Some aspects of Mahler Measure

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1. Mahler measure

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) = \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n$$

$$= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n}$$

By Jensen's formula:

$$\int_0^1 \log|\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha|\mathrm{d}\theta = \log^+|\alpha|$$

we obtain

$$m(P) = \log|a_d| + \sum_{n=1}^{d} \log^+|\alpha_n|$$

for

$$P(x) = a_d \prod_{n=1}^{d}(x - \alpha_n) \in \mathbb{C}[x]$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

2. Examples in several variables

Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathsf{L}(\chi_{-3},2) = \mathsf{L}'(\chi_{-3},-1)$$

$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

Boyd & Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathsf{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathsf{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathsf{L}'(A,0)$$

$$A: y^2 = x^3 - 44x + 112$$

3. Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$P_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k} \frac{2^j B_j}{j!} (\log|x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number, $\mathrm{Li}_0(x) \equiv -\frac{1}{2}$,

$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

P_k satisfies lots of functional equations

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1} P_k(x) \qquad P_k(\bar{x}) = (-1)^{k-1} P_k(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

Five-term relation

$$D(x)+D(1-xy)+D(y)+D\left(\frac{1-y}{1-xy}\right)+D\left(\frac{1-x}{1-xy}\right)=0$$

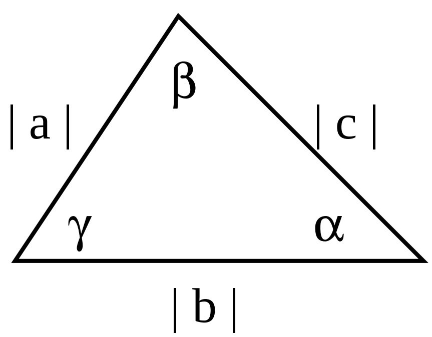
4. Mahler measure and hyperbolic volumes

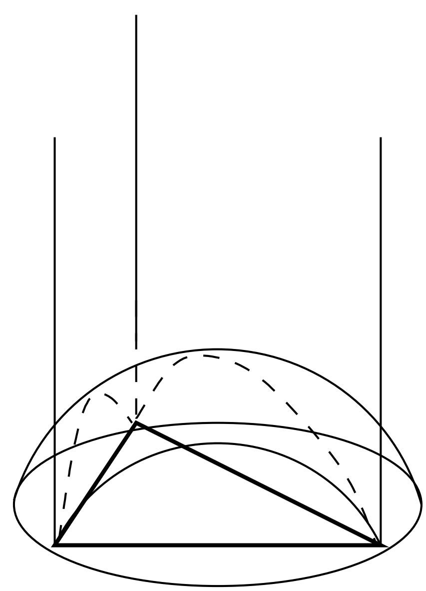
Cassaigne – Maillot (2000) for $a, b, c \in \mathbb{C}^*$,

$$\pi m(a + bx + cy)$$

$$= \begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{\mathrm{i}\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| & \triangle \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases}$$

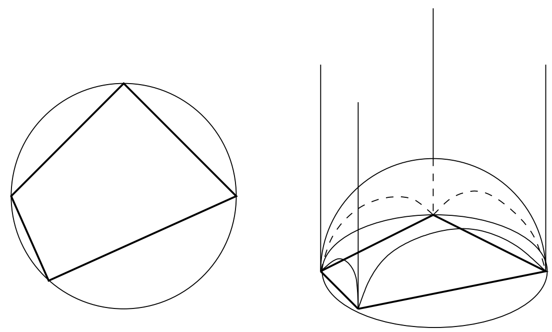
Ideal tetrahedron:





- Vandervelde (2003)

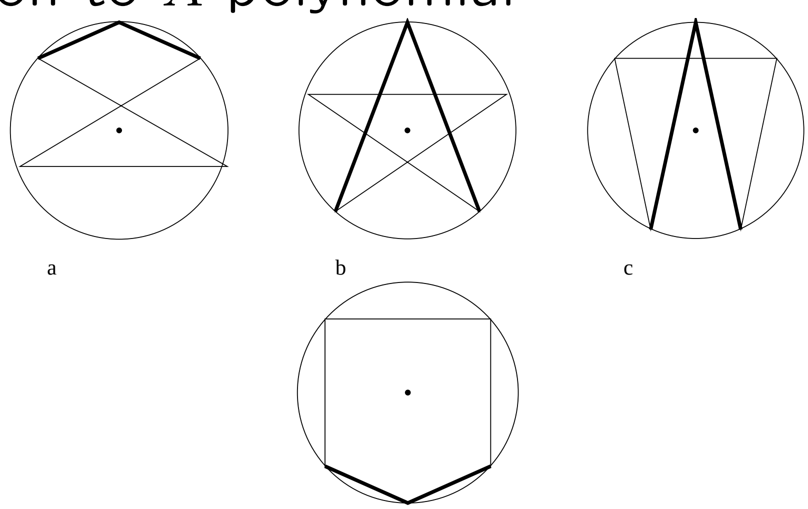
$$y = \frac{bx+d}{ax+c} \qquad \text{quadrilateral}$$



- L(2004)

$$y = \frac{x^n - 1}{t(x^m - 1)} = \frac{x^{n-1} + \ldots + 1}{t(x^{m-1} + \ldots + 1)} \quad \text{polyhedral}$$

+ relation to A-polynomial



5. More examples in several variables

L (2003)

$$\pi^n m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)z\right)$$

= combination of $\zeta(\text{odd})\,/\,\mathrm{L}(\chi_{-4},\text{even})$

$$\pi^n m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

= combination of $\zeta(\text{odd})\,/\,\mathrm{L}(\chi_{-4},\text{even})$, polylogarithms

$$\pi^n m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)x\right.$$

$$\left.+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)\right)y\right)$$

= combination of $\zeta(\text{odd})$

Examples

$$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right)$$

$$= 24\mathsf{L}(\chi_{-4},4)+\pi^2\mathsf{L}(\chi_{-4},2)$$

$$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_4}{1+x_4}\right)z\right)$$

$$= 62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

6. Examples from the world of resultants

D'Andrea & L (2003).

- $m(\mathrm{Res}_{\{0,m,n\}})$

$$= m(\mathrm{Res}_t(x + yt^m + t^n, z + wt^m + t^n)) =$$

$$\frac{2}{\pi^2}(-mP_3(\varphi^n) - nP_3(-\varphi^m) + mP_3(\phi^n) + nP_3(\phi^m))$$

$$0 \le \varphi \le 1 \quad \text{root of} \quad x^n + x^{n-m} - 1 = 0$$

$$1 \le \phi \quad \text{root of} \quad x^n - x^{n-m} - 1 = 0$$

- $m(\mathrm{Res}_{\{(0,0),(1,0),(0,1)\}}) = m\left(\begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix}\right)$

$$= m((1-x)(1-y) - (1-z)(1-w)) = \frac{9\zeta(3)}{2\pi^2}$$

7. Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X

- L-function

- Finitely-generated abelian group K

- Regulator map $\mathrm{reg} : K \to \mathbb{R}$

$$(K \,\mathrm{rank}\, 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

8. An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) : $P(x,y) \in \mathbb{C}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

$$\eta(x, 1-x) = \mathrm{d}D(x)$$

Need $\{x,y\} = 0$ in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$.

$$x \wedge y = \sum_j r_j \ \ z_j \wedge (1 - z_j)$$

in $\bigwedge^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$, then

$$\int_\gamma \eta(x,y) = \sum r_j D(z_j)|_{\partial\gamma}$$

Big picture

$$\ldots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(C, \partial\gamma) \to K_2(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x, y)$ is exact, then $\{x, y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

  $\rightsquigarrow$ dilogarithms, zeta function

- $\partial\gamma = \emptyset$, then $\{x, y\} \in K_2(C)$. We have $\eta(x, y)$ is not exact.

  $\rightsquigarrow$ L-series of a curve

We may get combinations of both situations.

9. The three-variable case

$$P(x,y,z) = (1-x)+(1-y)z \quad S = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi i)^3}\int_{\mathbb{T}^3} \log\left|z - \frac{1-x}{1-y}\right| \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}$$

$$= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} \log^+\left|\frac{1-x}{1-y}\right| \frac{dx}{x}\frac{dy}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \log|z| \frac{dx}{x}\frac{dy}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \eta(x,y,z)$$

$$\Gamma = S \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$\eta(x,y,z) = \log|x| \left(\frac{1}{3}\mathrm{d}\log|y|\,\mathrm{d}\log|z| - \mathrm{d}\arg y\,\mathrm{d}\arg z\right)$$

$$+ \log|y| \left(\frac{1}{3}\mathrm{d}\log|z|\,\mathrm{d}\log|x| - \mathrm{d}\arg z\,\mathrm{d}\arg x\right)$$

$$+ \log|z| \left(\frac{1}{3}\mathrm{d}\log|x|\,\mathrm{d}\log|y| - \mathrm{d}\arg x\,\mathrm{d}\arg y\right)$$

Theorem 1

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x,y)$$

where

$$\omega(x,y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x)$$

Maillot: if $P \in \mathbb{Q}[x,y,z]$,

$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$

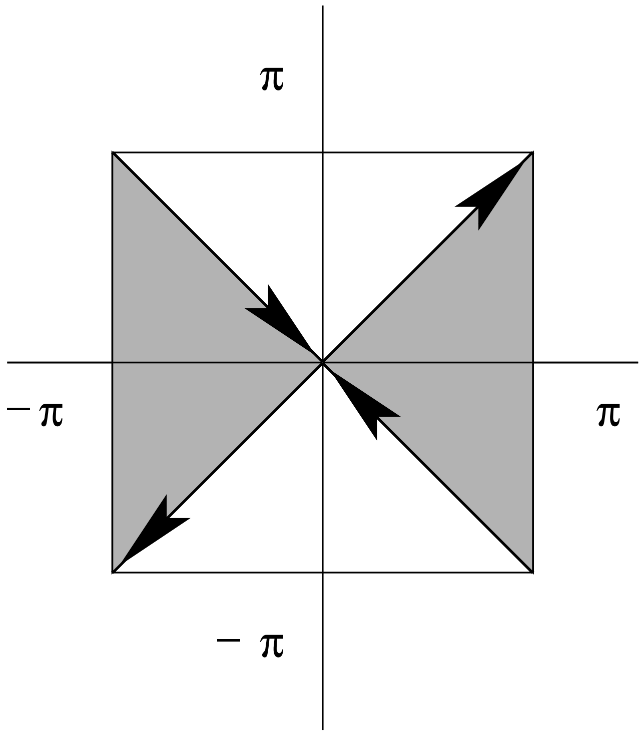
ω defined in

$$C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

Theorem 2

$$\omega(x,x) = \mathrm{d}P_3(x)$$

$$= \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_\Gamma \eta(x, y, z)$$

Need $\{x, y, z\} = 0$ in $K_3^M(\mathbb{C}(S)) \otimes \mathbb{Q}$.

$$x \wedge y \wedge z = \sum r_i \; x_i \wedge (1 - x_i) \wedge y_i$$

in $\bigwedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$, then

$$\int_\Gamma \eta(x, y, z) = \sum r_i \int_\Gamma \eta(x_i, 1 - x_i, y_i)$$

$$= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.

Then

$$\int_\gamma \omega(x,y) = \sum r_i \, P_3(x_i)|_{\partial\gamma}$$

10. A little bit of K-theory

F field, define subgroups $R_i(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$ as

$$R_1(F) := [x] + [y] - [xy]$$
$$R_2(F) := [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right]$$
$$R_3(F) := \text{functional equation of the trilogarithm}$$

$$B_i(F) := \mathbb{Z}[\mathbb{P}^1_F]/R_i(F)$$

$$\begin{aligned} B_F(3) &: \; B_3(F) \xrightarrow{\delta^3_1} B_2(F) \otimes F^* \xrightarrow{\delta^3_2} \wedge^3 F^* \\ B_F(2) &: \; B_2(F) \xrightarrow{\delta^2_1} \wedge^2 F^* \\ B_F(1) &: \; F^* \end{aligned}$$

($B_i(F)$ is placed in degree 1).

$\delta^3_1([x]_3) = [x]_2 \otimes x \qquad \delta^3_2([x]_2 \otimes y) = x \wedge (1-x) \wedge y$

$$\delta^2_1([x]_2) = x \wedge (1-x)$$

Proposition 3

$$\begin{aligned} H^1(B_F(1)) &\cong K_1(F) \\ H^1(B_F(2))_{\mathbb{Q}} &\cong K_3^{\text{ind}}(F)_{\mathbb{Q}} \\ H^2(B_F(2)) &\cong K_2(F) \\ H^3(B_F(3)) &\cong K_3^M(F) \end{aligned}$$

Goncharov conjectures:

$$H^i(B_F(3) \otimes \mathbb{Q}) \overset{?}{\cong} K_{6-i}^{[3-i]}(F)_{\mathbb{Q}}$$

Our first condition is $x \wedge y \wedge z$ is 0 in

$$H^3(B_{\mathbb{Q}(S)}(3) \otimes \mathbb{Q}) \cong K_3^M(\mathbb{Q}(S)) \otimes \mathbb{Q}$$

Our second condition is $[x_i]_2 \otimes y_i$ is 0 in

$$H^2(B_{\mathbb{Q}(C)}(3) \otimes \mathbb{Q}) \overset{?}{\cong} K_4^{[1]}(\mathbb{Q}(C))_{\mathbb{Q}}$$

Big picture II

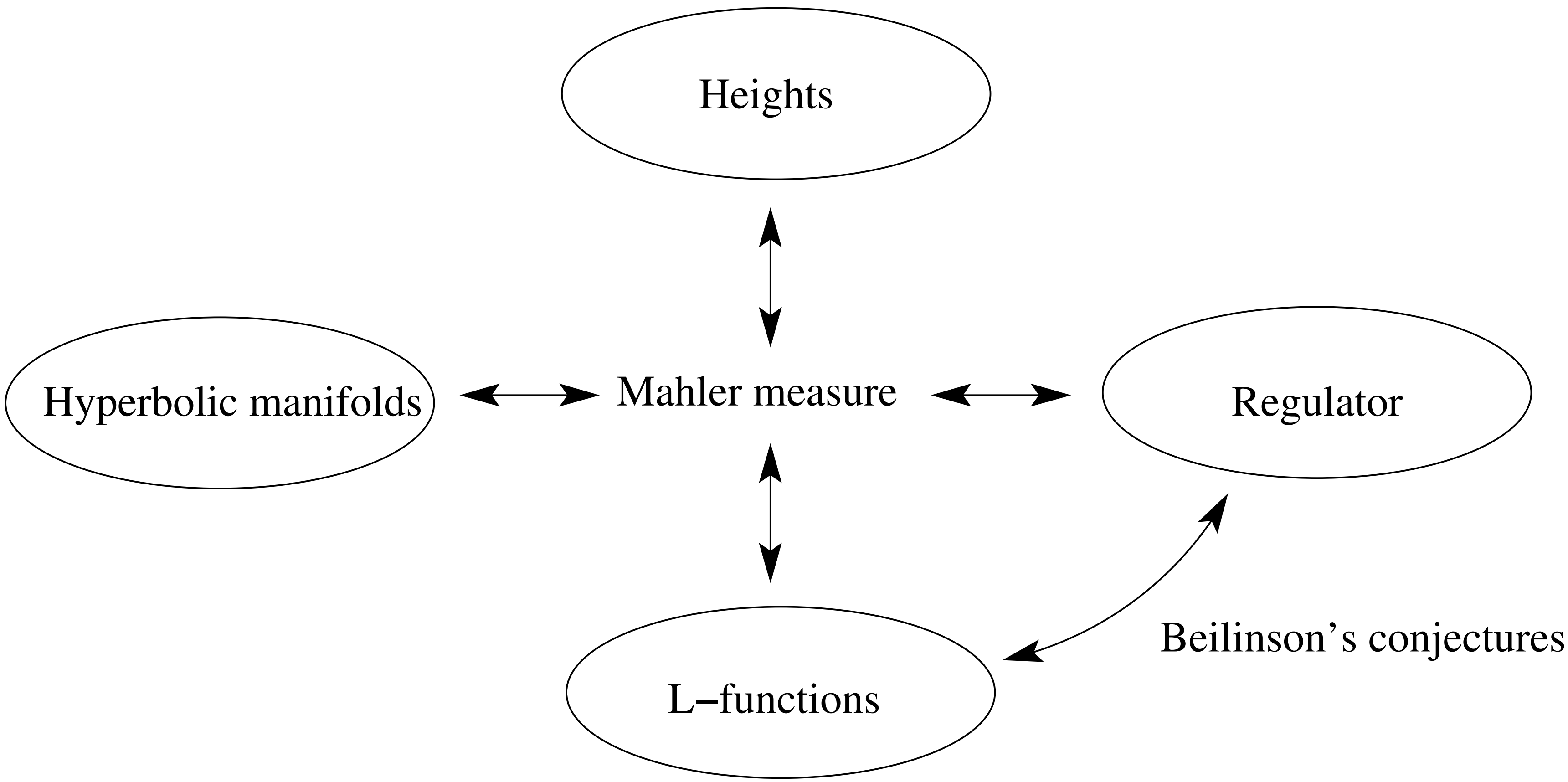
$$\ldots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \ldots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\ldots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.


Heights
Hyperbolic manifolds
Mahler measure
Regulator
L−functions
Beilinson's conjectures