Mahler measure of polynomials

Postdoc Seminar – Mathematical Sciences Research Institute

April 28th, 2006

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Mahler Measure and Lehmer's question

Looking for large primes, Pierce [15] proposed the following idea in 1918.
Consider $P \in \mathbb{Z}[x]$ monic, and write

$$P(x) = \prod_i (x - \alpha_i).$$

Then, let us look at

$$\Delta_n = \prod_i (\alpha_i^n - 1).$$

The α_i are integers over $\mathbb{Z}$. By applying Galois theory, it is easy to see that $\Delta_n \in \mathbb{Z}$. Note that if $P(x) = x - 2$, we get the sequence $\Delta_n = 2^n - 1$. Thus, we recover the example of Mersenne numbers. The idea is to look for primes among the factors of Δ_n. The prime divisors of such integers must satisfy some congruence conditions that are quite restrictive, hence they are easier to factorize than a randomly given number.

In order to minimize the number of trial divisions, the sequence Δ_n should grow slowly. Lehmer [13] studied $\frac{\Delta_{n+1}}{\Delta_n}$, observed that

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

and suggested the following definition.

Definition 1 *Given $P \in \mathbb{C}[x]$, such that*

$$P(x) = a \prod_i (x - \alpha_i),$$

define the Mahler measure [2] of P as

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}. \tag{1}$$

The logarithmic Mahler measure is defined as[3]

$$m(P) = \log M(P) = \log|a| + \sum_i \log^+ |\alpha_i|. \tag{2}$$

When does $M(P) = 1$ for $P \in \mathbb{Z}[x]$? We have

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[2] The name Mahler came later after the person who successfully extended this definition to the several-variable case.

[3] $\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

Lemma 2 *(Kronecker) Let* $P = \prod_i (x - \alpha_i) \in \mathbb{Z}[x]$*, if* $|\alpha_i| \leq 1$*, then the* α_i *are zero or roots of the unity.*

By Kronecker's Lemma, $P \in \mathbb{Z}[x]$, $P \neq 0$, then $M(P) = 1$ if and only if P is the product of powers of x and cyclotomic polynomials. This statement characterizes integral polynomials whose Mahler measure is 1.

Lehmer found the example

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = \log(1.176280818\ldots) = 0.162357612\ldots$$

and asked the following (Lehmer's question, 1933):

Is there a constant $C > 1$ *such that for every polynomial* $P \in \mathbb{Z}[x]$ *with* $M(P) > 1$*, then* $M(P) \geq C$*?*

Lehmer's question remains open nowadays. His 10-degree polynomial remains the best possible result.

The use of this polynomial has led to the discovery of the prime number $\sqrt{\Delta_{379}} = 1,794,327,140,357$ but bigger primes were discovered with the use of other polynomials (with $x^3 - x - 1$, $\Delta_{127} = 3,233,514,251,032,733$).

Here are some results in the direction of Lehmer's question.

- Dobrowolski [9] proved in 1979 that if P is monic, irreducible and noncyclotomic, and has degree d then

  $$M(P) > 1 + c\left(\frac{\log\log d}{\log d}\right)^3.$$

  This result has been improved by many people, but the nature remains the same.

- Breusch [5] in 1951 proved that if P is monic, irreducible, nonreciprocal,

  $$M(P) \geq 1.324717... = \text{real root of} \quad x^3 - x - 1$$

  (rediscovered by Smyth in 1971)

Mahler Measure in several variables

Definition 3 *For* $P \in \mathbb{C}[x_1, \ldots, x_n]$*, the (logarithmic)* Mahler measure *is defined by*

$$\begin{aligned} m(P) &:= \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \, \mathrm{d}\theta_1 \ldots \, \mathrm{d}\theta_n & (3) \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n}. & (4) \end{aligned}$$

where $\mathbb{T}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \,|\, |x_1| = \ldots = |x_n| = 1\}$.

It is possible to prove that this integral is not singular and that $m(P)$ always exists.

Because of Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \, \mathrm{d}\theta = \log^+ |\alpha|. \tag{5}$$

We recover the formula for the one-variable case.

Some properties

Proposition 4 *For* $P, Q \in \mathbb{C}[x_1, \dots, x_n]$

$$m(P \cdot Q) = m(P) + m(Q). \tag{6}$$

It is also true that $m(P) \geq 0$ if P has integral coefficients.

Mahler measure is related to heights. Indeed, if α is an algebraic number, and P_α is its minimal polynomial over $\mathbb{Q}$, then

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha),$$

where h is the logarithmic Weil height. Recall that the Weil height is defined as

$$h(\alpha) = \sum_v \log^+ |\alpha|_v$$

where $\alpha \in K$, K_v is the completion of K at the place v and

$$|\cdot|_v = ||\cdot||_v^{\frac{d_v}{d}}$$

where $||\cdot||_v$ restricts to the usual p-adic value in $\mathbb{Q}$.

This identity also extends to several-variable polynomials and heights in hypersurfaces.

Indeed the motivation for Mahler to define the measure for multivariable polynomials was to establish certain inequalities between heights of products of polynomials which are crucial in transcendence theory. They are

$$L(P) \leq \prod_{l=1}^{s} L(P_l) \leq 2^{m_1+\ldots+m_n} L(P)$$

$$H(P) \leq \prod_{l=1}^{s} H(P_l) \leq 2^{m_1+\ldots+m_n} H(P)$$

(they can be proved, for instance, by showing that $M(P) \leq L(P) \leq 2^{m_1+\ldots+m_n} M(P)$).

Let us also mention the following result:

Theorem 5 *(Boyd–Lawton) For* $P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2\to\infty} \dots \lim_{k_n\to\infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, \dots, x_n)). \tag{7}$$

In particular Lehmer's problem in several variables reduces to the one-variable case.

Examples

For one-variable polynomials, the Mahler measure has to do with the roots of the polynomial. However, it is very hard to compute explicit formulas for examples in several variables. The first and simplest ones were computed by Smyth:

- Smyth [17]

$$m(x + y + 1) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1), \tag{8}$$

where

$$\mathrm{L}(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \text{and} \quad \chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 3 \\ -1 & \text{if} \quad n \equiv -1 \bmod 3 \\ 0 & \text{if} \quad n \equiv 0 \bmod 3 \end{cases}$$

- Smyth [1]
$$m(x+y+z+1)=\frac{7}{2\pi^2}\zeta(3). \tag{9}$$

- Boyd [2], Deninger [8], Rodriguez-Villegas [16] studied cases of the following
$$\begin{aligned} m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) &\overset{?}{=} \frac{\mathrm{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N} \\ m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) &= 2\mathrm{L}'(\chi_{-4},-1) \\ m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) &= \mathrm{L}'(A,0) \end{aligned}$$

  The question mark stands for a numerical result (discovered by Boyd), B_k is a rational number, and E_k is the elliptic curve wich corresponds to the zero set of the polynomial. When $k=4$ the curve has genus zero. When $k=4\sqrt{2}$ the elliptic curve is
$$A: y^2=x^3-44x+112,$$
  which has complex multiplication (this case was proved by Rodriguez-Villegas).

Mahler measure and hyperbolic volumes

A generalization of Smyth's first result was due to Cassaigne and Maillot [14]: for $a,b,c\in\mathbb{C}^*$,
$$\pi m(a+bx+cy)=\begin{cases} D\left(\left|\frac{a}{b}\right|\mathrm{e}^{\mathrm{i}\gamma}\right)+\alpha\log|a|+\beta\log|b|+\gamma\log|c| & \triangle \\ \pi\log\max\{|a|,|b|,|c|\} & \text{not } \triangle \end{cases} \tag{10}$$
where $\triangle$ stands for the statement that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle, and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$, and $|c|$ respectively. The term with the Bloch-Wigner dilogarithm
$$D(x):=\mathrm{Im}(\mathrm{Li}_2(x))+\arg(1-x)\log|x|$$
can be interpreted as the volume of the ideal hyperbolic tetrahedron which has the triangle as basis and the fourth vertex is infinity. See figure 1. The five-term relation
$$D(x)+D(1-xy)+D(y)+D\left(\frac{1-y}{1-xy}\right)+D\left(\frac{1-x}{1-xy}\right)+D\left(\frac{1-y}{1-xy}\right). \tag{11}$$
guarantees the symmetry of the formula.

Boyd [1], [2] and Boyd and Rodriguez Villegas [4] found several examples where the Mahler measure of the A-polynomial of an orientable, complete, one-cusped, hyperbolic manifold M is related to the volume of the manifold. Boyd and Rodriguez Villegas found identities of the kind
$$\pi m(A)=\mathrm{Vol}(M)$$

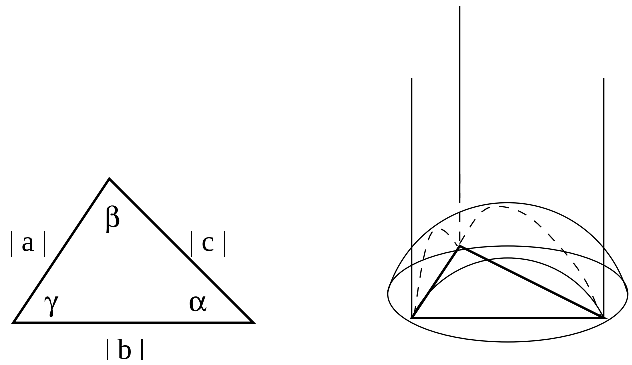


Figure 1: The main term in Cassaigne – Maillot formula is the volume of the ideal hyperbolic tetrahedron over the triangle.

The A-polynomial ($A(x, y) \in \mathbb{Q}[x, x^{-1}, y, y^{-1}]$) is an invariant from the space of representations $\rho : \pi_1(M) \to SL_2(\mathbb{C})$, more precisely, it is the minimal, nontrivial algebraic relation between two parameters x and y which have to do with $\rho(\lambda)$ and $\rho(\mu)$, where $\lambda, \mu \in \pi_1(\partial M)$ are the longitude and the meridian of the boundary torus.

An algebraic integration for Mahler measure

The appearance of the L-functions in Mahler measures formulas is a common phenomenon. Deninger [8] interpreted the Mahler measure as a Deligne period of a mixed motive. More specifically, in two variables, and under certain conditions, he proved that

$$m(P) = \mathrm{reg}(\xi_i),$$

where reg is the determinant of the regulator matrix, which we are evaluating in some class in an appropriate group in K-theory.

Rodriguez-Villegas [16] made explicit the relationship between Mahler measure and regulators by computing the regulator for the two-variable case, and using this machinery to explain the formulas for two variables.

For example, let us start with Smyth's example, $P(x, y) = y + x - 1$. Then its Mahler measure is

$$m(P) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log |y + x - 1| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}.$$

By Jensen's equality,

$$m(P) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{\mathrm{d}x}{x} = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

where $\gamma = \{P(x, y) = 0\} \cap \{|x| = 1, |y| \geq 1\}$ and

$$\eta(x, y) = \log |x| \, \mathrm{d} \arg y - \log |y| \, \mathrm{d} \arg x.$$

This is a closed differential form defined in $C = \{P(x, y) = 0\}$ minus the sets of zeros and poles of x and y. It is multiplicative and antisymmetric. Moreover

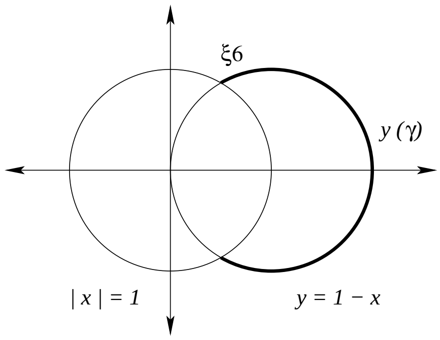
$$\eta(x, 1 - x) = \mathrm{d}D(x).$$

If we use Stokes Theorem, we get

$$m(P) = -\frac{1}{2\pi} D(\partial\gamma).$$

Now we parametrize

$$\gamma : x = e^{2\pi i\theta} \quad y(\gamma(\theta)) = 1 - e^{2\pi i\theta}, \quad \theta \in [1/6\,; 5/6] \qquad \partial\gamma = [\bar{\xi}_6] - [\xi_6]$$



Then we obtain

$$2\pi m(x+y+1) = D(\xi_6) - D(\bar{\xi}_6) = 2D(\xi_6) = \frac{3\sqrt{3}}{2} \mathrm{L}(\chi_{-3}, 2).$$

In general, given $P(x,y) \in \mathbb{C}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_\gamma \eta(x,y),$$

- If $\eta(x,y)$ is exact, and $\partial\gamma \neq 0$ and we use Stokes Theorem. We obtain an element in $K_3(\partial\gamma) \subset K_3(\bar{\mathbb{Q}})$. This leads to dilogarithms and then to zeta functions of number fields by Theorems of Borel, Zagier, and others.
- If $\partial\gamma = \emptyset$, then $\{x,y\} \in K_2(C)$. We have $\eta(x,y)$ is not exact. We get L-series of a curve and examples of Beilinson's conjectures.

Examples in three variables We have developed [12] a method for studying the three variable case. Here are some examples in three variables that can be also computed using regulators.

- Condon [6]:

$$\pi^2 m\left(z - \left(\frac{1-x}{1+x}\right)(1+y)\right) = \frac{28}{5}\zeta(3),$$

- D'Andrea & L. [7]:

$$\pi^2 m\left(z(1-xy)^2 - (1-x)(1-y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)},$$

- Boyd & L. (2005)

$$m(x^2+x+1+(x+1)y+z) = \frac{\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3}, 2) + \frac{19}{6\pi^2}\zeta(3).$$

- L. [11]

$$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right) = 24\mathrm{L}(\chi_{-4},4)+\pi^2\mathrm{L}(\chi_{-4},2) \quad (12)$$

$$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_4}{1+x_4}\right)z\right) = 62\zeta(5)+\frac{14}{3}\pi^2\zeta(3) \quad (13)$$

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = 93\zeta(5) \quad (14)$$

$$(15)$$

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