is due to Smyth [22], 1981:

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1) \tag{17}$$

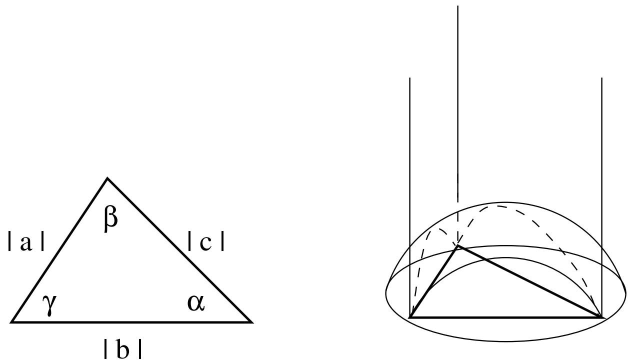

Figure 1: The main term in Caissaigne – Maillot formula is the volume of the ideal hyperbolic tetrahedron over the triangle.

- The above example can be extended to three variables, also due to Smyth:

$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) \tag{18}$$

More generally, Smyth [23] in 2002 computed the Mahler measure of $a + b\frac{1}{x} + cy + (a + bx + cy)z$, for a, b and c any real numbers, from which the above result can be deduced. Another corollary is the formula

$$m\left(1+\frac{1}{x}+y+(1+x+y)z\right) = \frac{14}{3\pi^2}\zeta(3) \tag{19}$$

- The dilogarithm occurs as the Mahler measure of certain polynomials in two variables. Perhaps the simplest example is Caissaigne – Maillot's in [16]: for $a, b, c \in \mathbb{C}$,

$$\pi m(a+bx+cy) = \begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{\mathrm{i}\gamma}\right) + \alpha \log|a| + \beta \log|b| + \gamma \log|c| & \triangle \\ \\ \pi \log \max\{|a|,|b|,|c|\} & \text{not } \triangle \end{cases} \tag{20}$$

Here $\triangle$ stands for the statement that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle, and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$, and $|c|$ respectively. See Figure 1.

The term with the dilogarithm can be interpreted as the volume of the ideal hyperbolic tetrahedron which has the triangle as basis and the fourth vertex is infinity (see [17], [27]).

Connections with hyperbolic geometry do not end here. Boyd [5] has found relations between Mahler measures of A-polynomials of knots and the hyperbolic volumes of their complements. These examples have been studied by Boyd and Rodriguez-Villegas.

- Boyd and Rodriguez-Villegas [6] studied polynomials of the form $P(x, y) = p(x)y - q(x)$ where p, q are cyclotomic and relatively prime. The Mahler measure turns out to be some combinations of dilogarithms which can also be interpreted in terms of $\zeta_F(2)$ for certain field F.

- Smyth [22] in 1981 found the approximation

$$m(x_1 + x_2 + \ldots + x_n) \\ = \frac{1}{2}\log n - \frac{1}{2}\gamma + O\left(\frac{\log N}{N}\right) \quad (21)$$

as $n \to \infty$, where γ is Euler constant. [4].

- Vandervelde [25] in 2002, studied the example of $axy + bx + cy + d$. He developed a general formula for this case. Some particular cases are:

$$m(1 + x + y + \mathrm{i}xy) = \frac{\sqrt{2}}{\pi}\mathrm{L}(\chi_{-8}, 2) = \frac{1}{4}\mathrm{L}'(\chi_{-8}, -1) \quad (22)$$

$$m(1 + x + y + \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}xy) = \frac{4\sqrt{2}}{15\pi}\mathrm{L}(\chi_{-8}, 2) = \frac{1}{15}\mathrm{L}'(\chi_{-8}, -1) \quad (23)$$

- Other examples with special values of L-series have been discovered by Ray, Smyth and others

$$m(1 + x + y - xy) = \frac{2}{\pi}\mathrm{L}(\chi_{-4}, 2) = \mathrm{L}'(\chi_{-4}, -1) \quad (24)$$

$$m(1 + x + x^2 + y) = \frac{2}{3}\mathrm{L}'(\chi_{-4}, -1) \quad (25)$$

$$m(1 + x + y + x^2y) = \frac{3}{2}\mathrm{L}'(\chi_{-3}, -1) \quad (26)$$

- Boyd [4] has computed numerically several examples involving L-series of elliptic curves, some of them were proved by Rodriguez-Villegas [20]. These formulas are supported by other conjectures and have been explained by Deninger [8] and Rodriguez-Villegas [20]. For instance

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) \stackrel{?}{=} \mathrm{L}'(E, 0) \quad (27)$$

where E is the elliptic curve of conductor 15 which is the projective closure of the curve $x + \frac{1}{x} + y + \frac{1}{y} + 1 = 0$, and $\mathrm{L}(E, s)$ is the L-function of E.

- Vandervelde also generalized Smyth's example. For $a \in \mathbb{R}_{>0}$,

$$\pi^2 m(1 + x + ay + az) = \begin{cases} 2(\mathrm{Li}_3(a) - \mathrm{Li}_3(-a)) & \text{if } a \leq 1 \\ \pi^2 \log a + 2\left(\mathrm{Li}_3\left(\frac{1}{a}\right) - \mathrm{Li}_3\left(\frac{-1}{a}\right)\right) & \text{if } a \geq 1 \end{cases} \quad (28)$$

This can be also proved by adapting the elementary proof given in Boyd [2]. For $0 \leq a \leq 1$:

$$\pi^2 m(1 + x + ay + az) = \pi^2 m(1 + ay + x(1 + aw)) = \pi^2 m\left(\frac{1 + ay}{1 + aw} + x\right)$$

[4] $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.577215664\ldots$

$$= \int_0^\pi \int_0^\pi \log^+ \left| \frac{1+a\mathrm{e}^{\mathrm{i}t}}{1+a\mathrm{e}^{\mathrm{i}s}} \right| \mathrm{d}s\,\mathrm{d}t = \int_{0\le t\le s\le \pi} \log|1+a\mathrm{e}^{\mathrm{i}t}| - \log|1+a\mathrm{e}^{\mathrm{i}s}|\,\mathrm{d}s\,\mathrm{d}t$$

$$= \int_0^\pi (\pi - t)\,\log|1+a\mathrm{e}^{\mathrm{i}t}|\mathrm{d}t - \int_0^\pi s\,\log|1+a\mathrm{e}^{\mathrm{i}s}|\mathrm{d}s = -2\int_0^\pi t\,\log|1+a\mathrm{e}^{\mathrm{i}t}|\mathrm{d}t$$

(here we have used that $0 \le a \le 1$, and Jensen's formula).

Now use that

$$\log|1+a\mathrm{e}^{\mathrm{i}t}| = \mathrm{Re}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, a^n \mathrm{e}^{\mathrm{i}nt} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(n\,t)}{n} a^n \tag{29}$$

and apply integration by parts,

$$\pi^2 m(1+x+ay+az) = -2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(n\,t)}{n^2}\, a^n t \Big|_0^\pi$$

$$+2\int_0^\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(n\,t)}{n^2}\, a^n \mathrm{d}t = 4\sum_{n=1\,(\mathrm{odd})}^{\infty} \frac{a^n}{n^3} = 2(\mathrm{Li}_3(a) - \mathrm{Li}_3(-a))$$

When $a \ge 1$, use that

$$m(1+x+ay+az) = \log a + m\left(\frac{1}{a} + \frac{x}{a} + y + z\right)$$

5. Examples of higher weight

- Smyth [24] in 2003, found for $n \ge 3$,

$$m((x_1 + x_1^{-1})\dots(x_{n-2} + x_{n-2}^{-1}) + 2^{n-3}(x_{n-1} + x_n))$$

$$= (n-3)\log 2 + \left(\frac{2}{\pi}\right)^{n-1} \cdot {}_{n+1}F_n\left(\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \dots, 1\right\}, \left\{\frac{3}{2}, \dots, \frac{3}{2}\right\}, 1\right) \tag{30}$$

where

$${}_rF_m(\{a_1, \dots, a_r\}, \{b_1, \dots, b_m\}, z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_r)_k}{(b_1)_k \dots (b_m)_k} \frac{z^k}{k!}$$

is a hypergeometric function and $(a)_k = a(a+1)\dots(a+k-1)$.

- We have obtained (see [13]) examples of polynomials in several variables whose Mahler measures depend on polylogarithms, special values of the Riemann zeta function and special values of a certain L-series. See Table 1 (the cases with more than three variables are new).

In addition to these formulas, we have proved

$\pi^2 m\left(\frac{1-x_1}{1+x_1}+\alpha\frac{1-y_1}{1+y_1}z\right)$	$7\,\zeta(3)$
$\pi^4 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}z\right)$	$62\zeta(5)+28\zeta(2)\zeta(3)$
$\pi^6 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}\frac{1-x_3}{1+x_3}+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}\frac{1-y_3}{1+y_3}z\right)$	$381\zeta(7)+372\zeta(2)\zeta(5)+336\zeta(4)\zeta(3)$
$\pi m\left((1+y)+\alpha(1-y)z\right)$	$2\mathrm{L}(\chi_{-4},2)$
$\pi^3 m\left(\frac{1-x_1}{1+x_1}(1+y)+\alpha\frac{1-y_1}{1+y_1}(1-y)z\right)$	$24\mathrm{L}(\chi_{-4},4)+6\zeta(2)\mathrm{L}(\chi_{-4},2)$
$\pi^5 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}(1+y)+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}(1-y)z\right)$	$160\mathrm{L}(\chi_{-4},6)+120\zeta(2)\mathrm{L}(\chi_{-4},4)+\frac{135}{2}\zeta(4)\mathrm{L}(\chi_{-4},2)$
$\pi^2 m\left((1+x)+\alpha(1+y)z\right)$	$\frac{7}{2}\zeta(3)$
$\pi^4 m\left(\frac{1-x_1}{1+x_1}(1+x)+\alpha\frac{1-y_1}{1+y_1}(1+y)z\right)$	$93\zeta(5)$
$\pi^6 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}(1+x)+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}(1+y)z\right)$	$\frac{15\cdot 127}{2}\zeta(7)+186\zeta(2)\zeta(5)$
$\pi^8 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}\frac{1-x_3}{1+x_3}(1+x)+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}\frac{1-y_3}{1+y_3}(1+y)z\right)$	$14\cdot 511\zeta(9)+30\cdot 127\zeta(2)\zeta(7)+48\cdot 31\zeta(4)\zeta(5)$
$\pi^3 m\left((1+w)(1+x)+\alpha(1-w)(1+y)z\right)$	$12\zeta(2)\mathrm{L}(\chi_{-4},2)+2\mathrm{i}\mathcal{L}_{3,1}(\mathrm{i},\mathrm{i})$
$\pi^5 m\left(\frac{1-x_1}{1+x_1}(1+w)(1+x)+\frac{1-y_1}{1+y_1}(1-w)(1+y)z\right)$	$144\zeta(2)\mathrm{L}(\chi_{-4},4)+90\zeta(4)\mathrm{L}(\chi_{-4},2)+16\mathrm{i}\mathcal{L}_{3,3}(\mathrm{i},\mathrm{i})+24\mathrm{i}\zeta(2)\mathcal{L}_{3,1}(\mathrm{i},\mathrm{i})$
$\pi^2 m((1+w)(1+y)+(1-w)(x-y))$	$\frac{7}{2}\zeta(3)+\frac{\pi^2}{2}\log 2$

Table 1: Here α is a nonzero complex number. The second column indicates the value of the first column for $\alpha=1$.

Theorem 15 *The Mahler measure of an n-variable polynomial in the first or in the third families is a homogeneous (of weight n) linear combination (with coefficients in $\mathbb{Q}[\pi]$) of special (odd) values of the Riemann zeta function. Analogously, the Mahler measure of an n-variable polynomial in the second family is a homogeneous (of weight n) linear combination (with coefficients in $\mathbb{Q}[\pi]$) of special (even) values of the L-series in the Dirichlet character of conductor 4.*

The idea behind these computations is the following. Let $P_\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ whose coefficients depend polynomially on a parameter $\alpha \in \mathbb{C}$. We replace α by $\alpha\frac{1-y}{1+y}$ and obtain a polynomial $\tilde{P}_\alpha \in \mathbb{C}[x_1, \ldots, x_n, y]$. The Mahler measure of the second polynomial is a certain integral of the Mahler measure of the first polynomial.

$$m(\tilde{P}_\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} m\left(P_{\alpha\frac{1-y}{1+y}}\right) \frac{\mathrm{d}y}{y}$$

The proof of these results uses multiple polylogarithms, which are several-variable versions of the polylogarithms and have analogous analytic continuations.

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