Some aspects of the multivariable Mahler measure

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Mahler measure of polynomials

Definition 1 *For $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic)* Mahler measure *is defined by*

$$m(P) \quad = \quad \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \tag{1}$$

This integral is not singular and $m(P)$ always exists.
Because of Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \,\mathrm{d}\theta = \log^+ |\alpha| \tag{2}$$

[2]we have a simple expression for the Mahler measure of one-variable polynomials:

$$m(P) = \log |a_d| + \sum_{n=1}^{d} \log^+ |\alpha_n| \quad \text{for} \quad P(x) = a_d \prod_{n=1}^{d} (x - \alpha_n)$$

Properties

Here are some general properties (see [6])

- For $P, Q \in \mathbb{C}[x_1, \dots, x_n]$, we have $m(P \cdot Q) = m(P) + m(Q)$. Because of that, it makes sense to talk about the Mahler measure of rational functions.
- $m(P) \geq 0$ if P has integral coefficients.
- Mahler measure is related to heights. Indeed, if α is an algebraic number, and P_α is its minimal polynomial over $\mathbb{Q}$, then

  $$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha),$$

  where h is the logarithmic Weil height.
- By Kronecker's Lemma, $P \in \mathbb{Z}[x]$, $P \neq 0$, then $m(P) = 0$ if and only if P is the product of powers of x and cyclotomic polynomials.
- For $P \in \mathbb{C}[x_1, \dots, x_n]$

  $$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots x^{k_n})) = m(P(x_1, \dots x_n)) \tag{3}$$

  (this result is due to Boyd and Lawton see [1], [14]).

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[2]$\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

- Lehmer [15] studied this example in 1933:

  $$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)=\log(1.176280818\ldots)=0.162357612\ldots$$

  This example is special because its Mahler measure is very small, but still greater than zero.

  The following questions are still open: Is there a lower bound for positive Mahler measure of polynomials in one variable with integral coefficients? Does this degree 10 polynomial reach the lowest bound?

  Boyd–Lawton result implies that Lehmer's problem in several variables reduces to the one variable case.

Examples

For one-variable polynomials, the Mahler measure has to do with the roots of the polynomial. However, it is very hard to compute explicit formulas for examples in several variables. The first and simplest ones were computed by Smyth:

- Smyth [18]

  $$m(x+y+1)=\frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2)=\mathrm{L}'(\chi_{-3},-1), \tag{4}$$

  where

  $$\mathrm{L}(\chi_{-3},s)=\sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \quad \text{and} \quad \chi_{-3}(n)=\begin{cases} 1 & \text{if} \quad n\equiv 1 \bmod 3 \\ -1 & \text{if} \quad n\equiv -1 \bmod 3 \\ 0 & \text{if} \quad n\equiv 0 \bmod 3 \end{cases}$$

- Smyth [1]

  $$m(x+y+z+1)=\frac{7}{2\pi^2}\zeta(3). \tag{5}$$

- Boyd [2], Deninger [5], Rodriguez-Villegas [16]

  $$\begin{aligned} m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) &\stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{s_k} \qquad k\in\mathbb{N} \\ m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) &= 2\mathrm{L}'(\chi_{-4},-1) \\ m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) &= \mathrm{L}'(A,0) \end{aligned}$$

  Where B_k is a rational number, and E_k is the elliptic curve with corresponds to the zero set of the polynomial. When $k=4$ the curve has genus zero. When $k=4\sqrt{2}$ the elliptic curve is

  $$A: y^2=x^3-44x+112,$$

  which has complex multiplication.

- D'Andrea & L. (2003):

$$m\left(z(1-xy)^2-(1-x)(1-y)\right)=\frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)\pi^2}$$

- Boyd & L. (2005):

$$m(x^2+1+(x+1)y+(x-1)z)=\frac{\mathrm{L}(\chi_{-4},2)}{\pi}+\frac{21}{8\pi^2}\zeta(3)$$

- Bertin (2003)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}-k\right)=$$

  Kronecker-Eisenstein series

  L.(2003):

- $$m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right)=\frac{24}{\pi^3}\mathrm{L}(\chi_{-4},4)+\frac{\mathrm{L}(\chi_{-4},2)}{\pi}$$

- $$m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)=\frac{62}{\pi^4}\zeta(5)+\frac{14}{3\pi^2}\zeta(3)$$

- $$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)=\frac{24}{\pi^3}\mathrm{L}(\chi_{-4},4)$$

- $$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=\frac{93}{\pi^4}\zeta(5)$$

Polylogarithms

Many examples should be understood in the context of polylogarithms.

Definition 2 *The* kth polylogarithm *is the function defined by the power series*

$$\mathrm{Li}_k(x):=\sum_{n=1}^{\infty}\frac{x^n}{n^k}\qquad x\in\mathbb{C},\quad |x|<1. \tag{6}$$

This function can be continued analytically to $\mathbb{C}\setminus[1,\infty)$.

In order to avoid discontinuities, and to extend polylogarithms to the whole complex plane, several modifications have been proposed. Zagier [20] considers the following version:

$$\mathcal{L}_k(x):=\mathrm{Re}_k\left(\sum_{j=0}^{k-1}\frac{2^jB_j}{j!}(\log|x|)^j\mathrm{Li}_{k-j}(x)\right), \tag{7}$$

where B_j is the jth Bernoulli number, and Re_k denotes Re or Im depending on whether k is odd or even.

This function is one-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ and continuous in $\mathbb{P}^1(\mathbb{C})$. Moreover, $\mathcal{L}_k$ satisfy very clean functional equations. The simplest ones are

$$\mathcal{L}_k\left(\frac{1}{x}\right) = (-1)^{k-1}\mathcal{L}_k(x) \qquad \mathcal{L}_k(\bar{x}) = (-1)^{k-1}\mathcal{L}_k(x).$$

There are also lots of functional equations which depend on the index k. For instance, for $k = 2$, we have the Bloch–Wigner dilogarithm,

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

which satisfies the well-known five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0. \tag{8}$$

Beilinson's conjectures

There are several theorems and conjectures which predict that one may obtain global information from local information and that that relation is made through values of L-functions. These statements include the Dirichlet class number formula, the Birch–Swinnerton-Dyer conjecture, and more generally, Bloch's and Beilinson's conjectures.

Typically, there are four elements involved in this setting:

- An arithmetic-geometric object X (typically, an algebraic variety)
- its L-function (which codify local information)
- a finitely generated abelian group K, and
- a regulator map $K \to \mathbb{R}$.

When K has rank 1, Beilinson's conjectures predict that the $\mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$.

For instance, for a number field F, Dirichlet class number formula states that

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2}h_F\mathrm{reg}_F}{\omega_F\sqrt{|D_F|}}.$$

Here, $X = \mathcal{O}_F$ (the ring of integers), $\mathrm{L}_X = \zeta_F$, and the group is $\mathcal{O}_F^*$. Hence, when F is a real quadratic field, Dirichlet class number formula may be written as $\zeta'_F(0)$ is equal to, up to a rational number, $\log|\epsilon|$, for some $\epsilon \in \mathcal{O}_F^*$.

In general, the regulator is a function $r_{\mathcal{D}} : K_{2j-i}^{\{j\}}(X) \to H_{\mathcal{D}}^i(X, \mathbb{R}(j))$ where X is a complex projective variety. It was defined by Beilinson and it is hard to compute explictly.

An algebraic integration for Mahler measure

The appearance of the L-functions in Mahler measures formulas is a common phenomenon. Deninger [5] interpreted the Mahler measure as a Deligne period of a mixed motive. More specifically, in two variables, and under certain conditions, he proved that

$$m(P) = \mathrm{reg}(\xi_i),$$

where reg is the determinant of the regulator matrix, which we are evaluating in some class in an appropriate group in K-theory.

Rodriguez-Villegas [16] made explicit the relationship between Mahler measure and regulators by computing the regulator for the two-variable case, and using this machinery to explain the formulas for two variables. More precisely, he wrote, for $P(x,y) \in \mathbb{C}[x,y]$,

$$m(P) = m(P^*) - \frac{1}{2\pi}\eta(x,y)$$

where $\eta(x,y) = log|x| \,\mathrm{d}\arg y - log|y| \,\mathrm{d}\arg x$ is a closed differential for defined in $C = \{P(x,y)=0\}$ minus the set of poles and zeros of x and y. It staisfies $\eta(x, 1-x) = \mathrm{d}D(x)$.

For the three variable case, let us start with Smyth example, $P(x,y,z) = (1-x)+(1-y)z$. Then,

$$\begin{aligned} m(P) &= m(1-y) + \frac{1}{(2\pi \mathrm{i})^3}\int_{\mathbb{T}^3} \log\left|z - \frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}z}{z} \\ &= \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \log^+\left|\frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = -\frac{1}{(2\pi)^2}\int_{\Gamma} \log|z| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} \\ &= -\frac{1}{(2\pi)^2}\int_{\Gamma}\eta(x,y,z) \end{aligned}$$

where $\Gamma = \{P(x,y,z)=0\} \cap \{|x|=|y|=1, |z|\geq 1\}$, and

$$\eta(x,y,z) = \log|x|\left(\frac{1}{3}\,\mathrm{d}\log|y| \wedge \,\mathrm{d}\log|z| - \,\mathrm{d}\arg y \wedge \,\mathrm{d}\arg z\right)$$

$$+\log|y|\left(\frac{1}{3}\,\mathrm{d}\log|z| \wedge \,\mathrm{d}\log|x| - \,\mathrm{d}\arg z \wedge \,\mathrm{d}\arg x\right) + \log|z|\left(\frac{1}{3}\,\mathrm{d}\log|x| \wedge \,\mathrm{d}\log|y| - \,\mathrm{d}\arg x \wedge \,\mathrm{d}\arg y\right).$$

This is a closed differential form defined in $S = \{P(x,y,z)=0\}$ minus the set of zeros and poles of x, y and z. Now note that

$$\eta(x, 1-x, y) = \,\mathrm{d}w(x,y) \tag{9}$$

where

$$\omega(x,y) = -D(x)\,\mathrm{d}\arg y + \frac{1}{3}\log|y|(\log|1-x|\,\mathrm{d}\log|x| - \log|x|\,\mathrm{d}\log|1-x|).$$

In our case,

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x).$$

The computation of $\gamma = \partial\Gamma$ can be made in an efficient way (for polynomials with real coefficients) by applying certain ideas of Maillot. If $P \in \mathbb{R}[x,y,z]$, we can think of

$$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x|=|y|=1\}.$$

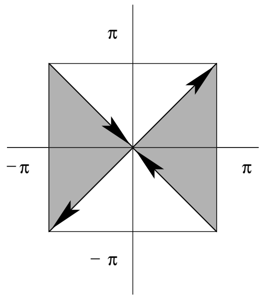
Now ω is defined in $C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$.

So,

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

leads to $C = \{x = y\} \cup \{xy = 1\}$ and

$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x).$$



The exactness of $\omega(x, y)$ is guaranteed by the condition

$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x) \tag{10}$$

where $\mathcal{L}_3(x)$ is Zagier's modified version of the trilogarithm:

$$\mathcal{L}_3(x) := \mathrm{Re}\left(\mathrm{Li}_3(x) - \log|x|\mathrm{Li}_2(x) + \frac{1}{3}\log^2|x|\mathrm{Li}_1(x)\right).$$

This leads to

$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}8(\mathcal{L}_3(1) - \mathcal{L}_3(-1)) = \frac{7}{2\pi^2}\zeta(3).$$

In general, let $P(x, y, z) \in \mathbb{C}[x, y, z]$, then we can write

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z) \tag{11}$$

where P^* is a two-variable polynomial which is the principal coefficient of $P \in \mathbb{C}[x, y][z]$.

In order to use equation (9), we need that $\{x, y, z\} = 0$ as an element of the Milnor K-theory group $K_3^M(\mathbb{C}(S))$ or

$$x \wedge y \wedge z = \sum r_i\ x_i \wedge (1-x_i) \wedge y_i \quad \text{in} \quad \bigwedge^3(\mathbb{C}(S)^*)\otimes\mathbb{Q}.$$

We obtain

$$\int_\Gamma \eta(x,y,z) = \sum r_i \int_{\partial\Gamma}\omega(x_i,y_i).$$

The problem is that $\omega(x, y)$ is only multiplicative in the second variable. For the first variable, its behavior is ruled by the five term relation:

$$R_2(x,y) = [x]+[y]+[1-xy]+\left[\frac{1-x}{1-xy}\right]+\left[\frac{1-y}{1-xy}\right]$$

in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.

In general, for a field F, define

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F]/\langle[0],[\infty],R_2(x,y)\rangle.$$

In order to achieve that $\omega(x,y)$ is exact, we need the condition

$$[x]_2 \otimes y = \sum r_i \; [x_i]_2 \otimes x_i$$

in $B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*$. By a conjecture of Goncharov [7], this translates into an element in $gr_3^\gamma K_4(\mathbb{C}(C))_\mathbb{Q}$ which has to be zero.

If the condition is satisfied, we get

$$\int_\gamma \omega(x,y) = \sum r_i \; \mathcal{L}_3(x_i)|_{\partial\gamma} \, .$$

We could summarize this picture as follows. We first integrate in this picture

$$\ldots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \ldots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

We have two situations. Either $\eta(x,y,z)$ is non-exact and $\partial\Gamma = 0$ or $\eta(x,y,z)$ is exact and $\partial\Gamma \neq 0$. All the examples we have talked about fit into the situation when $\eta(x,y,z)$ is exact and $\partial\Gamma \neq \emptyset$. Then we finish with an element in $K_4(\partial\Gamma)$.

Then we go to

$$\ldots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

Again we have two possibilities, but in our context, $\omega(x,y)$ is exact and we finish with an element in $K_5(\partial\gamma) \subset K_5(\bar{\mathbb{Q}})$ leading to trilogarithms and zeta functions, due to Zagier's conjecture and Borel's theorem.

The Mahler measure of $x + \frac{1}{x} + y + \frac{1}{y} - k$ for $k \neq 4$, studied by Boyd [1], Deninger [5], and Rodriguez-Villegas [16] is an example of the non-exact case.

Another application is seen in identities like the one discovered numerically by Boyd [2] and proved by Rodriguez-Villegas [17]

$$7m(y^2 + 2xy + y - x^3 - 2x^2 - x) = 5m(y^2 + 4xy + y - x^3 + x^2).$$

Using regulators in this way it is also possible to prove the following equation due to Rogers

$$m(4n^2) + m\left(\frac{4}{n^2}\right) = 2m\left(2n + \frac{2}{n}\right),$$

where

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} - k\right).$$

The idea of how the non-exact case works is as follows. Under certain circumstances (when the tame symbols are trivial), we have $\{x,y\} \in K_2(E)$. Then

$$r(\{x,y\}) = -\frac{1}{2\pi} \int_\gamma \eta(x,y)$$

where γ is a generator of $H_1(E,\mathbb{Z})^-$ (the subgroup of $H_1(E,\mathbb{Z})$ where complex conjugation acts by -1). The idea is to identify $\mathbb{T}^1$ with γ in the homology. In that way, the right-hand-side is the Mahler measure. On the other hand,

$$r(\{x,y\}) = D^E((x)\diamond(y))$$

if $(x),(y)$ supported on $E_{tors}(\bar{\mathbb{Q}})$. Writing $E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ we have $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$ where $z \bmod \Lambda = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ is identified with $\mathrm{e}^{\mathrm{i}\pi z}$. Then the elliptic dilogarithm is defined by

$$D^E(x) := \sum_{n\in\mathbb{Z}} D(xq^n)$$

where $q = \mathrm{e}^{\mathrm{i}\pi\tau}$, and D is the Bloch-Wigner dilogarithm.

Finally, the last step is to relate πD^E to $L(E,2)$ and that is HARD. It is known only in the cases of complex multiplication or when the curve is a modular curve, which are the cases for which Beilinson's conjectures are known.

Also can be proved using Rodriguez-Villegas result:

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+k\right) = \mathrm{Re}\left(\frac{16y_\mu}{\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})}\right)$$

Another identity discovered by Rogers is:

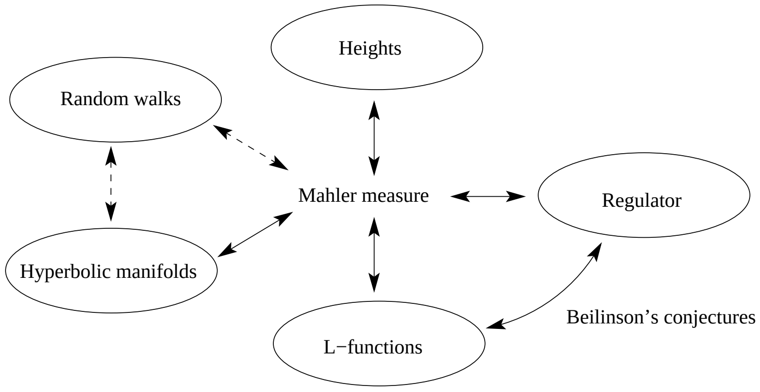
$$m\left(\frac{4}{n^2}\right) = m\left(2n+\frac{2}{n}\right) + m\left(2\mathrm{i}n+\frac{2}{\mathrm{i}n}\right)$$

These identitites can provide information that allows us to compute more Mahler measures. For instance,

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2})$$

The curve corresponding to $3\sqrt{2}$ is $X_0(24)$, then we obtain the values for $m(2)$, $m(4)$, previously unknown. We are working on understanding these identities directly from regulators.

The next picture shows how Mahler measure interacts with several elements (some have been discussed here and some have not). We can see the key role of Mahler measure in the relation among special values of L-functions and regulators (which are related via Beilinson's conjectures), heights, and hyperbolic manifolds (that are related by Beilinson's conjectures as well). It is our general goal to bring more light to the nature of these relationships.



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