

# Functional equations for Mahler measures of genus-one curves

(joint with Mat Rogers) Matilde N. Lalín

University of British Columbia and Pacific Institute for the Mathematical Sciences  
Simon Fraser University  
mlalin@math.ubc.ca  
<http://www.math.ubc.ca/~mlalin>

October 5th, 2006



# Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ , the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$



# The measures of a family of genus-one curves

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd 1998

$$m(k) \stackrel{?}{=} \frac{L'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

$E_k$  determined by  $x + \frac{1}{x} + y + \frac{1}{y} + k = 0$ .

Deninger 1997

L-functions ← Beilinson's conjectures

Kronecker-Eisenstein series for  $k = 1$



# The measures of a family of genus-one curves

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd 1998

$$m(k) \stackrel{?}{=} \frac{L'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

$E_k$  determined by  $x + \frac{1}{x} + y + \frac{1}{y} + k = 0$ .

Deninger 1997

L-functions ← Beilinson's conjectures

Kronecker-Eisenstein series for  $k = 1$



# The measures of a family of genus-one curves

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd 1998

$$m(k) \stackrel{?}{=} \frac{L'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

$E_k$  determined by  $x + \frac{1}{x} + y + \frac{1}{y} + k = 0$ .

Deninger 1997

L-functions  $\leftarrow$  Beilinson's conjectures

Kronecker-Eisenstein series for  $k = 1$



Rodriguez-Villegas 1997

$k = 4\sqrt{2}$  (CM case)

$$m(4\sqrt{2}) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 4\sqrt{2}\right) = L'(E_{4\sqrt{2}}, 0)$$

$k = 3\sqrt{2}$  (modular curve  $X_0(24)$ )

$$m(3\sqrt{2}) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 3\sqrt{2}\right) = qL'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$



## Theorem

(Rodriguez-Villegas)  $E_k \sim$  modular elliptic surface assoc  $\Gamma_0(4)$ .

$$\begin{aligned} m(k) &= \operatorname{Re} \left( \frac{16y_\mu}{\pi^2} \sum'_{m,n} \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})} \right) \\ &= \operatorname{Re} \left( -\pi i \mu + 2 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-4}(d) d^2 \frac{q^n}{n} \right) \end{aligned}$$

where  $j(E_k) = j\left(-\frac{1}{4\mu}\right)$

$$q = e^{2\pi i \mu} = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 1 - \frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{16}{k^2}\right)}\right)$$

and  $y_\mu$  is the imaginary part of  $\mu$ .

## Theorem

(Rogers, and also Kurokawa & Ochiai 2005)

For  $h \in \mathbb{R}^*$ ,

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).$$

(Rogers)

For  $|h| < 1$ ,  $h \neq 0$ ,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(ih + \frac{1}{ih}\right)\right) = m\left(\frac{4}{h^2}\right).$$

$$m\left(2\left(h + \frac{1}{h}\right)\right) - m\left(2\left(ih + \frac{1}{ih}\right)\right) = m(4h^2).$$





## Theorem

(Rogers, and also Kurokawa & Ochiai 2005)

For  $h \in \mathbb{R}^*$ ,

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).$$

(Rogers)

For  $|h| < 1$ ,  $h \neq 0$ ,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(ih + \frac{1}{ih}\right)\right) = m\left(\frac{4}{h^2}\right).$$

$$m\left(2\left(h + \frac{1}{h}\right)\right) - m\left(2\left(ih + \frac{1}{ih}\right)\right) = m(4h^2).$$



## Corollary

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2})$$

$$m(3\sqrt{2}) = qL'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$



## Corollary

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2})$$

$$m(3\sqrt{2}) = qL'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$



# Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object  $X$  (for instance,  $X = \mathcal{O}_F$ ,  $F$  a number field)
- L-function ( $L_F = \zeta_F$ )
- Finitely-generated abelian group  $K$  ( $K = \mathcal{O}_F^*$ )
- Regulator map  $\text{reg} : K \rightarrow \mathbb{R}$  ( $\text{reg} = \log |\cdot|$ )

$$(K \text{ rank } 1) \quad L'_X(0) \sim_{\mathbb{Q}^*} \text{reg}(\xi)$$

(Dirichlet class number formula, for  $F$  real quadratic,

$$\zeta'_F(0) \sim_{\mathbb{Q}^*} \log |\epsilon|, \epsilon \in \mathcal{O}_F^*$$



# The elliptic regulator

$F$  field. Matsumoto Theorem:

$$K_2(F) = \langle \{a, b\}, a, b \in F \rangle / \langle \text{bilinear}, \{a, 1 - a\} \rangle$$

$K_2(E) \otimes \mathbb{Q}$  subgroup of  $K_2(\mathbb{Q}(E)) \otimes \mathbb{Q}$  determined by kernels of tame symbols.

$x, y \in \mathbb{C}(E)$

$$\eta(x, y) := \log |x| d \arg y - \log |y| d \arg x$$

1-form on  $E(\mathbb{C}) \setminus S$

for any loop  $\gamma \in E(\mathbb{C}) \setminus S$

$$(\gamma, \eta(x, y)) = \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$



# The elliptic regulator

$F$  field. Matsumoto Theorem:

$$K_2(F) = \langle \{a, b\}, a, b \in F \rangle / \langle \text{bilinear}, \{a, 1 - a\} \rangle$$

$K_2(E) \otimes \mathbb{Q}$  subgroup of  $K_2(\mathbb{Q}(E)) \otimes \mathbb{Q}$  determined by kernels of tame symbols.

$x, y \in \mathbb{C}(E)$

$$\eta(x, y) := \log |x| d \arg y - \log |y| d \arg x$$

1-form on  $E(\mathbb{C}) \setminus S$

for any loop  $\gamma \in E(\mathbb{C}) \setminus S$

$$(\gamma, \eta(x, y)) = \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$



# The elliptic regulator

$F$  field. Matsumoto Theorem:

$$K_2(F) = \langle \{a, b\}, a, b \in F \rangle / \langle \text{bilinear}, \{a, 1 - a\} \rangle$$

$K_2(E) \otimes \mathbb{Q}$  subgroup of  $K_2(\mathbb{Q}(E)) \otimes \mathbb{Q}$  determined by kernels of tame symbols.

$x, y \in \mathbb{C}(E)$

$$\eta(x, y) := \log |x| d \arg y - \log |y| d \arg x$$

1-form on  $E(\mathbb{C}) \setminus S$

for any loop  $\gamma \in E(\mathbb{C}) \setminus S$

$$(\gamma, \eta(x, y)) = \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$



The regulator map (Beilinson, Bloch):

$$r : K_2(E) \otimes \mathbb{Q} \rightarrow H^1(E, \mathbb{R})$$

$$\{x, y\} \rightarrow \left\{ \gamma \rightarrow \int_{\gamma} \eta(x, y) \right\}$$

for  $\gamma \in H_1(E, \mathbb{Z})$ . ( $H^1(E, \mathbb{R})$  dual of  $H_1(E, \mathbb{Z})$ )

Follows from  $\eta(x, 1-x) = dD(x)$ ,

$$D(x) = \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1-x) \log|x|$$

is the Bloch-Wigner dilogarithm





# The relation with Mahler measures

Deninger

$$m(k) \sim_{\mathbb{Z}} \frac{1}{2\pi} r(\{x, y\})(\gamma)$$

In the example,

$$yP_k(x, y) = (y - y_{(1)}(x))(y - y_{(2)}(x)),$$

$$m(k) = \frac{1}{2\pi i} \int_{\mathbb{T}^1} (\log^+ |y_{(1)}(x)| + \log^+ |y_{(2)}(x)|) \frac{dx}{x}.$$

By Jensen's formula respect to  $y$ .

$$m(k) = \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log |y| \frac{dx}{x} = -\frac{1}{2\pi} \int_{\mathbb{T}^1} \eta(x, y),$$

$$\mathbb{T}^1 \in H_1(E, \mathbb{Z}).$$



# Computing the regulator

$$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$$

$z \bmod \Lambda = \mathbb{Z} + \tau\mathbb{Z}$  is identified with  $e^{2i\pi z}$ .

Bloch regulator function

$$R_{\tau} \left( e^{2\pi i(a+b\tau)} \right) = \frac{y_{\tau}^2}{\pi} \sum'_{m,n \in \mathbb{Z}} \frac{e^{2\pi i(bn-am)}}{(m\tau + n)^2(m\bar{\tau} + n)}$$

$y_{\tau}$  is the imaginary part of  $\tau$ .

Elliptic dilogarithm

$$D_{\tau}(z) := \sum_{n \in \mathbb{Z}} D(zq^n)$$

Regulator function given by

$$R_{\tau} = D_{\tau} - iJ_{\tau}$$



$$\mathbb{Z}[E(\mathbb{C})]^- = \mathbb{Z}[E(\mathbb{C})]/ \sim \quad [-P] \sim -[P].$$

$R_\tau$  is an odd function,

$$\mathbb{Z}[E(\mathbb{C})]^- \rightarrow \mathbb{C}.$$

$$(x) = \sum m_i(a_i), \quad (y) = \sum n_j(b_j).$$

$$\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \rightarrow \mathbb{Z}[E(\mathbb{C})]^-$$

$$(x) \diamond (y) = \sum m_i n_j (a_i - b_j).$$



## Theorem

(Beilinson)  $E/\mathbb{R}$  elliptic curve,  $x, y$  non-constant functions in  $\mathbb{C}(E)$ ,  
 $\omega \in \Omega^1$

$$\int_{E(\mathbb{C})} \bar{\omega} \wedge \eta(x, y) = \Omega_0 R_\tau((x) \diamond (y))$$

## Corollary

(after an idea of Deninger)  $x, y$  are non-constant functions in  $\mathbb{C}(E)$  with trivial tame symbols

$$-r\{x, y\} = - \int_\gamma \eta(x, y) = \text{Im} \left( \frac{\Omega}{y_\tau \Omega_0} R_\tau((x) \diamond (y)) \right)$$

where  $\Omega_0$  is the real period and  $\Omega = \int_\gamma \omega$ .



## Theorem

(Beilinson)  $E/\mathbb{R}$  elliptic curve,  $x, y$  non-constant functions in  $\mathbb{C}(E)$ ,  
 $\omega \in \Omega^1$

$$\int_{E(\mathbb{C})} \bar{\omega} \wedge \eta(x, y) = \Omega_0 R_\tau((x) \diamond (y))$$

## Corollary

(after an idea of Deninger)  $x, y$  are non-constant functions in  $\mathbb{C}(E)$  with trivial tame symbols

$$-r\{x, y\} = - \int_\gamma \eta(x, y) = \text{Im} \left( \frac{\Omega}{y_\tau \Omega_0} R_\tau((x) \diamond (y)) \right)$$

where  $\Omega_0$  is the real period and  $\Omega = \int_\gamma \omega$ .



# Idea of Proof

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

Weierstrass form:

$$x = \frac{kX - 2Y}{2X(X - 1)} \quad y = \frac{kX + 2Y}{2X(X - 1)}.$$

$$Y^2 = X \left( X^2 + \left( \frac{k^2}{4} - 2 \right) X + 1 \right).$$

$P = (1, \frac{k}{2})$ , torsion point of order 4.

$$(x) \diamond (y) = 4(P) - 4(-P) = 8(P).$$



$$P \equiv -\frac{1}{4} \pmod{\mathbb{Z} + \tau\mathbb{Z}} \quad k \in \mathbb{R}$$

$$\tau = iy_\tau \quad k \in \mathbb{R}, |k| > 4,$$

$$\tau = \frac{1}{2} + iy_\tau \quad k \in \mathbb{R}, |k| < 4$$

Understand cycle  $[|x| = 1] \in H_1(E, \mathbb{Z})$

$$\Omega = \tau\Omega_0 \quad k \in \mathbb{R}$$



$$-r\{x, y\} = - \int_{\gamma} \eta(x, y) = \operatorname{Im} \left( \frac{\Omega}{y_{\tau} \Omega_0} R_{\tau} ((x) \diamond (y)) \right)$$

$$m(k) = \frac{4}{\pi} \operatorname{Im} \left( \frac{\tau}{y_{\tau}} R_{\tau}(-i) \right), \quad k \in \mathbb{R}$$





# Modularity for the regulator

Let  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$  and let  $\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$ , such that

$$\begin{pmatrix} b' \\ a' \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}$$

Then:

$$R_{\tau'} \left( e^{2\pi i(a' + b'\tau')} \right) = \frac{1}{\gamma\bar{\tau} + \delta} R_{\tau} \left( e^{2\pi i(a + b\tau)} \right).$$



$$m(k) = \frac{4}{\pi} \operatorname{Im} \left( \frac{\tau}{y_\tau} R_\tau(-i) \right), \quad k \in \mathbb{R}$$

Take  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$ .

$$m(k) = -\frac{4|\tau|^2}{\pi y_\tau} J_{-\frac{1}{\tau}} \left( e^{-\frac{2\pi i}{4\tau}} \right)$$

If we let  $\mu = -\frac{1}{4\tau}$ , then

$$\begin{aligned} m(k) &= -\frac{1}{\pi y_\mu} J_{4\mu} \left( e^{2\pi i \mu} \right) \\ &= \operatorname{Re} \left( \frac{16y_\mu}{\pi^2} \sum'_{m,n} \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})} \right) \end{aligned}$$



# Functional equations for the regulator

From

$$J(z) = p \sum_{x^p=z} J(x)$$

Let  $p$  prime,

$$(1 + \chi_{-4}(p)p^2)J_{4\tau}(e^{2\pi i\tau}) = \sum_{j=0}^{p-1} pJ_{\frac{4(\tau+j)}{p}}\left(e^{\frac{2\pi i(\tau+j)}{p}}\right) + \chi_{-4}(p)J_{4p\tau}(e^{2\pi ip\tau})$$

- In particular,  $p = 2$ ,

$$J_{4\tau}(e^{2\pi i\tau}) = 2J_{2\tau}(e^{\pi i\tau}) + 2J_{2(\tau+1)}(e^{\pi i(\tau+1)})$$

- Also:

$$J_{\frac{2\tau+1}{2}}(e^{\pi i\tau}) = J_{2\tau}(e^{\pi i\tau}) - J_{2\tau}(-e^{\pi i\tau})$$



# Functional equations for the regulator

From

$$J(z) = p \sum_{x^p=z} J(x)$$

Let  $p$  prime,

$$(1 + \chi_{-4}(p)p^2)J_{4\tau}(e^{2\pi i\tau}) = \sum_{j=0}^{p-1} pJ_{\frac{4(\tau+j)}{p}}\left(e^{\frac{2\pi i(\tau+j)}{p}}\right) + \chi_{-4}(p)J_{4p\tau}(e^{2\pi ip\tau})$$

- In particular,  $p = 2$ ,

$$J_{4\tau}(e^{2\pi i\tau}) = 2J_{2\tau}(e^{\pi i\tau}) + 2J_{2(\tau+1)}(e^{\pi i(\tau+1)})$$

- Also:

$$J_{\frac{2\tau+1}{2}}(e^{\pi i\tau}) = J_{2\tau}(e^{\pi i\tau}) - J_{2\tau}(-e^{\pi i\tau})$$



$$J_{4\mu}(e^{2\pi i\mu}) = 2J_{2\mu}(e^{\pi i\mu}) + 2J_{2(\mu+1)}\left(e^{\frac{2\pi i(\mu+1)}{2}}\right)$$

$$\frac{1}{y_{4\mu}} J_{4\mu}(e^{2\pi i\mu}) = \frac{1}{y_{2\mu}} J_{2\mu}(e^{\pi i\mu}) + \frac{1}{y_{2\mu}} J_{2\mu}(-e^{\pi i\mu})$$

$$q = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 1 - \frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{16}{k^2}\right)}\right)$$

Second degree modular equation,  $|h| < 1$ ,  $h \in \mathbb{R}$ ,

$$q^2 \left( \left( \frac{2h}{1+h^2} \right)^2 \right) = q(h^4).$$

$h \rightarrow ih$

$$-q \left( \left( \frac{2h}{1+h^2} \right)^2 \right) = q \left( \left( \frac{2ih}{1-h^2} \right)^2 \right).$$



$$J_{4\mu}(e^{2\pi i\mu}) = 2J_{2\mu}(e^{\pi i\mu}) + 2J_{2(\mu+1)}\left(e^{\frac{2\pi i(\mu+1)}{2}}\right)$$

$$\frac{1}{y_{4\mu}} J_{4\mu}(e^{2\pi i\mu}) = \frac{1}{y_{2\mu}} J_{2\mu}(e^{\pi i\mu}) + \frac{1}{y_{2\mu}} J_{2\mu}(-e^{\pi i\mu})$$

$$q = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 1 - \frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{16}{k^2}\right)}\right)$$

Second degree modular equation,  $|h| < 1$ ,  $h \in \mathbb{R}$ ,

$$q^2\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q(h^4).$$

$h \rightarrow ih$

$$-q\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(\left(\frac{2ih}{1-h^2}\right)^2\right).$$



Then the equation with  $J$  becomes

$$m\left(q\left(\left(\frac{2h}{1+h^2}\right)^2\right)\right) + m\left(q\left(\left(\frac{2ih}{1-h^2}\right)^2\right)\right) = m(q(h^4)).$$

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(ih + \frac{1}{ih}\right)\right) = m\left(\frac{4}{h^2}\right).$$



$$J_{\frac{2\mu+1}{2}} \left( e^{\frac{2\pi i \mu}{2}} \right) = J_{2\mu} (e^{\pi i \mu}) - J_{2\mu} (-e^{\pi i \mu})$$

Set  $\tau = -\frac{1}{2\mu}$  and use  $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ .

$$D_{\frac{\tau-1}{2}}(-i) = D_{\tau}(-i) - \frac{1}{y_{2(\mu+1)}} J_{2(\mu+1)} \left( e^{\frac{2\pi i(\mu+1)}{2}} \right)$$

First equation was:

$$D_{\frac{\tau-1}{2}}(-i) = D_{\tau}(-i) + \frac{1}{y_{2(\mu+1)}} J_{2(\mu+1)} \left( e^{\frac{2\pi i(\mu+1)}{2}} \right)$$

Putting things together,

$$2D_{\tau}(-i) = D_{\frac{\tau}{2}}(-i) + D_{\frac{\tau-1}{2}}(-i)$$

this is:

$$2m \left( 2 \left( h + \frac{1}{h} \right) \right) = m(4h^2) + m \left( \frac{4}{h^2} \right).$$





# Hecke operators approach

$$\begin{aligned} m(k) &= \operatorname{Re} \left( -\pi i \mu + 2 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-4}(d) d^2 \frac{q^n}{n} \right) \\ &= \operatorname{Re} \left( -\pi i \mu - \pi i \int_{i\infty}^{\mu} (e(z) - 1) dz \right) \end{aligned}$$

where

$$e(\mu) = 1 - 4 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-4}(d) d^2 q^n$$

is an Eisenstein series. Hence the equations can be also deduced from identities of Hecke operators.



## Direct approach

Also some equations can be proved directly using isogenies:

$$\phi_1 : E_{2(h+\frac{1}{h})} \rightarrow E_{4h^2}, \quad \phi_2 : E_{2(h+\frac{1}{h})} \rightarrow E_{\frac{4}{h^2}}.$$

$$\phi_1 : (X, Y) \rightarrow \left( \frac{X(h^2X + 1)}{X + h^2}, -\frac{h^3Y(X^2 + 2h^2X + 1)}{(X + h^2)^2} \right)$$

$$\begin{aligned} m(4h^2) &= r_1(\{x_1, y_1\}) = \frac{1}{2\pi} \int_{|x_1|=1} \eta(x_1, y_1) \\ &= \frac{1}{4\pi} \int_{|X|=1} \eta(x_1 \circ \phi_1, y_1 \circ \phi_1) = \frac{1}{2} r(\{x_1 \circ \phi_1, y_1 \circ \phi_1\}) \end{aligned}$$



The identity with  $h = \frac{1}{\sqrt{2}}$

$$m(2) + m(8) = 2m(3\sqrt{2})$$

$$m(3\sqrt{2}) + m(i\sqrt{2}) = m(8)$$

$$f = \frac{\sqrt{2}Y - X}{2} \text{ in } \mathbb{C}(E_{3\sqrt{2}}).$$

$$(f) \diamond (1 - f) = 6(P) - 10(P + Q) \Rightarrow 6(P) \sim 10(P + Q).$$

$Q = (-\frac{1}{h^2}, 0)$  has order 2.

$$\phi : E_{3\sqrt{2}} \rightarrow E_{i\sqrt{2}} \quad (X, Y) \rightarrow (-X, iY)$$

$$r_{i\sqrt{2}}(\{x, y\}) = r_{3\sqrt{2}}(\{x \circ \phi, y \circ \phi\})$$



The identity with  $h = \frac{1}{\sqrt{2}}$

$$m(2) + m(8) = 2m(3\sqrt{2})$$

$$m(3\sqrt{2}) + m(i\sqrt{2}) = m(8)$$

$$f = \frac{\sqrt{2}Y - X}{2} \text{ in } \mathbb{C}(E_{3\sqrt{2}}).$$

$$(f) \diamond (1 - f) = 6(P) - 10(P + Q) \Rightarrow 6(P) \sim 10(P + Q).$$

$Q = (-\frac{1}{h^2}, 0)$  has order 2.

$$\phi : E_{3\sqrt{2}} \rightarrow E_{i\sqrt{2}} \quad (X, Y) \rightarrow (-X, iY)$$

$$r_{i\sqrt{2}}(\{x, y\}) = r_{3\sqrt{2}}(\{x \circ \phi, y \circ \phi\})$$



But

$$(x \circ \phi) \diamond (y \circ \phi) = 8(P + Q)$$

$$(x) \diamond (y) = 8(P)$$

$$6r_{3\sqrt{2}}(\{x, y\}) = 10r_{i\sqrt{2}}(\{x, y\})$$

and

$$3m(3\sqrt{2}) = 5m(i\sqrt{2}).$$

Consequently,

$$m(8) = \frac{8}{5}m(3\sqrt{2})$$

$$m(2) = \frac{2}{5}m(3\sqrt{2})$$



## Other families

- Hesse family

$$h(a^3) = m \left( x^3 + y^3 + 1 - \frac{3xy}{a} \right)$$

(studied by Rodriguez-Villegas 1997)

$$h(u^3) = \sum_{j=0}^2 h \left( 1 - \left( \frac{1 - \xi_3^j u}{1 + 2\xi_3^j u} \right)^3 \right) \quad |u| \text{ small}$$

- More complicated equations for examples studied by Stienstra 2005:

$$m \left( (x+1)(y+1)(x+y) - \frac{xy}{t} \right)$$

and Bertin 2004, Zagier < 2005, and Stienstra 2005:

$$m \left( (x+y+1)(x+1)(y+1) - \frac{xy}{t} \right)$$

