

On the friendship between Mahler measure and polylogarithms

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1. Mahler measure

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

$$m(P) = \log |a_d| + \sum_{n=1}^d \log^+ |\alpha_n|$$

for

$$P(x) = a_d \prod_{n=1}^d (x - \alpha_n) \in \mathbb{C}[x]$$

because of Jensen's formula:

$$\int_0^1 \log |e^{2\pi i\theta} - \alpha| d\theta = \log^+ |\alpha|$$

Jensen's formula \longrightarrow simple expression in one-variable case.

Several-variable case?

3. Examples in several variables

Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3)$$

Vandervelde (2002)

$$m(1 + x + y + ixy) = \frac{\sqrt{2}}{\pi} L(\chi_{-8}, 2) = \frac{1}{4} L'(\chi_{-8}, -1)$$

Condon (2003)

$$m(1 + x + (1 - x)(y + z)) = \frac{28}{5\pi^2} \zeta(3)$$

Boyd (1997)

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) \stackrel{?}{=} L'(E, 0)$$

4. Polylogarithms

The k th polylogarithm is

$$\operatorname{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad x \in \mathbb{C}, \quad |x| < 1$$

$$\operatorname{Li}_k(x) = - \int_{0 \leq s_1 \leq \dots \leq s_k \leq 1} \frac{ds_1}{s_1 - \frac{1}{x}} \frac{ds_2}{s_2} \dots \frac{ds_k}{s_k}$$

An analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$P_k(x) := \operatorname{Re}_k \left(\sum_{j=0}^k \frac{2^j B_j}{j!} (\log |x|)^j \operatorname{Li}_{k-j}(x) \right)$$

B_j is j th Bernoulli number, $\operatorname{Li}_0(x) \equiv -\frac{1}{2}$,

$\operatorname{Re}_k = \operatorname{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$,
continuous in $\mathbb{P}^1(\mathbb{C})$.

P_k satisfies lots of functional equations

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1}P_k(x) \quad P_k(\bar{x}) = (-1)^{k-1}P_k(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1 - x) \log |x|$$

Five-term relation

$$D(x) + D(1 - xy) + D(y) + D\left(\frac{1 - y}{1 - xy}\right) + D\left(\frac{1 - x}{1 - xy}\right) = 0$$

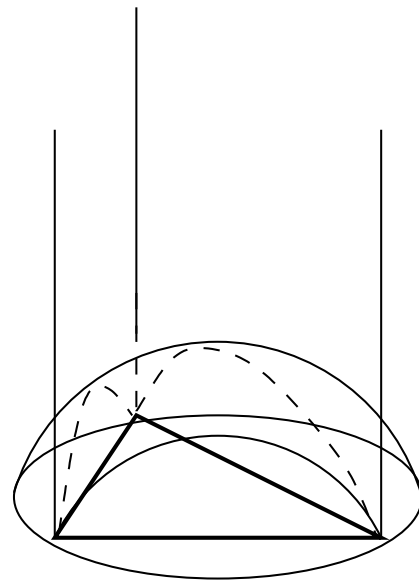
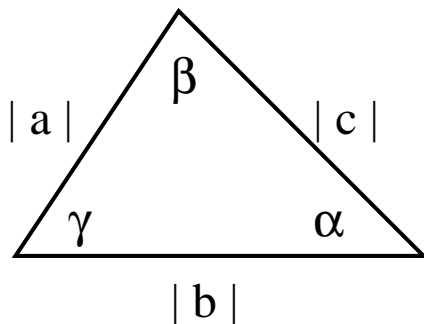
5. Mahler measure and hyperbolic volumes

Cassaigne – Maillot (2000) for $a, b, c \in \mathbb{C}^*$,

$$\pi m(a + bx + cy)$$

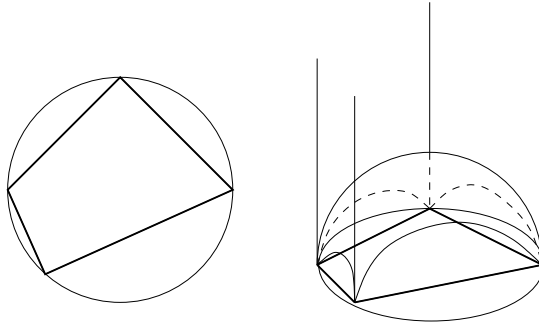
$$= \begin{cases} D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| & \triangle \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases}$$

Ideal tetrahedron:



- Vandervelde (2003)

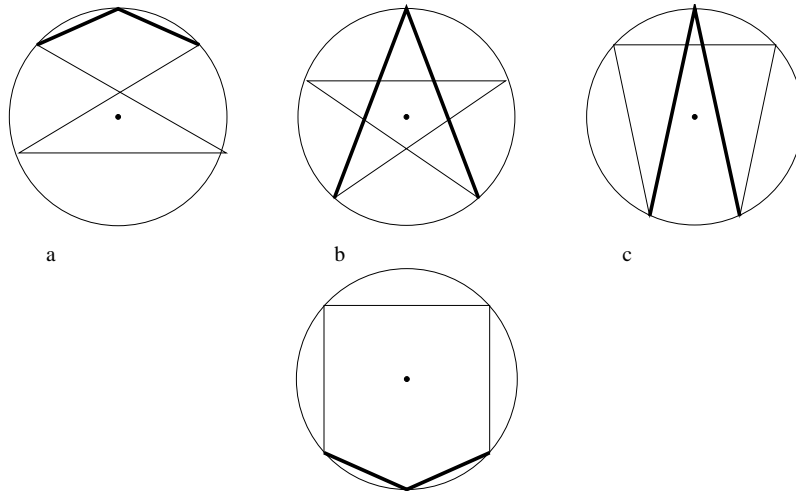
$$y = \frac{bx + d}{ax + c} \quad \text{quadrilateral}$$



- L(2004)

$$y = \frac{x^n - 1}{t(x^m - 1)} = \frac{x^{n-1} + \dots + 1}{t(x^{m-1} + \dots + 1)} \quad \text{polyhedral}$$

+ relation to A -polynomial



6. More examples in several variables

Theorem 1 *L(2004)*

$\pi^n m \left(1 + \left(\frac{1-x_1}{1+x_1} \right) \cdots \left(\frac{1-x_n}{1+x_n} \right) z \right) = \text{combination}$

of $\zeta(\text{odd}) / L(\chi_{-4}, \text{even})$ multiplied by powers of π

$$m \left(1 + x + \left(\frac{1-x_1}{1+x_1} \right) \cdots \left(\frac{1-x_n}{1+x_n} \right) (1 + y)z \right)$$

$$m \left(1 + \left(\frac{1-x_1}{1+x_1} \right) \cdots \left(\frac{1-x_n}{1+x_n} \right) x + \left(1 - \left(\frac{1-x_1}{1+x_1} \right) \cdots \left(\frac{1-x_n}{1+x_n} \right) \right) y \right)$$

Examples

$$\begin{aligned} \pi^3 m \left(1 + \left(\frac{1-x_1}{1+x_1} \right) \left(\frac{1-x_2}{1+x_2} \right) \left(\frac{1-x_3}{1+x_3} \right) z \right) \\ = 24L(\chi_{-4}, 4) + \pi^2 L(\chi_{-4}, 2) \end{aligned}$$

$$\begin{aligned} \pi^4 m \left(1 + \left(\frac{1-x_1}{1+x_1} \right) \cdots \left(\frac{1-x_4}{1+x_4} \right) z \right) \\ = 62\zeta(5) + \frac{14\pi^2}{3}\zeta(3) \end{aligned}$$

$$\pi^4 m \left(1 + x + \left(\frac{1-x_1}{1+x_1} \right) \left(\frac{1-x_2}{1+x_2} \right) (1+y)z \right) = 93\zeta(5)$$

Proof of Theorem (Idea).

- $P_\alpha \in \mathbb{C}[x_1, \dots, x_n]$, coefficients depend polynomially on a parameter $\alpha \in \mathbb{C}$.

For example, $P_\alpha(x) = 1 + \alpha x$.

$$m(P_\alpha) = \log^+ |\alpha|.$$

- $\alpha \rightarrow \alpha \frac{1-y}{1+y}$, get $\tilde{P}_\alpha \in \mathbb{C}[x_1, \dots, x_n, y]$

In the example,

$$\tilde{P}_\alpha(x, y) = 1 + y + \alpha(1 - y)x.$$

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$$m(\tilde{P}_\alpha) = \frac{1}{2\pi i} \int_{\mathbb{T}^1} m \left(P_{\alpha \frac{1-y}{1+y}} \right) \frac{dy}{y}$$

7. Examples from the world of resultants

D'Andrea & L (2003).

- $m(\text{Res}_{\{0,m,n\}})$

$$= m(\text{Res}_t(x + yt^m + t^n, z + wt^m + t^n)) =$$

$$\frac{2}{\pi^2}(-mP_3(\varphi^n) - nP_3(-\varphi^m) + mP_3(\phi^n) + nP_3(\phi^m))$$

$$0 \leq \varphi \leq 1 \quad \text{root of } x^n + x^{n-m} - 1 = 0$$

$$1 \leq \phi \quad \text{root of } x^n - x^{n-m} - 1 = 0$$

- $m(\text{Res}_{\{(0,0),(1,0),(0,1)\}}) = m \left(\begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix} \right)$

$$= m((1-x)(1-y) - (1-z)(1-w)) = \frac{9\zeta(3)}{2\pi^2}$$

8. An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) : $P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

$$\eta(x, 1 - x) = dD(x)$$

Need $\{x, y\} = 0$ in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$.

$$x \wedge y = \sum_j r_j z_j \wedge (1 - z_j)$$

in $\Lambda^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$, then

$$\int_{\gamma} \eta(x, y) = \sum r_j D(z_j)|_{\partial\gamma}$$

Big picture

$$\dots \rightarrow (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \rightarrow K_2(C, \partial\gamma) \rightarrow K_2(C) \rightarrow \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x, y)$ is exact, then $\{x, y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

\rightsquigarrow dilogarithms, zeta function

- $\partial\gamma = \emptyset$, then $\{x, y\} \in K_2(C)$. We have $\eta(x, y)$ is not exact.

\rightsquigarrow L -series of a curve

We may get combinations of both situations.

9. The three-variable case

$$\begin{aligned}\eta(x, y, z) &= \log |x| \left(\frac{1}{3} d \log |y| d \log |z| - d \arg y d \arg z \right) \\ &+ \log |y| \left(\frac{1}{3} d \log |z| d \log |x| - d \arg z d \arg x \right) \\ &+ \log |z| \left(\frac{1}{3} d \log |x| d \log |y| - d \arg x d \arg y \right)\end{aligned}$$

$$d\eta(x, y, z) = \operatorname{Re} \left(\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z} \right)$$

and

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z)$$

$$\Gamma = \{P(x, y, z) = 0\} \cap \{|x| = |y| = 1, |z| \geq 1\}$$

Smyth's case:

$$P(x, y, z) = (1 - x) + (1 - y)z$$

$$\begin{aligned} m(P) &= m(1 - y) + \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1 - x}{1 - y} \right| \frac{dx dy}{x y} \\ &= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{dx dy}{x y} \\ &= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z) \end{aligned}$$

$$\eta(x, y, z) = -\eta(x, 1 - x, y) - \eta(y, 1 - y, x)$$

Theorem 2

$$\eta(x, 1 - x, y) = d\omega(x, y)$$

where

$$\omega(x, y) = -D(x)d \arg y$$

$$+\frac{1}{3} \log |y| (\log |1 - x| d \log |x| - \log |x| d \log |1 - x|)$$

Need

$$x \wedge y \wedge z = \sum r_i x_i \wedge (1 - x_i) \wedge y_i$$

in $\Lambda^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$, then

$$\begin{aligned} \int_{\Gamma} \eta(x, y, z) &= \sum r_i \int_{\Gamma} \eta(x_i, 1 - x_i, y_i) \\ &= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i) \end{aligned}$$

in Smyth's case:

$$x \wedge y \wedge z = -x \wedge (1 - x) \wedge y - y \wedge (1 - y) \wedge x$$

Maillot: if $P \in \mathbb{Q}[x, y, z]$,

$$\partial\Gamma = \gamma = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

ω defined in

$$C = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

Theorem 3

$$\omega(x, x) = dP_3(x)$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$, then

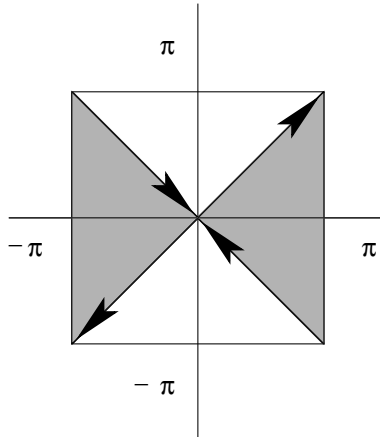
$$\int_{\gamma} \omega(x, y) = \sum r_i P_3(x_i)|_{\partial\gamma}$$

in Smyth's case

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$

$$-[x]_2 \otimes y - [y]_2 \otimes x = \pm 2[x]_2 \otimes x$$



$$m((1-x) + (1-y)z) = \frac{1}{4\pi^2} \int_{\gamma} \omega(x, y) + \omega(y, x)$$

$$= \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2} \zeta(3)$$

10. A little bit of K -theory

F field, define subgroups $R_i(F) \subset \mathbb{Z}[\mathbb{P}_F^1]$ as

$$R_1(F) := [x] + [y] - [xy]$$

$$R_2(F) := [x] + [y] + [1 - xy] + \left[\frac{1 - x}{1 - xy} \right] + \left[\frac{1 - y}{1 - xy} \right]$$

$$R_3(F) := \text{functional equation of the trilogarithm}$$

$$B_i(F) := \mathbb{Z}[\mathbb{P}_F^1] / R_i(F)$$

$$B_F(3) : B_3(F) \xrightarrow{\delta_1^3} B_2(F) \otimes F^* \xrightarrow{\delta_2^3} \wedge^3 F^*$$

$$B_F(2) : B_2(F) \xrightarrow{\delta_1^2} \wedge^2 F^*$$

$$B_F(1) : F^*$$

($B_i(F)$ is placed in degree 1).

$$\delta_1^3([x]_3) = [x]_2 \otimes x \quad \delta_2^3([x]_2 \otimes y) = x \wedge (1 - x) \wedge y$$

$$\delta_1^2([x]_2) = x \wedge (1 - x)$$

Proposition 4

$$\begin{aligned} H^1(B_F(1)) &\cong K_1(F) \\ H^1(B_F(2))_{\mathbb{Q}} &\cong K_3^{\text{ind}}(F)_{\mathbb{Q}} \\ H^2(B_F(2)) &\cong K_2(F) \\ H^3(B_F(3)) &\cong K_3^M(F) \end{aligned}$$

Goncharov conjectures:

$$H^i(B_F(3) \otimes \mathbb{Q}) \stackrel{?}{\cong} K_{6-i}^{[3-i]}(F)_{\mathbb{Q}}$$

Our first condition is $x \wedge y \wedge z$ is 0 in

$$H^3(B_{\mathbb{Q}(S)}(3) \otimes \mathbb{Q}) \cong K_3^M(\mathbb{Q}(S)) \otimes \mathbb{Q}$$

Our second condition is $[x_i]_2 \otimes y_i$ is 0 in

$$H^2(B_{\mathbb{Q}(C)}(3) \otimes \mathbb{Q}) \stackrel{?}{\cong} K_4^{[1]}(\mathbb{Q}(C))_{\mathbb{Q}}$$

Big picture II

$$\dots \rightarrow K_4(\partial\Gamma) \rightarrow K_3(S, \partial\Gamma) \rightarrow K_3(S) \rightarrow \dots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\dots \rightarrow (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \rightarrow K_4(C, \partial\gamma) \rightarrow K_4(C) \rightarrow \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.

11. Future research

- Mahler measure and regulators, Beilinson conjecture. Explain the n -variable results.
- More examples in n -variables.
- Understand Mahler measure of resultants.
- Multiple polylogarithms in roots of unity.
- Hyperbolic volumes in higher dimensions. 3-variable Mahler measures and volumes in the 5-dimensional hyperbolic space.