

Mahler measure and evaluation of regulators

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Mahler measure of polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

Given

$$P(x) = a_d \prod_{n=1}^d (x - \alpha_n) \in \mathbb{C}[x]$$

$$m(P) = \log |a_d| + \sum_{n=1}^d \log^+ |\alpha_n|$$

Jensen's formula:

$$\int_0^1 \log |e^{2\pi i \theta} - \alpha| d\theta = \log^+ |\alpha|$$

recovers one-variable case.

Properties

- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$
- α algebraic number, and P_α minimal polynomial over \mathbb{Q} ,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}] h(\alpha)$$

where h is the logarithmic Weil height.

Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

Lehmer's question

Lehmer (1933)

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$$= \log(1.176280818\dots) = 0.162357612\dots$$

Does there exist $C > 0$, for all $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is the above polynomial the best possible?

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Is the above polynomial the best possible?

Boyd & Lawton

$P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \rightarrow \infty} \dots \lim_{k_n \rightarrow \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, x_2, \dots, x_n))$$

Jensen's formula \longrightarrow simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)



$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$



$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3)$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - k\right) \stackrel{?}{=} \frac{L'(E_k, 0)}{B_k} \quad k \in \mathbb{N}, \quad k \neq 4$$

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\right) = 2L'(\chi_{-4}, -1)$$

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\sqrt{2}\right) = L'(A, 0)$$

$$A: Y^2 = X^3 - 44X + 112$$

Examples in three variables

- Condon (2003):

$$\pi^2 m \left(z - \left(\frac{1-x}{1+x} \right) (1+y) \right) = \frac{28}{5} \zeta(3)$$

- D'Andrea & L. (2003):

$$\pi^2 m (z(1-xy)^2 - (1-x)(1-y)) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}$$

- Boyd & L. (2005):

$$\pi^2 m(x^2 + 1 + (x+1)y + (x-1)z) = \pi L(\chi_{-4}, 2) + \frac{21}{8} \zeta(3)$$

Examples with more than three variables

L.(2003):

- $$\pi^3 m \left(1 + x + \left(\frac{1 - x_1}{1 + x_1} \right) (1 + y)z \right) = 24L(\chi_{-4}, 4)$$

- $$\pi^4 m \left(1 + \left(\frac{1 - x_1}{1 + x_1} \right) \dots \left(\frac{1 - x_4}{1 + x_4} \right) z \right) = 62\zeta(5) + \frac{14}{3}\pi^2\zeta(3)$$

- $$\pi^4 m \left(1 + x + \left(\frac{1 - x_1}{1 + x_1} \right) \left(\frac{1 - x_2}{1 + x_2} \right) (1 + y)z \right) = 93\zeta(5)$$

Known formulas for n .

Polylogarithms

The k th polylogarithm is

$$\operatorname{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$\mathcal{L}_k(x) := \operatorname{Re}_k \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \operatorname{Li}_{k-j}(x) \right)$$

B_j is j th Bernoulli number

$\operatorname{Re}_k = \operatorname{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

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\mathcal{L}_k satisfies lots of functional equations

$$\mathcal{L}_k\left(\frac{1}{x}\right) = (-1)^{k-1} \mathcal{L}_k(x) \quad \mathcal{L}_k(\bar{x}) = (-1)^{k-1} \mathcal{L}_k(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1-x) \log|x|$$

Five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

Philosophy of Beilinson's conjectures

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($L_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\text{reg} : K \rightarrow \mathbb{R}$ ($\text{reg} = \log |\cdot|$)

$$(K \text{ rank } 1) \quad L'_X(0) \sim_{\mathbb{Q}^*} \text{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic,

$$\zeta'_F(0) \sim_{\mathbb{Q}^*} \log |\epsilon|, \epsilon \in \mathcal{O}_F^*$$

An algebraic integration for Mahler measure

Deninger (1997): General framework

Rodriguez-Villegas (1997) : $P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

$$\eta(x, y) = \log |x| d \arg y - \log |y| d \arg x$$

$$\eta(x, 1-x) = dD(x) \quad d\eta(x, y) = \operatorname{Im} \left(\frac{dx}{x} \wedge \frac{dy}{y} \right)$$

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The three-variable case

$$P(x, y, z) = (1 - x) - (1 - y)z \quad X = \{P(x, y, z) = 0\}$$

$$m(P) = m(1 - y) + \frac{1}{(2\pi i)^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1 - x}{1 - y} \right| \frac{dx}{x} \frac{dy}{y} \frac{dz}{z}$$

$$= \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1 - x}{1 - y} \right| \frac{dx}{x} \frac{dy}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{dx}{x} \frac{dy}{y}$$

$$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z)$$

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$$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z)$$

$$\eta(x, y, z) = \log |x| \left(\frac{1}{3} d \log |y| \wedge d \log |z| - d \arg y \wedge d \arg z \right)$$

$$+ \log |y| \left(\frac{1}{3} d \log |z| \wedge d \log |x| - d \arg z \wedge d \arg x \right)$$

$$+ \log |z| \left(\frac{1}{3} d \log |x| \wedge d \log |y| - d \arg x \wedge d \arg y \right)$$

$$d\eta(x, y, z) = \operatorname{Re} \left(\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z} \right)$$

$$\eta(x, 1-x, y) = d\omega(x, y)$$

where

$$\begin{aligned}\omega(x, y) = & -D(x)d \arg y \\ & + \frac{1}{3} \log |y| (\log |1-x| d \log |x| - \log |x| d \log |1-x|)\end{aligned}$$

$$z = \frac{1-x}{1-y}$$

$$\eta(x, y, z) = -\eta(x, 1-x, y) - \eta(y, 1-y, x)$$

$$m(P) = \frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, 1-x, y) + \eta(y, 1-y, x) = \frac{1}{(2\pi)^2} \int_{\partial\Gamma} \omega(x, y) + \omega(y, x)$$

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$$\omega(x, x) = d\mathcal{L}_3(x)$$

$$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$$

Maillot: if $P \in \mathbb{R}[x, y, z]$,

$$\partial\Gamma = \gamma = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

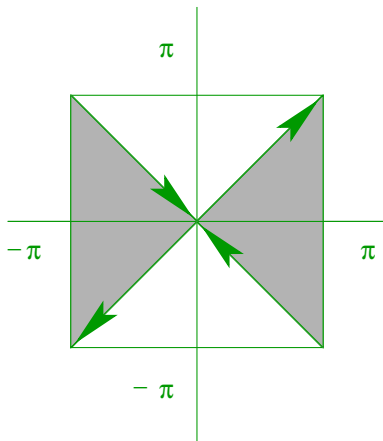
ω defined in

$$C = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x) - (1-y)z) = \frac{1}{4\pi^2} \int_{\gamma} \omega(x, y) + \omega(y, x)$$

$$\omega(x, x) = d\mathcal{L}_3(x)$$

$$= \frac{1}{4\pi^2} 8(\mathcal{L}_3(1) - \mathcal{L}_3(-1)) = \frac{7}{2\pi^2} \zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z)$$

Need

$$x \wedge y \wedge z = \sum r_i x_i \wedge (1 - x_i) \wedge y_i$$

in $\wedge^3(\mathbb{C}(X)^*) \otimes \mathbb{Q}$,

($\{x, y, z\} = 0$ in $K_3^M(\mathbb{C}(X)) \otimes \mathbb{Q}$) then

$$\begin{aligned} \int_{\Gamma} \eta(x, y, z) &= \sum r_i \int_{\Gamma} \eta(x_i, 1 - x_i, y_i) \\ &= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i) \end{aligned}$$

$$\omega(x, y) = -D(x)d \arg y \\ + \frac{1}{3} \log |y| (\log |1 - x| d \log |x| - \log |x| d \log |1 - x|)$$

Let

$$R_2(x, y) = [x] + [y] + [1 - xy] + \left[\frac{1 - x}{1 - xy} \right] + \left[\frac{1 - y}{1 - xy} \right]$$

in $\mathbb{Z} \left[\mathbb{P}_{\mathbb{C}(C)}^1 \right]$.
 F field,

$$B_2(F) := \mathbb{Z}[\mathbb{P}_F^1] / \langle [0], [\infty], R_2(x, y) \rangle$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.

Then

$$\int_{\gamma} \omega(x, y) = \sum r_i \mathcal{L}_3(x_i)|_{\partial \gamma}$$

$$\omega(x, y) = -D(x)d \arg y$$

$$+ \frac{1}{3} \log |y| (\log |1 - x| d \log |x| - \log |x| d \log |1 - x|)$$

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Then

$$\int_{\gamma} \omega(x, y) = \sum r_i \mathcal{L}_3(x_i)|_{\partial \gamma}$$

Big picture in three variables

$$\cdots \rightarrow K_4(\partial\Gamma) \rightarrow K_3(X, \partial\Gamma) \rightarrow K_3(X) \rightarrow \cdots$$

$$\partial\Gamma = X \cap \mathbb{T}^3$$

$$\cdots \rightarrow (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \rightarrow K_4(C, \partial\gamma) \rightarrow K_4(C) \rightarrow \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

Polylogarithmic motivic complexes

Beilinson (after work of Bloch, Deligne, Beilinson, etc)

$$r_{\mathcal{D}} : gr_j^{\gamma} K_{2j-i}(X)_{\mathbb{Q}} \rightarrow H_{\mathcal{D}}^i(X, \mathbb{R}(j))$$

Goncharov: possible "Explicit construction" of $r_{\mathcal{D}}$ and $gr_j^{\gamma} K_{2j-i}(X)_{\mathbb{Q}}$.

F field, define $\mathcal{R}_n(F) \subset \mathbb{Z}[\mathbb{P}_F^1]$ and

$$\mathbf{B}_n(F) := \mathbb{Z}[\mathbb{P}_F^1] / \mathcal{R}_n(F)$$

$$\mathcal{R}_1(F) := \langle \{x\} + \{y\} - \{xy\}, \quad x, y \in F^*, \{0\}, \{\infty\} \rangle$$

$$\mathbf{B}_n(F) := \mathbb{Z}[\mathbb{P}_F^1] / \mathcal{R}_n(F)$$

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$$\mathbb{Z}[\mathbb{P}_F^1] \xrightarrow{\delta_n} \begin{cases} \mathbf{B}_{n-1}(F) \otimes F^* & \text{if } n \geq 3 \\ \wedge^2 F^* & \text{if } n = 2 \end{cases}$$

$$\delta_n(\{x\}) = \begin{cases} \{x\}_{n-1} \otimes x & \text{if } n \geq 3 \\ (1-x) \wedge x & \text{if } n = 2 \\ 0 & \text{if } \{x\} = \{0\}, \{1\}, \{\infty\} \end{cases}$$

$$\mathcal{A}_n(F) := \ker \delta_n$$

$$\mathcal{R}_n(F) := \langle \alpha(0) - \alpha(1), \alpha(t) \in \mathcal{A}_n(F(t)) \rangle$$

$$\delta_n(\mathcal{R}_n(F)) = 0$$

$\mathbf{B}_F(n)$:

$$\mathbf{B}_n(F) \xrightarrow{\delta} \mathbf{B}_{n-1}(F) \otimes F^* \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathbf{B}_2(F) \otimes \wedge^{n-2} F^* \xrightarrow{\delta} \wedge^n F^*$$

$$\delta : \{x\}_p \otimes \bigwedge_{i=1}^{n-p} y_i \rightarrow \delta_p(\{x\}_p) \wedge \bigwedge_{i=1}^{n-p} y_i.$$

$$H^n(\mathbf{B}_F(n)) \cong K_n^M(F)$$

Conjecture

$$H^i(\mathbf{B}_F(n) \otimes \mathbb{Q}) \cong \text{gr}_n^\gamma K_{2n-i}(F)_\mathbb{Q}$$

(Goncharov) X complex algebraic variety. There exist $\eta_n(m)$ inducing a homomorphism of complexes

$$\begin{array}{ccccccc}
 \mathbf{B}_n(\mathbb{C}(X)) & \xrightarrow{\delta} & \mathbf{B}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \bigwedge^n \mathbb{C}(X)^* \\
 \downarrow \eta_n(1) & & \downarrow \eta_n(2) & & & & \downarrow \eta_n(n) \\
 \mathcal{A}^0(X)(n-1) & \xrightarrow{d} & \mathcal{A}^1(X)(n-1) & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathcal{A}^{n-1}(X)(n-1)
 \end{array}$$

$(\mathcal{A}^i(X)(j) = \text{smooth } i\text{-forms with values in } (2\pi i)^j \mathbb{R})$ such that

- $\eta_n(1)(\{x\}_n) = \widehat{\mathcal{L}}_n(x)$.
- $d\eta_n(n)(x_1 \wedge \cdots \wedge x_n) = \pi_n \left(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \right)$.
- $\eta_n(m)$ compatible with residues (residues are given by tame symbols).

Conjecture

"Image of $\eta_n(i)$ " coincides with image of regulator

Deninger(1997)

$$m(P) = m(P^*) + \frac{1}{(-2i\pi)^{n-1}} \int_{\Gamma} \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \dots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$$\begin{aligned} & \pi^{2n} m \left(1 + \left(\frac{1 - x_1}{1 + x_1} \right) \dots \left(\frac{1 - x_{2n}}{1 + x_{2n}} \right) z \right) \\ &= \sum_{h=1}^n c_{n,h} \pi^{2n-2h} \zeta(2h+1) \end{aligned}$$

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