Mahler measure as special values of L-functions

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Diophantine equations and zeta functions

$$2x^2 = 1 \qquad x \in \mathbb{Z}$$

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No solutions!!!
($2x^2$ is always even and 1 is odd.)

We are looking at "odd" and "even" numbers instead of integers (reduction modulo $p = 2$).

Local solutions = solutions modulo p, and in $\mathbb{R}$.

Global solutions = solutions in $\mathbb{Z}$

global solutions $\Rightarrow$ local solutions

local solutions $\not\Rightarrow$ global solutions

Zeta functions

Local info $\rightsquigarrow$ zeta functions

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Nice properties:

- Euler product
- Functional equation
- Riemann Hypothesis
- Special values

$$\zeta(1) \quad \text{pole}, \qquad \zeta(2) = \frac{\pi^2}{6}$$

Periods

Definition

A complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Example:

$$\pi = \int\int_{x^2+y^2\leq 1} dx\, dy = \int_{\mathbb{R}} \frac{dx}{1+x^2}$$

$$\zeta(3) = \int\int\int_{0<x<y<z<1} \frac{dx\, dy\, dz}{(1-x)yz}$$

algebraic numbers

$$\log(2) = \int_1^2 \frac{dx}{x}$$

$e = 2.718218\ldots$ does not seem to be a period

"Beilinson's type" statements: Special values of L, *zeta*-functions may be written in terms of certain periods called *regulators*.

Mahler measure of one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n\to\infty} \frac{|\alpha^{n+1}-1|}{|\alpha^n-1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a\prod_i (x-\alpha_i)$$

$$M(P) = |a|\prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log|a| + \sum_i \log^+ |\alpha_i|$$

Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

Jensen's formula implies

$$m(P) = \log |a| + \sum_i \log\{\max\{1, |\alpha_i|\}\} \quad \text{for} \quad P(x) = a \prod_i (x - \alpha_i)$$

Mahler measure is ubiquitous!

- Interesting questions about distribution of values
- Heights
- Volumes in hyperbolic space
- Entropy of certain arithmetic dynamical systems

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3},2) = L'(\chi_{-3},-1)$$

$$L(\chi_{-3},s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

More examples in several variables

- Boyd & L. (2005)

$$m(x^2+1+(x+1)y+(x-1)z) = \frac{1}{\pi}L(\chi_{-4},2) + \frac{21}{8\pi^2}\zeta(3)$$

- L. (2003)

$$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5)$$

- Rogers & Zudilim (2010)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+8\right) = \frac{24}{\pi^2}L(E_{24},2)$$

Why do we get nice numbers?

In many cases, the Mahler measure is the special period coming from Beilinson's conjectures!

Deninger (1997) General framework.

Rodriguez-Villegas (1997) 2-variable case.

An algebraic integration for Mahler measure

$$P(x,y) = y + x - 1 \qquad X = \{P(x,y) = 0\}$$

$$m(P) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log|y + x - 1| \frac{dx}{x}\frac{dy}{y}$$

By Jensen's equality:

$$= \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{dx}{x}$$

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$$= \frac{1}{2\pi i} \int_\gamma \log |y| \frac{dx}{x} = -\frac{1}{2\pi i} \int_\gamma \eta(x, y)$$

where

$$\gamma = X \cap \{|x| = 1, |y| \geq 1\}$$

$$\eta(x, y) = \log |x| di \arg y - \log |y| di \arg x$$

$$d \arg x = \mathrm{Im} \left(\frac{dx}{x} \right)$$

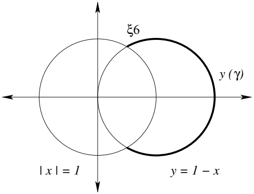
$$\eta(x, 1-x) = di D(x)$$

dilogarithm

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad |x| < 1$$

$$m(y + x - 1) = -\frac{1}{2\pi i} \int_{\gamma} \eta(x, y)$$

$$= -\frac{1}{2\pi} D(\partial\gamma) = \frac{1}{2\pi}(D(\xi_6) - D(\bar{\xi_6})) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$



The three-variable case

Theorem

L. (2005)
$P(x,y,z) \in \mathbb{Q}[x,y,z]$ *irreducible, nonreciprocal,*

$$X = \{P(x,y,z) = 0\}, \qquad C = \{\mathrm{Res}_z(P(x,y,z), P(x^{-1},y^{-1},z^{-1})) = 0\}$$

$$x \wedge y \wedge z = \sum_i r_i x_i \wedge (1-x_i) \wedge y_i \qquad \textit{in} \qquad \bigwedge^3 (\mathbb{C}(X)^*) \otimes \mathbb{Q},$$

$$\{x_i\}_2 \otimes y_i = \sum_j r_{i,j} \{x_{i,j}\}_2 \otimes x_{i,j} \qquad \textit{in} \qquad (\mathcal{B}_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$$

Then

$$4\pi^2 (m(P^*) - m(P)) = \mathcal{L}_3(\xi) \qquad \xi \in \mathcal{B}_3(\bar{\mathbb{Q}})_{\mathbb{Q}}$$

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$P(x, y, z) \in \mathbb{Q}[x, y, z]$ *irreducible, nonreciprocal,*

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$$\{x, y, z\} = 0 \quad \textit{in} \quad K_3^M(\mathbb{C}(X)) \otimes \mathbb{Q}$$

$$\{x_i\}_2 \otimes y_i \quad \textit{trivial in} \quad gr_3^{\gamma} K_4(\mathbb{C}(C)) \otimes \mathbb{Q}\,(?)$$

Then

$$4\pi^2(m(P^*) - m(P)) = \mathcal{L}_3(\xi) \qquad \xi \in \mathcal{B}_3(\bar{\mathbb{Q}})_{\mathbb{Q}}$$

- Explains all the known cases involving $\zeta(3)$ (by Borel's Theorem).
- It is constructive (no need of "happy idea" integrals).
- Integration sets hard to describe.
- Conjecture for n-variables using Goncharov's regulator currents. Provides motivation for Goncharov's construction.

Elliptic curves

$$E : Y^2 = X^3 + aX + b$$

Group structure!

Example:

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

$$x = \frac{kX - 2Y}{2X(X-1)} \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

L-function

$$L(E,s) = \prod_{\text{good } p} (1 - a_p p^{-s} + p^{1-2s})^{-1} \prod_{\text{bad } p} (1 - a_p p^{-s})^{-1}$$

$$a_p = 1 + p - \#E(\mathbb{F}_p)$$

Back to Mahler measure in two variables

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd (1998)

$$m\,(k) \stackrel{?}{=} s_k L'(E_{N(k)}, 0) \quad k \in \mathbb{N} \neq 0, 4$$

$$m\left(4\sqrt{2}\right) = L'(E_{64}, 0)$$

- Kurokawa & Ochiai (2005)
  For $h \in \mathbb{R}^*$,

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).$$

- Rogers & L. (2006)
  For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(ih + \frac{1}{ih}\right)\right) = m\left(\frac{4}{h^2}\right).$$

- Rogers & L. (2006)

$$m(8) = 4m(2)$$

- L. (2008)

$$m(5) = 6m(1)$$

- Regulator $\int_\gamma \eta(x, y)$ is given by a Kronecker–Eisenstein series that depends on the divisors (zeros and poles) of x, y.

- The relation between Mahler measures can be read from relations of the divisors.

Rogers (2007)

$$g(p) = \frac{1}{3} n\left(\frac{p+4}{p^{2/3}}\right) + \frac{4}{3} n\left(\frac{p-2}{p^{1/3}}\right),$$

where

$$g(k) = m\left((1+x)(1+y)(x+y) - kxy\right)$$
$$n(k) = m(x^3 + y^3 + 1 - kxy).$$

Using hypergeometric series.
In progress: understanding the relations using regulators and isogenies.

Rogers & Zudilim (late 2010)

$$m(8) = \frac{24}{\pi^2} L(E_{24}, 2) = 4L'(E_{24}, 0)$$

- $$m(k) = \mathrm{Re}\left(\log(k) - \frac{2}{k^2}\, {}_4F_3\left(\begin{matrix} \frac{3}{2}, \frac{3}{2}, 1, 1 \\ 2, 2, 2 \end{matrix} \,\middle|\, \frac{16}{k^2}\right)\right) \quad k \in \mathbb{C}$$
$$= \frac{k}{4} \mathrm{Re}\ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \,\middle|\, \frac{k^2}{16}\right) \quad k \geq 0$$
- $$m(8) = \frac{24}{\pi^2} F(2,3)$$
  where
$$F(2,3)$$
$$= 144 \sum_{\substack{n_i=-\infty \\ i=1,2,3,4}}^{\infty} \frac{(-1)^{n_1+\cdots+n_4}}{((6n_1+1)^2 + 2(6n_2+1)^2 + 3(6n_3+1)^2 + 6(6n_4+1)^2)^2}$$
  which can be in turn related the special values of L-functions of elliptic curves via modularity.

A three-variable example

Boyd (2005)

$$m(z+(x+1)(y+1)) \stackrel{?}{=} 2L'(E_{15},-1).$$

Eliminating z in

$$\begin{cases} z = (x+1)(y+1) \\ z^{-1} = (x^{-1}+1)(y^{-1}+1) \end{cases}$$

$$E_{15} : Y^2 = X^3 - 7X^2 + 16X.$$

They can also be related to regulators and some complicated generalizations of Kronecker-Eisenstein series due to Goncharov.

Boyd & L. (in progress)
This relationship may be used to compare with other Mahler measure formulas.

$$m(x+1+(x^2+x+1)y+(x+1)^2z) \stackrel{?}{=} \frac{1}{3}L'(\chi_{-3},-1)+\frac{13}{3\pi^2}\zeta(3) = m_1+m_2$$

with the exotic relation

$$m_1 - m_2 \stackrel{?}{=} 3L'(\chi_{-3},-1) - L'(E_{15},-1) \tag{1}$$

$$m(z+(x+1)(y+1)) \stackrel{?}{=} 2L'(E_{15},-1) \tag{2}$$

We can prove that the coefficient of $L'(E_{15},-1)$ in (1) is $-\frac{1}{2}$ of the coefficient in (2)

Higher Mahler measure

For $k \in \mathbb{Z}_{\geq 0}$, the k-higher Mahler measure of P is

$$m_k(P) := \int_0^1 \log^k \left| P\left(e^{2\pi i\theta}\right)\right| d\theta.$$

$$k = 1: \qquad m_1(P) = m(P),$$

and

$$m_0(P) = 1.$$

The simplest examples

Kurokawa, L.& Ochiai (2008)

$$\begin{aligned}
m_2(x-1) &= \frac{\zeta(2)}{2} = \frac{\pi^2}{12}. \\
m_3(x-1) &= -\frac{3\zeta(3)}{2}. \\
m_4(x-1) &= \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}. \\
m_5(x-1) &= -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}. \\
m_6(x-1) &= \frac{45}{2}\zeta(3)^2 + \frac{275}{1344}\pi^6.
\end{aligned}$$

Sinha& L. (2010)

$$m_3\left(\frac{x^n-1}{x-1}\right)=\frac{3}{2}\zeta(3)\left(\frac{-2+3n-n^3}{n^2}\right)+\frac{3\pi}{2}\sum_{\substack{j=1\\ n\nmid j}}^{\infty}\frac{\cot\left(\pi\frac{j}{n}\right)}{j^2}.$$

Examples

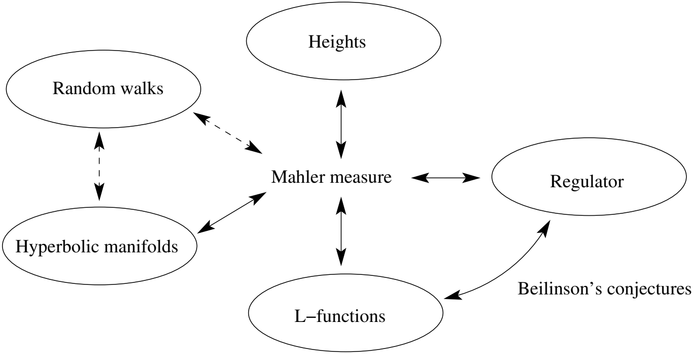
$$m_3\left(x^2+x+1\right)=-\frac{10}{3}\zeta(3)+\frac{\sqrt{3}\pi}{2}L(2,\chi_{-3}).$$

$$m_3\left(x^3+x^2+x+1\right)=-\frac{81}{16}\zeta(3)+\frac{3\pi}{2}L(2,\chi_{-4}).$$

Regulator interpretation?

In conclusion...

- Natural examples of Beilinson conjectures in action
- Examples of nontrivial identities between periods
- Hope of better understanding of special values of L-functions



Merci!

Thank you!