On the Mahler measure of resultants in small dimensions

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Mahler measure of multivariate polynomials

Definition 1 *For* $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, *the (logarithmic)* Mahler measure *is defined by*

$$m(P) \;=\; \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n},$$

$$\mathbb{T}^n = S^1 \times \dots \times S^1.$$

For example, Smyth [Smy1] computed

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1)$$

$$\mathrm{L}(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}.$$

The mixed sparse resultant

Let $\mathcal{A}_0, \dots, \mathcal{A}_n \subset \mathbb{Z}^n$ be finite sets of integer vectors, $\mathcal{A}_i := \{a_{ij}\}_{j=1,\dots,k_i}$, which jointly span the lattice $\mathbb{Z}^n$.

Consider the system

$$F_i(t_1, \dots, t_n) := \sum_{j=1}^{k_i} x_{ij} \mathbf{t}^{a_{ij}} = 0 \quad i = 0, \dots, n \tag{1}$$

of Laurent polynomials, where $\mathbf{t}^a = t_1^{a_1} t_2^{a_2} \dots t_n^{a_n}$ for $a = (a_1, \dots, a_n)$.

The associated mixed sparse resultant $\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n} \in \mathbb{Z}[X_0, \dots, X_n]$ is an irreducible polynomial in $n+1$ groups $X_i := \{x_{ij};\, 1 \le j \le k_i\}$ of k_i variables each. The resultant vanishes on a specialization of the x_{ij} in an algebraically closed field K iff the system (1) has a common solution in $(K \setminus \{0\})^n$. See definitions in [CLO, Stu],

Examples

- If we choose, $\mathcal{A}_0 = \{0, \dots, d_0\}$, $\mathcal{A}_1 = \{0, \dots, d_1\} \subset \mathbb{Z}$, then $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1}$ is the Sylvester resultant of two univariate polynomials of degree d_0 and d_1.

- If we choose $\mathcal{A}_0 = \mathcal{A}_1 = \ldots = \mathcal{A}_n = \{(0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$, then we obtain a system of linear equations

$$x_{i0} + x_{i1}t_1 + \ldots + x_{in}t_n = 0 \quad i = 0, \ldots, n$$

and $\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n} = \det(x_{ij})$.

Previous work

Some previous work include

- **Theorem 2** *(Sombra [Som])*

$$h(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}), m(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}) \leq \frac{1}{[\mathbb{Z}^n : L_{\mathcal{A}}]} \sum_{i=0}^{n} MV(Q_0, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n) \log(\#\mathcal{A}_i)$$

Where $L_{\mathcal{A}} \subset \mathbb{Z}^n$ denotes the $\mathbb{Z}$-module spanned by the pointwise sum $\sum_{i=0}^{n} \mathcal{A}_i$, Q_i is the convex hull of $\mathcal{A}_i$, and MV denotes the mixed volume function.

Implies, for the Sylvester resultant,

$$H(\mathrm{Res}(f_{(m)}, g_{(n)})) \leq (m+1)^n (n+1)^m$$

- D'Andrea and Hare [DH] computed the height of the Sylvester resultant of two polynomials, $H(\mathrm{Res}(f, g))$, when one of the polynomials is quadratic and found a tight estimate when one of the polynomials is cubic. In particular,

$$H(\mathrm{Res}(f_0 + f_1 x + f_2 x^2, g_{(n)})) \sim \frac{2.3644}{\sqrt{n\pi}} 1.6180^n - O\left(\frac{1.6180^n}{n\sqrt{n}}\right)$$

$$H(\mathrm{Res}(f_0 + f_1 x + f_2 x^2 + f_3 x^3, g_{(n)})) \sim \frac{8.13488}{n\pi} 1.83928^n - O\left(\frac{1.83928^n}{n^2}\right)$$

Mahler measure and heights differ

Mahler measures and heights behave completely different in resultants. For instance, take $n = 1$, $\mathcal{A}_0 = \{0, 1\}$, $\mathcal{A}_1 = \{0, 1, \ldots, \ell\}$, then

$$\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1} = \pm \sum_{j=0}^{\ell} (-1)^j x_{1j} x_{00}^{\ell-j} x_{01}^j,$$

so $h(\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1}) = 0$.

But setting $y_j = (-1)^j x_{1j} x_{00}^{\ell-j} x_{01}^j$,

$$m(\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1}) = m\left(\sum_{j=0}^{\ell} y_j\right).$$

$m(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n})$ when the Newton polytope has low dimension

Our work focuses on explicit computations for cases when $N(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n})$ has low dimension. We use the following Theorem

Theorem 3 *(Sturmfelds, [Stu])*

$$\dim(N(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n})) = k - 2n - 1,$$

where $k = \sum_{i=0}^{n} k_i$.

Using this Theorem we can prove

- **Theorem 4**

$$m(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}) = 0 \iff \dim(N(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n})) = 1.$$

  (Because the system must be like $x_{j1}t_j^{\eta_j} - x_{j2}$ *for* $j = 0, \dots, n$*).*

- When $\dim(N(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n})) = 2$, we can assume $k_0 = 3, k_1 = \dots = k_n = 2$ by the Theorem above. We can think of the system as

$$\begin{array}{rcl} F_0(t_1,\dots,t_n) & = & x_{01}\mathbf{t}^{a_{01}} + x_{02}\mathbf{t}^{a_{02}} + x_{03}\mathbf{t}^{a_{03}} \\ F_1(t_1,\dots,t_n) & = & x_{11}{t_1}^{\eta_1} - x_{12}, \\ \dots & \dots & \dots \\ F_n(t_1,\dots,t_n) & = & x_{n1}{t_n}^{\eta_n} - x_{n2}. \end{array} \tag{2}$$

  Let $\eta := \eta_1 + \eta_2 + \dots + \eta_n$, then

  Theorem 5 *For systems having support as in (2),*

$$m(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}) = \eta\, \mathrm{L}'(\chi_{-3}, -1).$$

  Theorem 6 *With the notation established above, for systems as*

$$\begin{array}{rcl} F_0 & = & x_{01}\mathbf{t}^{a_{01}} + x_{02}\mathbf{t}^{a_{02}} + \dots + x_{0\ell}\mathbf{t}^{a_{0l}}, \\ F_1 & = & x_{11}{t_1}^{\eta_1} - x_{12}, \\ \dots & \dots & \dots \\ F_n & = & x_{n1}{t_n}^{\eta_n} - x_{n2}. \end{array} \tag{3}$$

  we have

$$m(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}) = \eta\, m(1 + s_1 + s_2 + \dots + s_{\ell-1}).$$

  Asymptotics for these Mahler measures were studied in [Smy1, R-VTV].

- Finally, we study the case where $\dim(N(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n})) = 3$. We have two possibilities:

  1. $k_0 = 4$, $k_1 = k_2 = \dots = k_n = 2$. This is a system of the form (3), and hence we have that
$$m(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}) = \eta\, \frac{7}{2\pi^2}\zeta(3),$$
  2. $k_0 = k_1 = 3$, $k_2 = k_3 = \dots = k_n = 2$.

$$
\begin{array}{lcl}
F_0 & = & x_{01}\mathbf{t}^{a_{01}} + x_{02}\mathbf{t}^{a_{02}} + x_{03}\mathbf{t}^{a_{03}}, \\
F_1 & = & x_{11}\mathbf{t}^{a_{11}} + x_{12}\mathbf{t}^{a_{12}} + x_{13}\mathbf{t}^{a_{13}}, \\
F_2 & = & x_{21}{t_2}^{\eta_2} - x_{22}, \\
\dots & \dots & \dots \\
F_n & = & x_{n1}{t_n}^{\eta_n} - x_{n2}.
\end{array}
\tag{4}
$$

This case can be reduced to

Theorem 7

$$m(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}) = \eta\, m(\mathrm{Res}_{\mathcal{A}'_0,\mathcal{A}'_1})$$

where

$$\mathcal{A}'_0 := \{\alpha_{01},\alpha_{02},\alpha_{03}\}, \mathcal{A}'_1 := \{\alpha_{11},\alpha_{12},\alpha_{13}\}$$

and $\alpha_{ij} \in \mathbb{Z}$ *is the first coordinate of the vector* $a_{ij},\ i = 0,1,\ j = 1,2,3.$

$m(\mathrm{Res}_{\mathcal{A}'_0,\mathcal{A}'_1})$ seems to be hard to compute. We have obtained the following partial result:

Theorem 8 *Suppose that* $\mathcal{A}'_0 = \mathcal{A}'_1$*, both having cardinality three, w.l.o.g. we can suppose that* $\mathcal{A}'_0 = \{0,p,q\}$*, with* $p < q$ *and* $\gcd(p,q) = 1$.

Then,

$$m(\mathrm{Res}_{\mathcal{A}'_0,\mathcal{A}'_1}) = \frac{2}{\pi^2}\left(-p\mathcal{L}_3(\varphi^q) - q\mathcal{L}_3(-\varphi^p) + p\mathcal{L}_3\left(\phi^q\right) + q\mathcal{L}_3\left(\phi^p\right)\right)$$

where φ *is the real root of* $x^q + x^{q-p} - 1 = 0$ *such that* $0 \le \varphi \le 1$*, and* ϕ *is the real root of* $x^q - x^{q-p} - 1 = 0$ *such that* $1 \le \phi$*. Finally,*

$$\mathcal{L}_3(z) = \mathrm{Re}\left(\mathrm{Li}_3(z) - \log|z|\mathrm{Li}_2(z) + \frac{1}{3}\log^2|z|\mathrm{Li}_1(z)\right).$$

is a modified version of the trilogarithm.

In particular,

$$m(\mathrm{Res}_{\{\{0,1,2\},\{0,1,2\}\}}) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\pi^2\zeta(3)}$$

The proof rests in writing the resultant as $z - \frac{(1-x)^p(1-y)^{q-p}}{(1-xy)^q}$.

- We studied also an example in dimension 4. Take $n = 2$ and

$$\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A} := \{(0,0),(1,0),(0,1)\}.$$

Then the resultant is the 3x3 determinant. We have

Theorem 9

$$m(\mathrm{Res}_{\mathcal{A},\mathcal{A},\mathcal{A}}) = \frac{9\zeta(3)}{2\pi^2}.$$

The proof consists in writing the resultant as $(x-1)(y-1)-(z-1)(w-1)$.

Why should we expect such values?

Let $\mathcal{X}$ be the irreducible surface in $\mathbb{C}^k$ defined by $\mathcal{X} := \{\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n} = 0\}$.

Theorem 10 *The symbol*

$$\{x_{01},\dots,x_{0k_0},\dots,x_{n1},\dots,x_{nk_n}\} \in K_k^M(\mathbb{C}(\mathcal{X}))_{\mathbb{Q}} \tag{5}$$

is trivial.

This implies that the tame symbols of the facets are trivial and that the first regulator is exact. This is the first step that may led to a Mahler measure involving special values of polylogarithms [RV, Lal2].

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