On the Mahler measure of resultants in small dimensions

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Mahler measure of multivariate polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) \; = \; \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n}$$

$$\mathbb{T}^n = S^1 \times \cdots \times S^1$$

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Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1)$$

$$\mathrm{L}(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

Mixed sparse resultant

$\mathcal{A}_0, \ldots, \mathcal{A}_n \subset \mathbb{Z}^n$, $\mathcal{A}_i := \{a_{ij}\}_{j=1,\ldots,k_i}$.

$$F_i(t_1, \ldots, t_n) := \sum_{j=1}^{k_i} x_{ij} \mathbf{t}^{a_{ij}} = 0 \quad i = 0, \ldots, n \tag{1}$$

$\mathbf{t}^a = t_1^{a_1} t_2^{a_2} \ldots t_n^{a_n}$ for $a = (a_1, \ldots, a_n)$.

$\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n} \in \mathbb{Z}[x_{ij}]$ irreducible polynomial.

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$\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n} \in \mathbb{Z}[x_{ij}]$ irreducible polynomial.

It vanishes on a specialization of the x_{ij} in K (alg. closed) if the system (1) has a common solution in $(K \setminus \{0\})^n$.

Examples

- $\mathcal{A}_0 = \{0, \ldots, d_0\}$, $\mathcal{A}_1 = \{0, \ldots, d_1\} \subset \mathbb{Z}$

  $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1}$ = Sylvester resultant of two polynomials of degree d_0 and d_1.

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- $\mathcal{A}_0 = \mathcal{A}_1 = \cdots = \mathcal{A}_n = \{(0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$

  $$x_{i0} + x_{i1} t_1 + \cdots + x_{in} t_n = 0 \quad i = 0, \ldots, n$$

  $\mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n} = \det(x_{ij})$.

Some previous work

- Sombra (2004) bounds $h(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}), m(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n})$ in terms of mixed volumes.

$$H(\mathrm{Res}(f_{(m)}, g_{(n)})) \leq (m+1)^n (n+1)^m$$

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- D'Andrea and Hare (2004) computed $H(\mathrm{Res}(f,g))$ when $\deg(f) = 2$ and found a tight estimate when $\deg(f) = 3$.

$$H(\mathrm{Res}(f_0 + f_1 x + f_2 x^2, g_{(n)})) \sim \frac{2.3644}{\sqrt{n\pi}} 1.6180^n - O\left(\frac{1.6180^n}{n\sqrt{n}}\right)$$

$$H(\mathrm{Res}(f_0 + f_1 x + f_2 x^2 + f_3 x^3, g_{(n)})) \sim \frac{8.13488}{n\pi} 1.83928^n - O\left(\frac{1.83928^n}{n^2}\right)$$

$m(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n})$ when $\dim(N(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}))$ is low.

Theorem

(Sturmfelds (1994))

$$\dim(N(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n})) = k - 2n - 1,$$

where $k = \sum_{i=0}^{n} k_i$.

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Theorem

$$m(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}) = 0 \iff \dim(N(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n})) = 1.$$

$$x_{j1} t_j^{\eta_j} - x_{j2} \qquad j = 0, \ldots, n$$

$\dim(N(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n})) = 2$

$k_0 = 3, k_1 = \dots = k_n = 2.$

$$\begin{array}{rcl} F_0(t_1,\dots,t_n) & = & x_{01}\mathbf{t}^{a_{01}} + x_{02}\mathbf{t}^{a_{02}} + x_{03}\mathbf{t}^{a_{03}} \\ F_1(t_1,\dots,t_n) & = & x_{11}{t_1}^{\eta_1} - x_{12}, \\ \dots & \dots & \dots \\ F_n(t_1,\dots,t_n) & = & x_{n1}{t_n}^{\eta_n} - x_{n2}. \end{array}$$

Theorem

$$m(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}) = \eta\, \mathrm{L}'(\chi_{-3}, -1),$$

where $\eta := \eta_1 + \eta_2 + \ldots + \eta_n.$

$$
\begin{array}{lcl}
F_0 & = & x_{01}\mathbf{t}^{a_{01}} + x_{02}\mathbf{t}^{a_{02}} + \ldots + x_{0\ell}\mathbf{t}^{a_{0l}}, \\
F_1 & = & x_{11}\,{t_1}^{\eta_1} - x_{12}, \\
\ldots & \ldots & \ldots \\
F_n & = & x_{n1}\,{t_n}^{\eta_n} - x_{n2}.
\end{array}
$$

Theorem

$$m(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}) = \eta\, m(1 + s_1 + s_2 + \ldots + s_{\ell-1}).$$

$\dim(N(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n})) = 3$

1. $k_0 = 4,\ k_1 = k_2 = \ldots = k_n = 2$. As before,

$$m(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}) = \eta m(1 + s_1 + s_2 + s_3) = \eta \frac{7}{2\pi^2}\zeta(3).$$

2. $k_0 = k_1 = 3,\ k_2 = k_3 = \ldots = k_n = 2$.

$$
\begin{array}{lcl}
F_0 & = & x_{01}\mathbf{t}^{a_{01}} + x_{02}\mathbf{t}^{a_{02}} + x_{03}\mathbf{t}^{a_{03}}, \\
F_1 & = & x_{11}\mathbf{t}^{a_{11}} + x_{12}\mathbf{t}^{a_{12}} + x_{13}\mathbf{t}^{a_{13}}, \\
F_2 & = & x_{21}\, t_2{}^{\eta_2} - x_{22}, \\
\dots & \dots & \dots \\
F_n & = & x_{n1}\, t_n{}^{\eta_n} - x_{n2}.
\end{array}
$$

Can be reduced to

Theorem

$$m(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}) = \eta\, m(\mathrm{Res}_{\mathcal{A}'_0,\mathcal{A}'_1})$$

where

$$\mathcal{A}'_0 := \{\alpha_{01}, \alpha_{02}, \alpha_{03}\}, \mathcal{A}'_1 := \{\alpha_{11}, \alpha_{12}, \alpha_{13}\}$$

and $\alpha_{ij} \in \mathbb{Z}$ *is the first coordinate of the vector* $a_{ij},\ i = 0, 1,\ j = 1, 2, 3.$

Theorem

Let $\mathcal{A}'_0 = \mathcal{A}'_1 = \{0, p, q\}$, *with* $p < q$ *and* $\gcd(p, q) = 1$.
Then,

$$m(\mathrm{Res}_{\mathcal{A}'_0,\mathcal{A}'_1}) = \frac{2}{\pi^2}\left(-p\mathcal{L}_3(\varphi^q) - q\mathcal{L}_3(-\varphi^p) + p\mathcal{L}_3\left(\phi^q\right) + q\mathcal{L}_3\left(\phi^p\right)\right)$$

φ *root of* $x^q + x^{q-p} - 1 = 0$ *s.t.* $0 \leq \varphi \leq 1$,
ϕ *root of* $x^q - x^{q-p} - 1 = 0$ *s.t.* $1 \leq \phi$.

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$$\mathcal{L}_3(z) = \mathrm{Re}\left(\mathrm{Li}_3(z) - \log|z|\mathrm{Li}_2(z) + \frac{1}{3}\log^2|z|\mathrm{Li}_1(z)\right).$$

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty}\frac{x^n}{n^k}$$

$$m(\mathrm{Res}_{\{\{0,1,2\},\{0,1,2\}\}}) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\pi^2\zeta(3)}$$

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Use

$$z - \frac{(1-x)^p(1-y)^{q-p}}{(1-xy)^q}$$

An example in dimension 4

$$\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A} := \{(0,0),(1,0),(0,1)\}.$$

(3x3 determinant)

Theorem

$$m(\mathrm{Res}_{\mathcal{A},\mathcal{A},\mathcal{A}}) = \frac{9\zeta(3)}{2\pi^2}$$

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Use

$$(x-1)(y-1)-(z-1)(w-1)$$

Why should we expect such values?

$\mathcal{X} := \{\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n} = 0\} \subset \mathbb{C}^k$.

Theorem

The symbol

$$\{x_{01},\dots,x_{0k_0},\dots,x_{n1},\dots,x_{nk_n}\} \in K_k^M(\mathbb{C}(\mathcal{X}))_{\mathbb{Q}}$$

is trivial.