Mahler measure as special values of $L$-functions

Matilde N. Lalín

Université de Montréal
mlalin@dms.umontreal.ca
http://www.dms.umontreal.ca/~mlalin

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Diophantine equations and zeta functions

\[ 2x^2 = 1 \quad x \in \mathbb{Z} \]
Diophantine equations and zeta functions

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No solutions!!!
(\(2x^2\) is always even and 1 is odd.)

We are looking at “odd” and “even” numbers instead of integers (reduction modulo \(p = 2\)).
Local solutions $=$ solutions modulo $p$, and in $\mathbb{R}$.

Global solutions $=$ solutions in $\mathbb{Z}$

$\text{global solutions} \implies \text{local solutions}$

$\text{local solutions} \not\implies \text{global solutions}$
Zeta functions

Local info $\leadsto$ zeta functions

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \]

Nice properties:
- Euler product
- Functional equation
- Riemann Hypothesis
- Special values

\[ \zeta(1) \text{ pole, } \quad \zeta(2) = \frac{\pi^2}{6} \]
Definition

A complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Example:

$$\pi = \int \int_{x^2+y^2 \leq 1} dx \; dy = \int_{\mathbb{R}} \frac{dx}{1 + x^2}$$

$$\zeta(3) = \int \int \int_{0 < x < y < z < 1} \frac{dx \; dy \; dz}{(1 - x)yz}$$
algebraic numbers

\[ \log(2) = \int_{1}^{2} \frac{dx}{x} \]

\[ e = 2.718218 \ldots \text{ does not seem to be a period} \]

“Beilinson’s type” statements: Special values of \( L, \) \( zeta \)-functions may be written in terms of certain periods called \textit{regulators}.
Mahler measure of one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_{i}(x - \alpha_i)$$

$$\Delta_n = \prod_{i}(\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$
Lehmer (1933):

\[
\lim_{n \to \infty} \frac{\Delta_{n+1}}{\Delta_n} \prod_i \max\{1, |\alpha_i|\} \leq \frac{|\alpha|^{n+1} - 1}{|\alpha^n - 1|} = \begin{cases} 
|\alpha| & \text{if } |\alpha| > 1 \\
1 & \text{if } |\alpha| < 1
\end{cases}
\]

For

\[P(x) = a \prod_i (x - \alpha_i)\]

\[M(P) = |a| \prod_i \max\{1, |\alpha_i|\}\]

\[m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|\]
Mahler measure of multivariable polynomials

\[ P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \text{ the (logarithmic) Mahler measure is:} \]

\[
m(P) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| \, d\theta_1 \cdots d\theta_n
\]

\[
= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}
\]

Jensen’s formula implies

\[
m(P) = \log |a| + \sum_i \log \{\max\{1, |\alpha_i|\}\} \quad \text{for} \quad P(x) = a \prod_i (x - \alpha_i)
\]
Mahler measure is ubiquitous!

- Interesting questions about distribution of values
- Heights
- Volumes in hyperbolic space
- Entropy of certain arithmetic dynamical systems
Examples in several variables

Smyth (1981)

\[ m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \]

\[ L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \chi_{-3}(n) = \begin{cases} 
1 & n \equiv 1 \mod 3 \\
-1 & n \equiv -1 \mod 3 \\
0 & n \equiv 0 \mod 3 
\end{cases} \]

\[ m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3) \]
More examples in several variables


\[
m(x^2 + 1 + (x + 1)y + (x - 1)z) = \frac{1}{\pi} L(\chi_{-4}, 2) + \frac{21}{8\pi^2} \zeta(3)
\]


\[
m \left( 1 + x + \left( \frac{1 - x_1}{1 + x_1} \right) \left( \frac{1 - x_2}{1 + x_2} \right) (1 + y)z \right) = \frac{93}{\pi^4} \zeta(5)
\]

- Rogers & Zudilim (2010)

\[
m \left( x + \frac{1}{x} + y + \frac{1}{y} + 8 \right) = \frac{24}{\pi^2} L(E_{24}, 2)
\]
Why do we get nice numbers?

In many cases, the Mahler measure is the special period coming from Beilinson’s conjectures!


An algebraic integration for Mahler measure

\[ P(x, y) = y + x - 1 \quad X = \{ P(x, y) = 0 \} \]

\[ m(P) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |y + x - 1| \frac{dx \, dy}{x \, y} \]

By Jensen’s equality:

\[ = \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{dx}{x} \]
\[
\begin{align*}
&= \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{dx}{x} \\
&= \frac{1}{2\pi i} \int_{\gamma} \log |y| \frac{dx}{x} = -\frac{1}{2\pi i} \int_{\gamma} \eta(x, y)
\end{align*}
\]

where

\[
\gamma = X \cap \{|x| = 1, |y| \geq 1\}
\]

\[
\eta(x, y) = \log |x| di \arg y - \log |y| di \arg x
\]

\[
d \arg x = \text{Im} \left( \frac{dx}{x} \right)
\]
\[ \eta(x, 1 - x) = diD(x) \]

dilogarithm

\[ \text{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad |x| < 1 \]

\[ m(y + x - 1) = -\frac{1}{2\pi i} \int_{\gamma} \eta(x, y) \]

\[ = -\frac{1}{2\pi} D(\partial \gamma) = \frac{1}{2\pi} (D(\xi_6) - D(\bar{\xi}_6)) = \frac{3\sqrt{3}}{4\pi} L(\chi_3, 2) \]
The three-variable case

Theorem
L. (2005)

\[ P(x, y, z) \in \mathbb{Q}[x, y, z] \text{ irreducible, nonreciprocal}, \]

\[ X = \{ P(x, y, z) = 0 \}, \quad C = \{ \text{Res}_z(P(x, y, z), P(x^{-1}, y^{-1}, z^{-1})) = 0 \} \]

\[ x \wedge y \wedge z = \sum_i r_i x_i \wedge (1 - x_i) \wedge y_i \quad \text{in} \quad \bigwedge^3 (\mathbb{C}(X)^*) \otimes \mathbb{Q}, \]

\[ \{ x_i \}_2 \otimes y_i = \sum_j r_{i,j} \{ x_{i,j} \}_2 \otimes x_{i,j} \quad \text{in} \quad (\mathcal{B}_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}} \]

Then

\[ 4\pi^2 (m(P^*) - m(P)) = \mathcal{L}_3(\xi) \quad \xi \in \mathcal{B}_3(\bar{\mathbb{Q}})_{\mathbb{Q}} \]
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\]

\[
\{x, y, z\} = 0 \quad \text{in} \quad K_3^M (\mathbb{C}(X)) \otimes \mathbb{Q}
\]

\[
\{x_i\} \otimes y_i \quad \text{trivial in} \quad \text{gr}_3^\gamma K_4 (\mathbb{C}(C)) \otimes \mathbb{Q} (?)
\]

Then

\[
4\pi^2 (m(P^*) - m(P)) = \mathcal{L}_3(\xi) \quad \xi \in \mathcal{B}_3(\bar{\mathbb{Q}})_{\mathbb{Q}}
\]
• Explains all the known cases involving $\zeta(3)$ (by Borel’s Theorem).

• It is constructive (no need of “happy idea” integrals).

• Integration sets hard to describe.

• Conjecture for $n$-variables using Goncharov’s regulator currents. Provides motivation for Goncharov’s construction.
Elliptic curves

\[ E : Y^2 = X^3 + aX + b \]

Group structure!

Example:

\[
\begin{align*}
  x + \frac{1}{x} + y + \frac{1}{y} + k &= 0 \\
  x &= \frac{kX - 2Y}{2X(X - 1)} \\
  y &= \frac{kX + 2Y}{2X(X - 1)}.
\end{align*}
\]

\[ Y^2 = X \left( X^2 + \left( \frac{k^2}{4} - 2 \right) X + 1 \right). \]
\[ L(E, s) = \prod_{\text{good } p} (1 - a_p p^{-s} + p^{1-2s})^{-1} \prod_{\text{bad } p} (1 - a_p p^{-s})^{-1} \]

\[ a_p = 1 + p - \#E(\mathbb{F}_p) \]
Back to Mahler measure in two variables

\[ m(k) := m \left( x + \frac{1}{x} + y + \frac{1}{y} + k \right) \]

Boyd (1998)

\[ m(k) \equiv s_k L'(E_{N(k)}, 0) \quad k \in \mathbb{N} \neq 0, 4 \]

\[ m \left( 4\sqrt{2} \right) = L'(E_{64}, 0) \]
• Kurokawa & Ochiai (2005)
For $h \in \mathbb{R}^*$,

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).$$

• Rogers & L. (2006)
For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(\frac{1}{ih} + \frac{1}{ih}\right)\right) = m\left(\frac{4}{h^2}\right).$$
• Rogers & L. (2006)
  \[ m(8) = 4m(2) \]

• L. (2008)
  \[ m(5) = 6m(1) \]
• Regulator $\int_\gamma \eta(x, y)$ is given by a Kronecker–Eisenstein series that depends on the divisors (zeros and poles) of $x, y$.

• The relation between Mahler measures can be read from relations of the divisors.
Rogers (2007)

\[
g(p) = \frac{1}{3} n \left( \frac{p + 4}{p^{2/3}} \right) + \frac{4}{3} n \left( \frac{p - 2}{p^{1/3}} \right),
\]

where

\[
g(k) = m ((1 + x)(1 + y)(x + y) - kxy)
\]

\[
n(k) = m(x^3 + y^3 + 1 - kxy).
\]

Using hypergeometric series.
In progress: understanding the relations using regulators and isogenies.
Rogers & Zudilim (late 2010)

\[ m(8) = \frac{24}{\pi^2} L(E_{24}, 2) = 4L'(E_{24}, 0) \]
\[ m(k) = \text{Re} \left( \log(k) - \frac{2}{k^2} \; _4F_3 \left( \begin{array}{c} \frac{3}{2}, \frac{3}{2}, 1, 1 \\ 2, 2, 2 \end{array} \; \bigg| \frac{16}{k^2} \right) \right) \quad k \in \mathbb{C} \]

\[ = \frac{k}{4} \text{Re} \; _3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \; \bigg| \frac{k^2}{16} \right) \quad k \geq 0 \]

\[ m(8) = \frac{24}{\pi^2} F(2, 3) \]

where

\[ F(2, 3) = \sum_{n_i=-\infty}^{\infty} \frac{(-1)^{n_1+\cdots+n_4}}{((6n_1 + 1)^2 + 2(6n_2 + 1)^2 + 3(6n_3 + 1)^2 + 6(6n_4 + 1)^2)^2} \]

which can be in turn related the special values of \( L \)-functions of elliptic curves via modularity.
A three-variable example

Boyd (2005)

\[ m(z + (x + 1)(y + 1)) \equiv 2L'(E_{15}, -1). \]

Eliminating \( z \) in

\[
\begin{align*}
  z &= (x + 1)(y + 1) \\
  z^{-1} &= (x^{-1} + 1)(y^{-1} + 1)
\end{align*}
\]

\( E_{15} : Y^2 = X^3 - 7X^2 + 16X. \)

They can also be related to regulators and some complicated generalizations of Kronecker-Eisenstein series due to Goncharov.
Boyd & L. (in progress)
This relationship may be used to compare with other Mahler measure formulas.

\[ m(x+1+(x^2+x+1)y+(x+1)^2z) = \frac{1}{3} L'(\chi_3, -1) + \frac{13}{3\pi^2} \zeta(3) = m_1 + m_2 \]

with the exotic relation

\[ m_1 - m_2 = 3L'(\chi_3, -1) - L'(E_{15}, -1) \quad (1) \]

\[ m(z + (x + 1)(y + 1)) = 2L'(E_{15}, -1) \quad (2) \]

We can prove that the coefficient of \( L'(E_{15}, -1) \) in (1) is \(-\frac{1}{2}\) of the coefficient in (2)
Higher Mahler measure

For $k \in \mathbb{Z}_{\geq 0}$, the $k$-higher Mahler measure of $P$ is

$$m_k(P) := \int_{0}^{1} \log^k \left| P \left( e^{2\pi i \theta} \right) \right| d\theta.$$ 

$k = 1 : \quad m_1(P) = m(P),$ 

and

$$m_0(P) = 1.$$
The simplest examples

Kurokawa, L. & Ochiai (2008)

\[ m_2(x - 1) = \frac{\zeta(2)}{2} = \frac{\pi^2}{12}. \]

\[ m_3(x - 1) = -\frac{3\zeta(3)}{2}. \]

\[ m_4(x - 1) = \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}. \]

\[ m_5(x - 1) = -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}. \]

\[ m_6(x - 1) = \frac{45}{2} \zeta(3)^2 + \frac{275}{1344} \pi^6. \]

\[ m_3 \left( \frac{x^n - 1}{x - 1} \right) = \frac{3}{2} \zeta(3) \left( \frac{-2 + 3n - n^3}{n^2} \right) + \frac{3\pi}{2} \sum_{\substack{j=1 \\ n \nmid j}}^{\infty} \cot \left( \frac{\pi j}{n} \right) j^2. \]

Examples

\[ m_3 \left( x^2 + x + 1 \right) = -\frac{10}{3} \zeta(3) + \frac{\sqrt{3}\pi}{2} L(2, \chi_{-3}). \]

\[ m_3 \left( x^3 + x^2 + x + 1 \right) = -\frac{81}{16} \zeta(3) + \frac{3\pi}{2} L(2, \chi_{-4}). \]

Regulator interpretation?
In conclusion...

- Natural examples of Beilinson conjectures in action
- Examples of nontrivial identities between periods
- Hope of better understanding of special values of $L$-functions
Merci!

Thank you!