Some aspects of higher Mahler measure

Matilde Lalín

Université de Montréal
mlalin@dms.umontreal.ca
http://www.dms.umontreal.ca/~mlalin

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Mahler measure of one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n\to\infty} \frac{|\alpha^{n+1}-1|}{|\alpha^n-1|} = \begin{cases} |\alpha| & \text{if } |\alpha|>1 \\ 1 & \text{if } |\alpha|<1 \end{cases}$$

For

$$P(x) = a\prod_i (x-\alpha_i)$$

$$M(P) = |a|\prod_i \max\{1,|\alpha_i|\}$$

$$m(P) = \log M(P) = \log|a| + \sum_i \log^+|\alpha_i|$$

Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^n \prod \phi_i(x)$$

Lehmer's question

Lehmer (1933):
Given $\varepsilon > 0$*, can we find a polynomial* $P(x) \in \mathbb{Z}[x]$ *such that* $0 < m(P) < \varepsilon$*?*

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \cong 0.162357612\ldots$$

Is the above polynomial the best possible?.

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

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$P \in \mathbb{C}[x]$ reciprocal iff

$$P(x) = \pm x^{\deg P} P\left(x^{-1}\right).$$

Smyth (1971):
$P \in \mathbb{Z}[x]$ nonreciprocal,

$$m(P) \geq m(x^3 - x - 1) \cong 0.2811995743\ldots.$$

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Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

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Properties

- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$
- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

where h is the logarithmic Weil height.

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Boyd & Lawton Theorem

$P \in \mathbb{C}[x_1, \ldots, x_n]$

$$\lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \ldots, x^{k_n})) = m(P(x_1, x_2, \ldots, x_n))$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

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Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$
- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

L. (2006)

- $$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5)$$
- Known formulas for
$$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

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Why do we get nice numbers?

The relationship with regulators

Deninger (1997)

$$m(P) = \text{easy term} + \frac{1}{(2\mathrm{i}\pi)^{n-1}} \int_\Gamma \eta_n(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \dots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$\eta_n(x_1, \dots, x_n)$ is a $\mathbb{R}(n-1)$-valued smooth $n-1$-form in $\{P = 0\}$.

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Regulators

Encode special values of L-functions.

Example: Dirichlet class number formula

$$\lim_{s\to 1}(s-1)\zeta_F(s)=\frac{2^{r_1}(2\pi)^{r_2}h_F\,\mathrm{reg}_F}{\omega_F\sqrt{|D_F|}},$$

$$\lim_{s\to 0} s^{1-r_1-r_2}\zeta_F(s)=-\frac{h_F\,\mathrm{reg}_F}{\omega_F}.$$

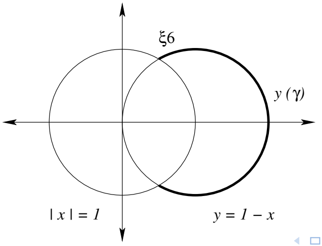
$$m(x+y+1) = -\frac{1}{2\pi}\int_\gamma \eta(x,y)$$

$$\eta(x,y) = \log|x| \mathrm{d} \arg y - \log|y| \mathrm{d} \arg x$$

$$\eta(x, 1-x) = \mathrm{d}D(x)$$

Use Stokes Theorem:

$$2\pi m(x+y+1) = D(\xi_6) - D(\bar{\xi_6}) = \frac{3\sqrt{3}}{2}\mathrm{L}(\chi_{-3}, 2)$$



An algebraic integration for Mahler measure

Rodriguez-Villegas (1997), L. (2007):

- Explains most known cases involving ζ, $\mathrm{L}(\chi)$ using $\mathrm{Li}_k(x)$ and Borel's Theorem in K-theory.

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$$

- It is constructive (no need of "happy idea" integrals).
- Conjecture for n-variables using Goncharov's regulator currents. Provides motivation for Goncharov's construction.
- Key use of Jensen's formula

$$m(x-\alpha) = \log^+ |\alpha|$$

Higher Mahler measure

For $k \in \mathbb{Z}_{\geq 0}$, and $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the k-higher *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log^k |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log^k |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

$$k = 1: \qquad m_1(P) = m(P),$$

and

$$m_0(P) = 1.$$

The simplest examples

Kurokawa, L., Ochiai (2008):

$$
\begin{aligned}
m_2(x-1) &= \frac{\zeta(2)}{2} = \frac{\pi^2}{12}.\\
m_3(x-1) &= -\frac{3\zeta(3)}{2}.\\
m_4(x-1) &= \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}.\\
m_5(x-1) &= -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}.\\
m_6(x-1) &= \frac{45}{2}\zeta(3)^2 + \frac{275}{1344}\pi^6.
\end{aligned}
$$

$$\sum_{k=0}^{\infty}\frac{m_k(x-1)}{k!}s^k = \int_0^1 \left|e^{2\pi i\theta}-1\right|^s d\theta$$

$$= \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2} = \exp\left(\sum_{k=2}^{\infty}\frac{(-1)^k(1-2^{1-k})\zeta(k)}{k}s^k\right).$$

Transform the integral into a Beta function, then use the Weierstrass product of the Γ-function.

Multiple Mahler measure

Let $P_1, \dots, P_k \in \mathbb{C}[x^{\pm}]$ be non-zero Laurent polynomials. Their multiple higher Mahler measure is defined by

$$m(P_1, \dots, P_k) = \int_0^1 \log \left| P_1\left(\mathrm{e}^{2\pi \mathrm{i}\theta}\right)\right| \cdots \log \left| P_k\left(\mathrm{e}^{2\pi \mathrm{i}\theta}\right)\right| \mathrm{d}\theta.$$

$$m_2\left(\prod_i (x - r_i)\right) = \sum_{i,j} m(x - r_i, x - r_j).$$

Kurokawa, L., Ochiai (2008):
For $0 \leq \alpha \leq 1$

$$m(x-1, x-\mathrm{e}^{2\pi \mathrm{i}\alpha}) = \frac{\pi^2}{2}\left(\alpha^2 - \alpha + \frac{1}{6}\right).$$

m_2 of cyclotomic polynomials

L.,Sinha (2010):

- For any two positive integers a and b,

$$m(x^a-1,x^b-1)=\frac{\pi^2}{12}\frac{(a,b)^2}{ab}.$$

- For a positive integer n, let $\phi_n(x)$ denote the n-th cyclotomic polynomial and φ Euler's function. Then

$$m_2(\phi_n(x))=\frac{\pi^2}{12}\frac{\varphi(n)2^{r(n)}}{n},$$

where $r(n)$ denotes the number of distinct prime divisors of n.

An example in two variables

Kurokawa, L., Ochiai (2008):

$$m_2(x+y+1) = \frac{7\pi^2}{54} = \frac{7}{9}\zeta(2)$$

Smyth (1981):

$$m(x+y+1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

The case $P = x + y + c$

Let $c \geq 2$.

$$\int_0^1 \int_0^1 |e^{2\pi i\theta} + e^{2\pi i\tau} + c|^s d\theta d\tau = c^s \sum_{j=0}^{\infty} \binom{s/2}{j}^2 \frac{1}{c^{2j}} \binom{2j}{j}.$$

$$= c^s {}_3F_2\left(\begin{array}{c} -\frac{s}{2}, -\frac{s}{2}, \frac{1}{2} \\ 1, 1 \end{array} \middle| \frac{4}{c^2} \right),$$

where the generalized hypergeometric series ${}_3F_2$ is defined by

$${}_3F_2\left(\begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \middle| z \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j (a_3)_j}{(b_1)_j (b_2)_j j!} z^j,$$

with the Pochhammer symbol defined by $(a)_j = a(a+1)\cdots(a+j-1)$.

In particular, we obtain the special values

- $$m_2(x+y+2) = \frac{\zeta(2)}{2}.$$
- $$m_3(x+y+2) = \frac{9}{2}(\log 2)\zeta(2) - \frac{15}{4}\zeta(3).$$

Why do we get these vales?

Massey product in Deligne cohomology?

Lehmer's question for even higher measure

L., Sinha (2010):

If $P(x) \in \mathbb{Z}[x]$, then for any $h \geq 1$,

$$m_{2h}(P) \geq \begin{cases} \left(\frac{\pi^2}{12}\right)^h, & \text{if } P(x) \text{ is reciprocal,} \\ \left(\frac{\pi^2}{48}\right)^h, & \text{if } P(x) \text{ is non-reciprocal.} \end{cases}$$

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- $m_2(P)$ for P reciprocal is minimized when P is a product of monomials and cyclotomic polynomials.
- $m_2(P) \geq \frac{\pi^2}{12}$ for P a product of monomials and cyclotomic polynomials.
- For P nonreciprocal, take $P(x)x^{\deg P}P\left(x^{-1}\right)$.
- For $m_{2h}(P)$, use Hölder's inequality.

m_2 of noncyclotomic polynomials

$P(x)$	$m(P)$	$m_2(P)$
$x^8+x^5-x^4+x^3+1$	0.2473585132	1.0980813745
$x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$	0.1623576120	1.7447964556
$x^{10}-x^6+x^5-x^4+1$	0.1958888214	1.2863292447
$x^{10}+x^7+x^5+x^3+1$	0.2073323581	1.2320444893
$x^{10}-x^8+x^5-x^2+1$	0.2320881973	1.1704950485
$x^{10}+x^8+x^7+x^5+x^3+x^2+1$	0.2368364616	1.1914083866
$x^{10}+x^9-x^5+x+1$	0.2496548880	1.0309287773
$x^{12}+x^{11}+x^{10}-x^8-x^7-x^6-x^5-x^4+x^2+x+1$	0.2052121880	1.4738375004
$x^{12}+x^{11}+x^{10}+x^9-x^6+x^3+x^2+x+1$	0.2156970336	1.5143823478
$x^{12}+x^{11}-x^7-x^6-x^5+x+1$	0.2239804947	1.2059443050
$x^{12}+x^{10}+x^7-x^6+x^5+x^2+1$	0.2345928411	1.2434560052
$x^{12}+x^{10}+x^9+x^8+2x^7+x^6+2x^5+x^4+x^3+x^2+1$	0.2412336268	1.6324129051
$x^{14}+x^{11}-x^{10}-x^7-x^4+x^3+1$	0.1823436598	1.3885013172
$x^{14}-x^{12}+x^7-x^2+1$	0.1844998024	1.3845721865
$x^{14}-x^{12}+x^{11}-x^9+x^7-x^5+x^3-x^2+1$	0.2272100851	1.4763006621
$x^{14}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+1$	0.2351686174	1.4352060397
$x^{14}+x^{13}-x^8-x^7-x^6+x+1$	0.2368858459	1.2498299096
$x^{14}+x^{13}+x^{12}-x^9-x^8-x^7-x^6-x^5+x^2+x+1$	0.2453300143	1.3362661982
$x^{14}+x^{13}-x^{11}-x^7-x^3+x+1$	0.2469561884	1.3898540050

Known noncyclotomic $P \in \mathbb{Z}[x]$, $\deg P \leq 14$, $m(P) < 0.25$.

(Mossinghoff: `http://www.cecm.sfu.ca/~mjm/Lehmer/search/`)

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m_3

Lemma (Kurokawa, L., Ochiai)

$$
\begin{aligned}
m(x-1, x-e^{2\pi i\alpha}, x-e^{2\pi i\beta}) = & -\frac{1}{4}\sum_{1\le k,l}\frac{\cos 2\pi((k+l)\beta - l\alpha)}{kl(k+l)}\\
& -\frac{1}{4}\sum_{1\le k,m}\frac{\cos 2\pi((k+m)\alpha - m\beta)}{km(k+m)}\\
& -\frac{1}{4}\sum_{1\le l,m}\frac{\cos 2\pi(l\alpha + m\beta)}{lm(l+m)}.
\end{aligned}
$$

$$m_3\left(\frac{x^n-1}{x-1}\right)=\frac{3}{2}\zeta(3)\left(\frac{-2+3n-n^3}{n^2}\right)+\frac{3\pi}{2}\sum_{\substack{j=1\\ n\nmid j}}^{\infty}\frac{\cot\left(\pi\frac{j}{n}\right)}{j^2}.$$

Examples

$$m_3\left(x^2+x+1\right)=-\frac{10}{3}\zeta(3)+\frac{\sqrt{3}\pi}{2}L(2,\chi_{-3}).$$

$$m_3\left(x^3+x^2+x+1\right)=-\frac{81}{16}\zeta(3)+\frac{3\pi}{2}L(2,\chi_{-4}).$$

Theorem

Let $P_n(x) = \frac{x^n - 1}{x - 1}$. *For* $h \geq 1$ *fixed,*

$$\lim_{n \to \infty} m_{2h+1}(P_n) = 0.$$

Moreover, the sequence $m_{2h+1}(P_n)$ *is nonconstant.*

- $$\lim_{n\to\infty} m_{2h+1}\left(\frac{x^n-1}{x-1}\right) = m_{2h+1}\left(\frac{y-1}{x-1}\right) = 0.$$
- $$m_{2h+1}\left(\frac{x^n-1}{x-1}\right) = C(h)\frac{\log^{2h-1} n}{n}\left(1+O\left(\log^{-1} n\right)\right).$$

  (suggested by Soundararajan)

Summing up

- m_k yields interesting special values.
- Lehmer's question FALSE for m_{2h}, $h \geq 1$.
- Lehmer's question TRUE for m_{2h+1}, $h \geq 1$.
- $m_k(P)$ is interesting for P cyclotomic and that answers Lehmer's question.

Further questions

- $m_3(P)$ for P noncyclotomic?
- Lehmer's question for P noncyclotomic.
- Bounds for $m_{2h}(P)$, $h \geq 2$.
- Explain zeta values!!!