Mahler measure under variations of the base group

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Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

By Jensen's formula,

$$m\left(a \prod (x - \alpha_i)\right) = \log |a| + \sum \log \max\{1, |\alpha_i|\}.$$

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Lehmer's question

Lehmer (1933)

$$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)$$

$= \log(1.176280818\dots) = 0.162357612\dots$

$$\text{Does there exist} \quad C>0, \quad \text{for all} \quad P(x)\in\mathbb{Z}[x]$$

$$m(P)=0 \quad \text{or} \quad m(P)>C??$$

Is the above polynomial the best possible?

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1)$$
- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}, \quad k\neq 4$$

E_k determined by $x+\frac{1}{x}+y+\frac{1}{y}-k=0$.

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right)=\mathrm{L}'(E_{4\sqrt{2}},0)$$

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The general technique

Rodriguez-Villegas (1997)

$$P_\lambda(x,y) = 1 - \lambda P(x,y) \qquad P(x,y) = x + \frac{1}{x} + y + \frac{1}{y}$$

$$P(x,y) = \overline{P(x^{-1}, y^{-1})}$$

$$m(P,\lambda) := m(P_\lambda)$$

$$m(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log|1 - \lambda P(x,y)| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}.$$

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Note

$$|\lambda P(x,y)| < 1, \qquad \lambda \quad \text{small}, \quad x, y \in \mathbb{T}^2$$

$$\tilde{m}(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log(1 - \lambda P(x,y)) \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$\frac{\mathrm{d}\tilde{m}(P,\lambda)}{\mathrm{d}\lambda} = -\frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \frac{P(x,y)}{1 - \lambda P(x,y)} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

Let

$$u(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \frac{1}{1-\lambda P(x,y)}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= \sum_{n=0}^{\infty} \lambda^n \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} P(x,y)^n \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = \sum_{n=0}^{\infty} a_n \lambda^n$$

Where

$$\frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} P(x,y)^n \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = [P(x,y)^n]_0 = a_n$$

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$$= -\int_0^{\lambda}(u(P,t)-1)\frac{\mathrm{d}t}{t} = -\sum_{n=1}^{\infty}\frac{a_n\lambda^n}{n}$$

In the case $P = x + \frac{1}{x} + y + \frac{1}{y}$,

$$a_n = 0 \qquad n \quad \text{odd}$$

$$a_{2m} = \binom{2m}{m}^2$$

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Definition

$\mathbb{F}_{x_1,\dots,x_l}$ *free group in* $x_1, \dots, x_l$,

$N \triangleleft \mathbb{F}_{x_1,\dots,x_l}$, $\Gamma = \mathbb{F}_{x_1,\dots,x_l}/N$

$$Q = Q(x_1, \dots, x_l) = \sum_{g \in \Gamma} c_g g \in \mathbb{C}\Gamma,$$

$$Q^* = \sum_{g \in \Gamma} \overline{c_g} g^{-1} \in \mathbb{C}\Gamma \textit{ reciprocal.}$$

$P = P(x_1, \dots, x_l) \in \mathbb{C}\Gamma$, $P = P^*$, $|\lambda|^{-1} >$ *length of* P,

$$m_\Gamma(P, \lambda) = -\sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n},$$

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$$m_\Gamma(P, \lambda) = -\sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n},$$

$$a_n = [P(x_1, \ldots, x_l)^n]_0 .$$

We also write

$$u_\Gamma(P,\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$$

for the generating function of the a_n.

$Q(x_1,\ldots,x_l) \in \mathbb{C}\Gamma$

$$QQ^* = \frac{1}{\lambda}\left(1-(1-\lambda QQ^*)\right)$$

for λ real and positive and $1/\lambda$ larger than the length of QQ^*.

$$m_\Gamma(Q) = -\frac{\log\lambda}{2} - \sum_{n=1}^{\infty} \frac{b_n}{2n}, \qquad b_n = \left[(1-\lambda QQ^*)^n\right]_0.$$

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Volume of hyperbolic knots

K knot: smooth embedding $S^1 \subset S^3$.

$$\Gamma = \pi_1(S^3 \setminus K) = \langle x_1, \dots, x_g \mid r_1, \dots, r_{g-1} \rangle$$

For any group Γ, let

$$\epsilon : \mathbb{Z}\Gamma \to \mathbb{Z} \qquad \sum_g c_g g \to \sum_g c_g .$$

Derivation: mapping $\mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$

- $D(u+v) = Du + Dv$.
- $D(u \cdot v) = D(u)\epsilon(v) + uD(v)$

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Fox (1953) $\{x_1, \dots\}$ generators, there is $\frac{\partial}{\partial x_i}$ such that

$$\frac{\partial x_j}{\partial x_i} = \delta_{i,j}.$$

Back to knots,
Let

$$F = \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_g} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_{g-1}}{\partial x_1} & \cdots & \frac{\partial r_{g-1}}{\partial x_g} \end{pmatrix} \in M^{(g-1)\times g}(\mathbb{C}\Gamma)$$

Fox matrix.

Delete a column $F \rightsquigarrow A \in M^{(g-1)\times(g-1)}(\mathbb{C}\Gamma)$.

Theorem (Lück, 2002)

Suppose K is a hyperbolic knot. Then, for c sufficiently large

$$\frac{1}{3\pi}\mathrm{Vol}(S^3 \setminus K) = 2(g-1)\ln(c) - \sum_{n=1}^{\infty} \frac{1}{n}\mathrm{tr}_{\mathbb{C}\Gamma}\left((1 - c^{-2}AA^*)^n\right).$$

$A \in M^g\mathbb{C}[t, t^{-1}]$ the right-hand side is $2m(\det(A))$.

Cayley Graphs

Γ of order m

$$\alpha : \Gamma \to \mathbb{C} \qquad \alpha(g) = \overline{\alpha(g^{-1})} \quad \forall g \in \Gamma$$

Weighted Cayley graph:

- Vertices $g_1, \ldots, g_m$.
- (directed) Edge between g_i and g_j has weight $\alpha(g_i^{-1} g_j)$.

$$A(\Gamma, \alpha) = \{\alpha(g_i g_j^{-1})\}_{i,j}$$

Weighted adjacency matrix

Let $\chi_1, \ldots, \chi_h$ be the irreducible characters of Γ of degrees $n_1, \ldots, n_h$.

Theorem (Babai, 1979)

The spectrum of $A(\Gamma, \alpha)$ *can be arranged as*

$$\mathcal{S} = \{\sigma_{i,j} : i = 1, \ldots, h; j = 1, \ldots, n_i\}.$$

such that $\sigma_{i,j}$ *has multiplicity* n_i *and*

$$\sigma_{i,1}^t + \cdots + \sigma_{i,n_i}^t = \sum_{g_1,\ldots,g_t \in \Gamma} \left(\prod_{s=1}^t \alpha(g_s) \right) \chi_i \left(\prod_{s=1}^t g_s \right).$$

The Mahler measure over finite groups

$$P = \sum_i (\delta_i S_i + \overline{\delta_i} S_i^{-1}) + \sum_j \eta_j T_j \in \mathbb{C}\Gamma$$

$S_i \neq S_i^{-1}$, $T_j = T_j^{-1}$, $\delta_i \in \mathbb{C}$, $\eta_j \in \mathbb{R}$, and $S_i, T_j \in \Gamma$,
Assume monomials generate Γ.

Theorem

For Γ finite

$$m_\Gamma(P,\lambda) = \frac{1}{|\Gamma|} \log \det(I - \lambda A),$$

A is the adjacency matrix of the Cayley graph (with weights) and $\frac{1}{\lambda} > \rho(A)$.

Analytic continuation for $m_\Gamma(P,\lambda)$ to $\mathbb{C} \setminus \mathrm{Spec}(A)$.

Abelian Groups

Γ finite abelian group

$$\Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_l\mathbb{Z}$$

Corollary

$$m_\Gamma(P,\lambda) = \frac{1}{|\Gamma|} \log \left(\prod_{j_1,\dots,j_l} \left(1 - \lambda P(\xi_{m_1}^{j_1}, \dots, \xi_{m_l}^{j_l})\right) \right)$$

where ξ_k is a primitive root of unity.

Theorem

For small λ,

$$\lim_{m_1,\dots,m_l\to\infty} m_{\mathbb{Z}/m_1\mathbb{Z}\times\cdots\times\mathbb{Z}/m_l\mathbb{Z}}(P,\lambda) = m_{\mathbb{Z}^l}(P,\lambda).$$

Where the limit is with $m_1,\dots,m_l$ going to infinity independently.

Dihedral groups

$$\Gamma = D_m = \langle \rho, \sigma \mid \rho^m, \sigma^2, \sigma\rho\sigma\rho \rangle.$$

Theorem

Let $P \in \mathbb{C}[D_m]$ be reciprocal. Then

$$[P^n]_0 = \frac{1}{2m} \sum_{j=1}^{m} \left(P^n \left(\xi_m^j, 1 \right) + P^n \left(\xi_m^j, -1 \right) \right),$$

where P^n is expressed as a sum of monomials ρ^k, $\sigma\rho^k$ before being evaluated.

For $\Gamma = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle x, y \,|\, x^m, y^2, [x,y] \rangle$,

$$[P^n]_0 = \frac{1}{2m} \sum_{j=1}^{m} \left(P\left(\xi_m^j, 1\right)^n + P\left(\xi_m^j, -1\right)^n \right).$$

Compare D_m and $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $x = \rho$ and $y = \sigma$ in D_m.

Theorem

Let

$$P = \sum_{k=0}^{m-1} \alpha_k x^k + \sum_{k=0}^{m-1} \beta_k y x^k$$

with real coefficients and reciprocal in $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (therefore it is also reciprocal in D_m). Then

$$m_{\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P,\lambda) = m_{D_m}(P,\lambda).$$

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$$m_{\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}(P, \lambda) = m_{D_m}(P, \lambda).$$

Corollary

Let $P \in \mathbb{R}[\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$ *be reciprocal. Then*

$$m_{\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P,\lambda) = m_{D_\infty}(P,\lambda),$$

where $D_\infty = \langle \rho, \sigma \mid \sigma^2, \sigma\rho\sigma\rho \rangle$.

Quotient approximations of the Mahler measure

Γ_m are quotients of Γ:

Theorem

Let $P \in \Gamma$ reciprocal.

- *For* $\Gamma = D_\infty$, $\Gamma_m = D_m$,

$$\lim_{m \to \infty} m_{D_m}(P, \lambda) = m_{D_\infty}(P, \lambda).$$

- *For* $\Gamma = PSL_2(\mathbb{Z}) = \langle x, y \,|\, x^2, y^3 \rangle$, $\Gamma_m = \langle x, y \,|\, x^2, y^3, (xy)^m \rangle$,

$$\lim_{m \to \infty} m_{\Gamma_m}(P, \lambda) = m_{PSL_2(\mathbb{Z})}(P, \lambda).$$

- *For* $\Gamma = \mathbb{Z} * \mathbb{Z} = \langle x, y \rangle$, $\Gamma_m = \langle x, y \,|\, [x, y]^m \rangle$,

$$\lim_{m \to \infty} m_{\Gamma_m}(P, \lambda) = m_{\mathbb{Z} * \mathbb{Z}}(P, \lambda).$$

$x + x^{-1} + y + y^{-1}$ revisited

Now $P = x + x^{-1} + y + y^{-1}$.

$$u_{\mathbb{Z}\times\mathbb{Z}}(P,\lambda) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \lambda^{2n} = F\left(\frac{1}{2}, \frac{1}{2}; 1, 16\lambda^2\right)$$

$$u_{\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P,\lambda) = \sum_{n=0}^{\infty} \binom{4n}{2n} \lambda^{2n}$$

$$u_{\mathbb{Z}*\mathbb{Z}}(P,\lambda) = \frac{3}{1 + 2\sqrt{1 - 12\lambda^2}}$$

Arbitrary number of variables

For $P_{1,l} = x_1 + x_1^{-1} + \cdots + x_l + x_l^{-1}$,

$$u_{\mathbb{F}_l}(P_{1,l}, \lambda) = g_{2l}(\lambda).$$

where

$$g_d(\lambda) = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)\lambda^2}}.$$

is the generating function of the circuits of a d-regular tree (Bartholdi, 1999).

For $P_{2,l} = (1 + x_1 + \cdots + x_{l-1})\left(1 + x_1^{-1} + \cdots + x_{l-1}^{-1}\right)$,

$$u_{\mathbb{F}_{l-1}}(P_{2,l}, \lambda) = g_l(\lambda).$$

In particular,

$$m_{\mathbb{F}_l}(P_{1,l}, \lambda) = m_{\mathbb{F}_{2l-1}}(P_{2,2l}, \lambda).$$

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Abelian case.

For $P_{1,l} = x_1 + x_1^{-1} + \cdots + x_l + x_l^{-1}$,

$$\left[P_{1,l}^n\right]_0 = \sum_{a_1+\cdots+a_l=n} \frac{(2n)!}{(a_1!)^2 \ldots (a_l!)^2},$$

For $P_{2,l} = (1 + x_1 + \cdots + x_{l-1}) \left(1 + x_1^{-1} + \cdots + x_{l-1}^{-1}\right)$,

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$$\left[P_{1,l}^{2n}\right]_0 = \binom{2n}{n} \left[P_{2,l}^n\right]_0$$

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$$\left[P_{1,l}^{2n}\right]_0 = \binom{2n}{n} \left[P_{2,l}^n\right]_0$$

Lück–Fuglede–Kadison determinant

Very general picture

- Γ discrete group.
- $l^2(\Gamma)$ Hilbert space
- $\mathcal{N}(\Gamma)$ algebra of Γ-equivariant bounded operators $l^2(\Gamma) \to l^2(\Gamma)$.
- M finite-dimensional Hilbert $\mathcal{N}(\Gamma)$-module.
- $A : M \to M$ selfadjoint, Lück–Fuglede–Kadison determinant:

$$\det(A) := \exp\left(\int_0^\infty \log(\lambda)\mathrm{d}F\right),$$

where F is the spectral density function.
For any T, $\det(T) := \det(TT^*)^{\frac{1}{2}}$.

If T is invertible, the classical Fuglede–Kadison determinant:

$$\det(T) = \exp\left(\frac{1}{2}\mathrm{tr}(\log(TT^*))\right),$$

where $tr(A) = \langle A(e), e\rangle$.

- Γ finite.

$$\mathbb{C}\Gamma = l^2(\Gamma) = \mathcal{N}(\Gamma).$$

$$T : U \to V$$

$0 < \lambda_1 \leq, \dots, \leq \lambda_r$ eigenvalues of TT^*. Then

$$\det(T) = \left(\prod_{i=1}^{r} \lambda_i\right)^{\frac{1}{2|\Gamma|}}.$$

- $\Gamma = \mathbb{Z}^n$
  Fourier transform:

$$l^2(\mathbb{Z}^n) \cong L^2(\mathbb{T}^n)$$

$$\mathcal{N}(\mathbb{Z}^n) \cong L^\infty(\mathbb{T}^n)$$

$f \in L^\infty(\mathbb{T}^n) \rightsquigarrow M_f : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$, where $M_f(g) = g \cdot f$.

$$\det(f) = \exp\left(\int_{\mathbb{T}^n} \log|f(z)| \chi_{\{u \in S^1 \mid f(u) \neq 0\}} \mathrm{dvol}_z\right).$$

Further Study: recurrence for coefficients

- $\mathbb{Z}^l$

$$u(\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \frac{1}{1-\lambda P(x,y)} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y},$$

and

$$u'(\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \frac{P(x,y)}{(1-\lambda P(x,y))^2} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y},$$

and $u''(\lambda)$ has a similar form.
u, u', u'' periods of a holomorphic differential in the curve defined by $1 = \lambda P(x,y)$.By
Griffiths (1969)

$$A(\lambda)u'' + B(\lambda)u' + C(\lambda)u = 0,$$

Recurrence of the coefficients.

- $\mathbb{F}_l$
  Haiman (1993): $u(\lambda)$ is algebraic.
  Algebraic functions in non-commuting variables.

- What happens in "between"? Is there a recurrence for the coefficients?

Further study: Tree entropy and Volume Conjecture

$m\left(P, \frac{1}{l^1(P)}\right)$ related to $h(G)$

where G is the Cayley graph and h is the tree entropy

$$h(G) := \log \deg_G(o) - \sum_{n=1}^{\infty} \frac{p_n(o, G)}{n},$$

- o fixed vertex
- $p_n(o, G)$ is the probability that a simple random walk started at o on G is again at o after n steps.

Lyons (2005)
G_n are finite graphs that tend to a fixed transitive infinite graph G, then

$$h(G) = \lim_{n\to\infty} \frac{\log \tau(G_n)}{|V(G_n)|},$$

where $\tau(G)$ is the complexity, i.e., the number of spanning trees.
Compare to

Conjecture ((Volume Conjecture) Kashaev, H. Murakami, J. Murakami (1997))

Let K be a hyperbolic knot, and $J_n(K,q)$ its normalized colored Jones polynomial. Then

$$\frac{1}{2\pi}\mathrm{Vol}(S^3 \setminus K) = \lim_{n\to\infty} \frac{\log \left| J_n\left(K, \mathrm{e}^{\frac{2\pi i}{n}}\right)\right|}{n}$$