There's something about Mahler measure

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Mahler measure

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) = \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n$$

$$= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n}$$

Examples in several variables L (2003)

$$\pi^n m \left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_n}{1+x_n}\right) z\right)$$

= combination of ζ(odd) / L(χ_{-4}, even)

$$\pi^n m \left(1 + x + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_n}{1+x_n}\right) (1+y)z\right)$$

= combination of ζ(odd) / L(χ_{-4}, even), polylogarithms

$$\pi^n m \left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_n}{1+x_n}\right) x \right.$$

$$\left. + \left(1 - \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_n}{1+x_n}\right)\right) y\right)$$

= combination of ζ(odd)

Examples

$$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right)$$

$$= 24\mathsf{L}(\chi_{-4},4)+\pi^2\mathsf{L}(\chi_{-4},2)$$

$$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_4}{1+x_4}\right)z\right)$$

$$= 62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

Polylogarithms

The k*th polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$P_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number, $\mathrm{Li}_0(x) \equiv -\frac{1}{2}$,

$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

P_k satisfies lots of functional equations

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1} P_k(x) \qquad P_k(\bar{x}) = (-1)^{k-1} P_k(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1 - x) \log |x|$$

Five-term relation

$$D(x)+D(1-xy)+D(y)+D\left(\frac{1-y}{1-xy}\right)+D\left(\frac{1-x}{1-xy}\right)=0$$

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X

- L-function

- Finitely-generated abelian group $K = H^i_{\mathcal{M}}(X, \mathbb{Q}(j))$

- Regulator map

$$r_{\mathcal{D}} : H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \to H^i_{\mathcal{D}}(X, \mathbb{R}(j))$$

  $H^i_{\mathcal{D}}(X, \mathbb{R}(i))$ differential forms.

$$\text{reg} : K \to \mathbb{R} \qquad \text{reg}(\xi) = \int_{\text{cycle}} r_{\mathcal{D}}(\xi)$$

$$(K\,\text{rank}\,1) \qquad \text{L}'_X(0) \sim_{\mathbb{Q}^*} \text{reg}(\xi)$$

Dirichlet class number formula

F number field

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F \mathrm{reg}_F}{\omega_F \sqrt{|D_F|}}.$$

- $X = \mathcal{O}_F$ (the ring of integers)
- $\mathsf{L}_X = \zeta_F$
- $K = \mathcal{O}_F^*$

If F real quadratic field,

$$\zeta_F'(0) \sim_{\mathbb{Q}^*} \log|\epsilon| \qquad \epsilon \in \mathcal{O}_F^*$$

An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) : $P(x,y) \in \mathbb{C}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

$$\eta(x,y) = \log|x| \mathrm{d} \arg y - \log|y| \mathrm{d} \arg x$$

$$\eta(x, 1-x) = \mathrm{d} D(x)$$

Need $\{x,y\} = 0$ in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$. In $\bigwedge^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$,

$$x \wedge y = \sum_j r_j \ \ z_j \wedge (1 - z_j).$$

$$\int_\gamma \eta(x,y) = \sum r_j D(z_j)|_{\partial\gamma}$$

Big picture

$$\ldots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(C, \partial\gamma) \to K_2(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x, y)$ is exact, then $\{x, y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

  $\rightsquigarrow$ dilogarithms, zeta function

- $\partial\gamma = \emptyset$, then $\{x, y\} \in K_2(C)$. We have $\eta(x, y)$ is not exact.

  $\rightsquigarrow$ L-series of a curve

We may get combinations of both situations.

The three-variable case

$$P(x,y,z) = (1-x)+(1-y)z \qquad S = \{P(x,y,z)=0\}$$

$$m(P) = m(1-y)+\frac{1}{(2\pi \mathrm{i})^3}\int_{\mathbb{T}^3} \log\left|z - \frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \log^+\left|\frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \log|z| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \eta(x,y,z)$$

$$\Gamma = S \cap \{|x|=|y|=1, |z|\geq 1\}$$

$$\eta(x,y,z) = \log|x| \left(\frac{1}{3}\mathrm{d}\log|y|\,\mathrm{d}\log|z| - \mathrm{d}\arg y\,\mathrm{d}\arg z\right)$$

$$+\log|y| \left(\frac{1}{3}\mathrm{d}\log|z|\,\mathrm{d}\log|x| - \mathrm{d}\arg z\,\mathrm{d}\arg x\right)$$

$$+\log|z| \left(\frac{1}{3}\mathrm{d}\log|x|\,\mathrm{d}\log|y| - \mathrm{d}\arg x\,\mathrm{d}\arg y\right)$$

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} \wedge \frac{\mathrm{d}z}{z}\right)$$

Theorem 1

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x,y)$$

where

$$\omega(x,y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x)$$

Maillot: if $P \in \mathbb{Q}[x,y,z]$,

$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$

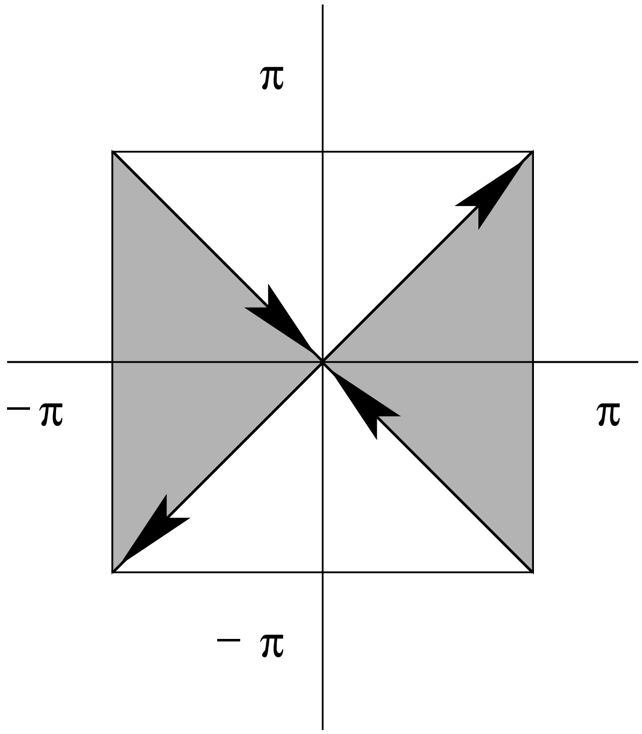
ω defined in

$$C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

Theorem 2

$$\omega(x,x) = \mathrm{d}P_3(x)$$

$$= \frac{1}{4\pi^2}8(P_3(1)-P_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_\Gamma \eta(x, y, z)$$

Need $\{x, y, z\} = 0$ in $K_3^M(\mathbb{C}(S)) \otimes \mathbb{Q}$.

$$x \wedge y \wedge z = \sum r_i \ \ x_i \wedge (1 - x_i) \wedge y_i$$

in $\bigwedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$, then

$$\int_\Gamma \eta(x, y, z) = \sum r_i \int_\Gamma \eta(x_i, 1 - x_i, y_i)$$

$$= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.

Then

$$\int_\gamma \omega(x,y) = \sum r_i \, P_3(x_i)|_{\partial\gamma}$$

The condition is $[x_i]_2 \otimes y_i$ is 0 in

$$H^2(B_{\mathbb{Q}(C)}(3) \otimes \mathbb{Q}) \overset{?}{\cong} K_4^{\{3\}}(\mathbb{C}(C))_{\mathbb{Q}}$$

Big picture II

$$\ldots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \ldots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\ldots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.

Studied examples

Smyth(1981):

$$\pi^2 m(1+x+y+z) = \frac{7}{2}\zeta(3)$$

Smyth(2002):

$$\pi^2 m(1+x+y^{-1}+(1+x+y)z) = \frac{14}{3}\zeta(3)$$

L (2003):

$$\pi^2 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)z\right) = 7\zeta(3)$$

$$\pi^2 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\right)y\right)$$

$$= \frac{7}{2}\zeta(3)+\frac{\pi^2\log 2}{2}$$

Condon (2003):

$$\pi^2 m\left(z-\left(\frac{1-x}{1+x}\right)(1+y)\right)=\frac{28}{5}\zeta(3)$$

D'Andrea & L (2003):

$$\pi^2 m\left(z(1-xy)^{m+n}-(1-x)^m(1-y)^n\right)$$

$$=2n(P_3(\phi_1^m)-P_3(-\phi_2^m))+2m(P_3(\phi_2^n)-P_3(-\phi_1^n))$$

$$\pi^2 m((1-x)(1-y)-(1-w)(1-z))=\frac{9}{2}\zeta(3)$$

L (2003):

$$\pi^2 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)x\right.$$

$$\left.+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\right)y\right)$$

$$=\frac{21}{4}\zeta(3)+\frac{\pi^2\log 2}{2}$$