

# Some aspects of Mahler Measure

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## 1. Mahler measure

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ , the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

By Jensen's formula:

$$\int_0^1 \log |e^{2\pi i\theta} - \alpha| d\theta = \log^+ |\alpha|$$

we obtain

$$m(P) = \log |a_d| + \sum_{n=1}^d \log^+ |\alpha_n|$$

for

$$P(x) = a_d \prod_{n=1}^d (x - \alpha_n) \in \mathbb{C}[x]$$

Jensen's formula  $\longrightarrow$  simple expression in one-variable case.

Several-variable case?

## 2. Examples in several variables

Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3)$$

Boyd & Rodriguez-Villegas (1997)

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - k\right) \stackrel{?}{=} \frac{L'(E_k, 0)}{B_k} \quad k \in \mathbb{N}$$

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\right) = 2L'(\chi_{-4}, -1)$$

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\sqrt{2}\right) = L'(A, 0)$$

$$A : y^2 = x^3 - 44x + 112$$

### 3. Polylogarithms

The  $k$ th polylogarithm is

$$\text{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to  $\mathbb{C} \setminus [1, \infty)$ .

Zagier:

$$P_k(x) := \text{Re}_k \left( \sum_{j=0}^k \frac{2^j B_j}{j!} (\log |x|)^j \text{Li}_{k-j}(x) \right)$$

$B_j$  is  $j$ th Bernoulli number,  $\text{Li}_0(x) \equiv -\frac{1}{2}$ ,

$\text{Re}_k = \text{Re}$  or  $\text{Im}$  if  $k$  is odd or even.

One-valued, real analytic in  $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ ,  
continuous in  $\mathbb{P}^1(\mathbb{C})$ .

$P_k$  satisfies lots of functional equations

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1} P_k(x) \quad P_k(\bar{x}) = (-1)^{k-1} P_k(x)$$

Bloch–Wigner dilogarithm ( $k = 2$ )

$$D(x) := \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1 - x) \log |x|$$

Five-term relation

$$D(x) + D(1 - xy) + D(y) + D\left(\frac{1 - y}{1 - xy}\right) + D\left(\frac{1 - x}{1 - xy}\right) = 0$$

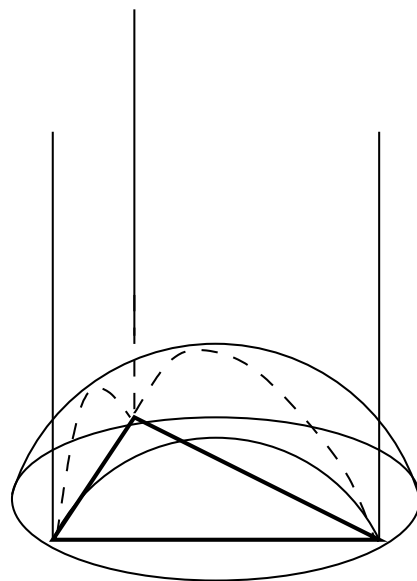
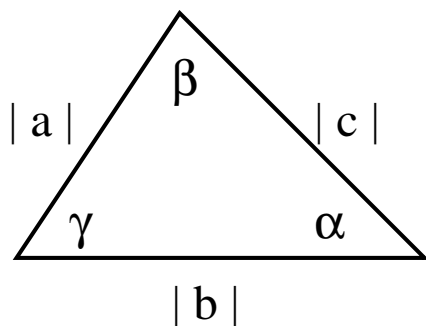
## 4. Mahler measure and hyperbolic volumes

Cassaigne – Maillot (2000) for  $a, b, c \in \mathbb{C}^*$ ,

$$\pi m(a + bx + cy)$$

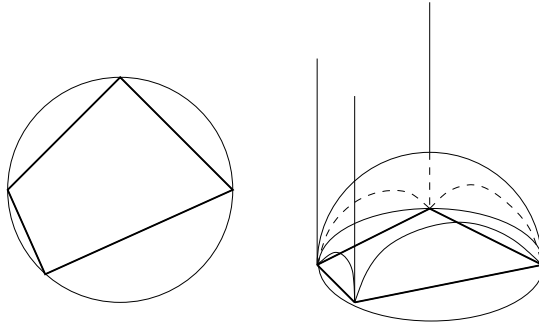
$$= \begin{cases} D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| & \triangle \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases}$$

Ideal tetrahedron:



- Vandervelde (2003)

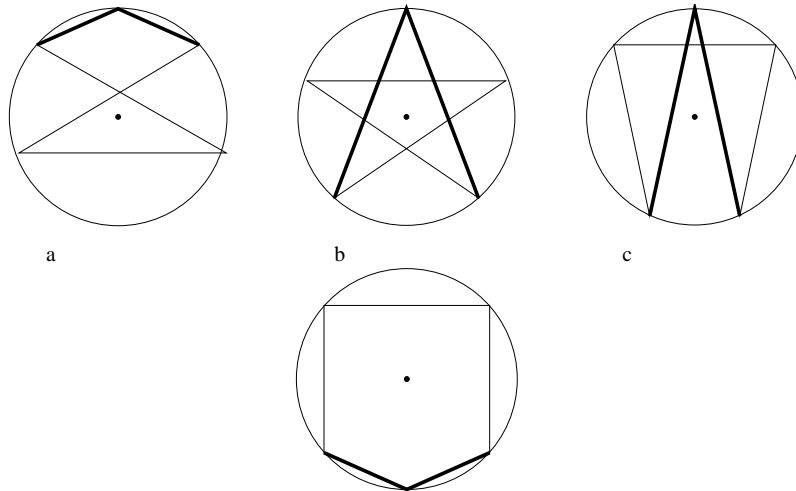
$$y = \frac{bx + d}{ax + c} \quad \text{quadrilateral}$$



- L(2004)

$$y = \frac{x^n - 1}{t(x^m - 1)} = \frac{x^{n-1} + \dots + 1}{t(x^{m-1} + \dots + 1)} \quad \text{polyhedral}$$

+ relation to  $A$ -polynomial





## 5. More examples in several variables

L (2003)

$$\pi^n m \left( 1 + \left( \frac{1 - x_1}{1 + x_1} \right) \cdots \left( \frac{1 - x_n}{1 + x_n} \right) z \right)$$

= combination of  $\zeta(\text{odd}) / L(\chi_{-4}, \text{even})$

$$\pi^n m \left( 1 + x + \left( \frac{1 - x_1}{1 + x_1} \right) \cdots \left( \frac{1 - x_n}{1 + x_n} \right) (1 + y)z \right)$$

= combination of  $\zeta(\text{odd}) / L(\chi_{-4}, \text{even})$ ,  
polylogarithms

$$\pi^n m \left( 1 + \left( \frac{1 - x_1}{1 + x_1} \right) \cdots \left( \frac{1 - x_n}{1 + x_n} \right) x \right)$$

$$+ \left( 1 - \left( \frac{1 - x_1}{1 + x_1} \right) \cdots \left( \frac{1 - x_n}{1 + x_n} \right) \right) y$$

= combination of  $\zeta(\text{odd})$

## Examples

$$\begin{aligned} \pi^3 m \left( 1 + \left( \frac{1-x_1}{1+x_1} \right) \left( \frac{1-x_2}{1+x_2} \right) \left( \frac{1-x_3}{1+x_3} \right) z \right) \\ = 24L(\chi_{-4}, 4) + \pi^2 L(\chi_{-4}, 2) \end{aligned}$$

$$\begin{aligned} \pi^4 m \left( 1 + \left( \frac{1-x_1}{1+x_1} \right) \cdots \left( \frac{1-x_4}{1+x_4} \right) z \right) \\ = 62\zeta(5) + \frac{14}{3}\pi^2\zeta(3) \end{aligned}$$

$$\pi^4 m \left( 1 + x + \left( \frac{1-x_1}{1+x_1} \right) \left( \frac{1-x_2}{1+x_2} \right) (1+y)z \right) = 93\zeta(5)$$

## 6. Examples from the world of resultants

D'Andrea & L (2003).

- $m(\text{Res}_{\{0,m,n\}})$ 

$$= m(\text{Res}_t(x + yt^m + t^n, z + wt^m + t^n)) =$$

$$\frac{2}{\pi^2}(-mP_3(\varphi^n) - nP_3(-\varphi^m) + mP_3(\phi^n) + nP_3(\phi^m))$$

$0 \leq \varphi \leq 1$  root of  $x^n + x^{n-m} - 1 = 0$   
 $1 \leq \phi$  root of  $x^n - x^{n-m} - 1 = 0$
- $m(\text{Res}_{\{(0,0),(1,0),(0,1)\}}) = m \left( \begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix} \right)$ 

$$= m((1-x)(1-y) - (1-z)(1-w)) = \frac{9\zeta(3)}{2\pi^2}$$

## 7. Philosophy of Beilinson's conjectures

Global information from local information  
through L-functions

- Arithmetic-geometric object  $X$
- L-function
- Finitely-generated abelian group  $K$
- Regulator map  $\text{reg} : K \rightarrow \mathbb{R}$

$$(K \text{ rank } 1) \quad L'_X(0) \sim_{\mathbb{Q}^*} \text{reg}(\xi)$$

## 8. An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) :  $P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

$$\eta(x, 1 - x) = dD(x)$$

Need  $\{x, y\} = 0$  in  $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$ .

$$x \wedge y = \sum_j r_j z_j \wedge (1 - z_j)$$

in  $\Lambda^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$ , then

$$\int_{\gamma} \eta(x, y) = \sum r_j D(z_j)|_{\partial\gamma}$$

Big picture

$$\dots \rightarrow (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \rightarrow K_2(C, \partial\gamma) \rightarrow K_2(C) \rightarrow \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x, y)$  is exact, then  $\{x, y\} \in K_3(\partial\gamma)$ . We have  $\partial\gamma \neq \emptyset$  and we use Stokes' Theorem.

$\rightsquigarrow$  dilogarithms, zeta function

- $\partial\gamma = \emptyset$ , then  $\{x, y\} \in K_2(C)$ . We have  $\eta(x, y)$  is not exact.

$\rightsquigarrow$   $L$ -series of a curve

We may get combinations of both situations.

## 9. The three-variable case

$$P(x, y, z) = (1-x) + (1-y)z \quad S = \{P(x, y, z) = 0\}$$

$$\begin{aligned} m(P) &= m(1-y) + \frac{1}{(2\pi i)^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{dx dy dz}{x y z} \\ &= \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{dx dy}{x y} \\ &= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{dx dy}{x y} \\ &= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z) \end{aligned}$$

$$\Gamma = S \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$\begin{aligned}
\eta(x, y, z) = & \log |x| \left( \frac{1}{3} d \log |y| d \log |z| - d \arg y d \arg z \right) \\
& + \log |y| \left( \frac{1}{3} d \log |z| d \log |x| - d \arg z d \arg x \right) \\
& + \log |z| \left( \frac{1}{3} d \log |x| d \log |y| - d \arg x d \arg y \right)
\end{aligned}$$

### **Theorem 1**

$$\eta(x, 1 - x, y) = d \omega(x, y)$$

where

$$\omega(x, y) = -D(x) d \arg y$$

$$+ \frac{1}{3} \log |y| (\log |1 - x| d \log |x| - \log |x| d \log |1 - x|)$$



$$\eta(x, y, z) = -\eta(x, 1 - x, y) - \eta(y, 1 - y, x)$$

Maillot: if  $P \in \mathbb{Q}[x, y, z]$ ,

$$\partial\Gamma = \gamma = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

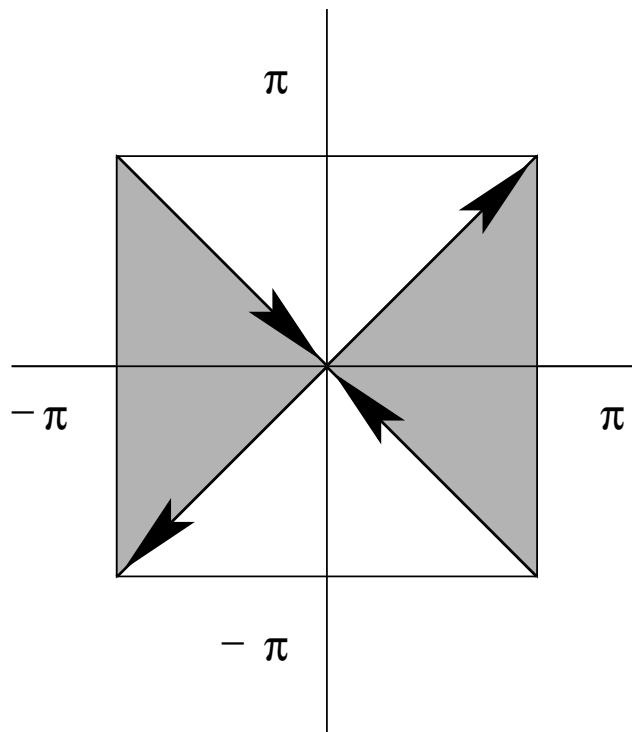
$\omega$  defined in

$$C = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x) + (1-y)z) = \frac{1}{4\pi^2} \int_{\gamma} \omega(x, y) + \omega(y, x)$$

## Theorem 2

$$\omega(x, x) = dP_3(x)$$

$$= \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2} \zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z)$$

Need  $\{x, y, z\} = 0$  in  $K_3^M(\mathbb{C}(S)) \otimes \mathbb{Q}$ .

$$x \wedge y \wedge z = \sum r_i x_i \wedge (1 - x_i) \wedge y_i$$

in  $\Lambda^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$ , then

$$\begin{aligned} \int_{\Gamma} \eta(x, y, z) &= \sum r_i \int_{\Gamma} \eta(x_i, 1 - x_i, y_i) \\ &= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i) \end{aligned}$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in  $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$ .

Then

$$\int_{\gamma} \omega(x, y) = \sum r_i P_3(x_i)|_{\partial\gamma}$$

## 10. A little bit of $K$ -theory

$F$  field, define subgroups  $R_i(F) \subset \mathbb{Z}[\mathbb{P}_F^1]$  as

$$R_1(F) := [x] + [y] - [xy]$$

$$R_2(F) := [x] + [y] + [1 - xy] + \left[ \frac{1 - x}{1 - xy} \right] + \left[ \frac{1 - y}{1 - xy} \right]$$

$R_3(F) :=$  functional equation of the trilogarithm

$$B_i(F) := \mathbb{Z}[\mathbb{P}_F^1] / R_i(F)$$

$$B_F(3) : B_3(F) \xrightarrow{\delta_1^3} B_2(F) \otimes F^* \xrightarrow{\delta_2^3} \wedge^3 F^*$$

$$B_F(2) : B_2(F) \xrightarrow{\delta_1^2} \wedge^2 F^*$$

$$B_F(1) : F^*$$

( $B_i(F)$  is placed in degree 1).

$$\delta_1^3([x]_3) = [x]_2 \otimes x \quad \delta_2^3([x]_2 \otimes y) = x \wedge (1 - x) \wedge y$$

$$\delta_1^2([x]_2) = x \wedge (1 - x)$$

### Proposition 3

$$\begin{aligned} H^1(B_F(1)) &\cong K_1(F) \\ H^1(B_F(2))_{\mathbb{Q}} &\cong K_3^{\text{ind}}(F)_{\mathbb{Q}} \\ H^2(B_F(2)) &\cong K_2(F) \\ H^3(B_F(3)) &\cong K_3^M(F) \end{aligned}$$

Goncharov conjectures:

$$H^i(B_F(3) \otimes \mathbb{Q}) \stackrel{?}{\cong} K_{6-i}^{[3-i]}(F)_{\mathbb{Q}}$$

Our first condition is  $x \wedge y \wedge z$  is 0 in

$$H^3(B_{\mathbb{Q}(S)}(3) \otimes \mathbb{Q}) \cong K_3^M(\mathbb{Q}(S)) \otimes \mathbb{Q}$$

Our second condition is  $[x_i]_2 \otimes y_i$  is 0 in

$$H^2(B_{\mathbb{Q}(C)}(3) \otimes \mathbb{Q}) \stackrel{?}{\cong} K_4^{[1]}(\mathbb{Q}(C))_{\mathbb{Q}}$$

Big picture II

$$\dots \rightarrow K_4(\partial\Gamma) \rightarrow K_3(S, \partial\Gamma) \rightarrow K_3(S) \rightarrow \dots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\dots \rightarrow (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \rightarrow K_4(C, \partial\gamma) \rightarrow K_4(C) \rightarrow \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.

