Mahler measure and special values of L-functions

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Diophantine equations and zeta functions

$$2x^2 - 1 = 0 \qquad x \in \mathbb{Z}$$

No solutions!!!

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Global solutions = solutions in $\mathbb{Z}$

global solutions $\Rightarrow$ local solutions

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Zeta functions

Local info $\rightsquigarrow$ zeta functions

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Nice properties:

- Euler product
- Functional equation
- Riemann Hypothesis
- Special values

$$\zeta(1) \quad \text{pole}, \qquad \zeta(2) = \frac{\pi^2}{6}$$

Periods

Definition

A complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Example:

$$\pi = \int\int_{x^2+y^2\leq 1} dx\, dy = \int_{\mathbb{R}} \frac{dx}{1+x^2}$$

$$\zeta(3) = \int\int\int_{0<x<y<z<1} \frac{dx\, dy\, dz}{(1-x)yz}$$

algebraic numbers

$$\log(2) = \int_1^2 \frac{dx}{x}$$

$\mathrm{e} = 2.718218\ldots$ does not seem to be a period

"Beilinson's type" conjectures: Special values of *zeta*-functions may be written in terms of certain periods called *regulators*.

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Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula implies

$$m(P) = \log|a| + \sum_i \log\{\max\{1, |\alpha_i|\}\} \quad \text{for} \quad P(x) = a \prod_i (x - \alpha_i)$$

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Mahler measure is ubiquitous!

- Interesting questions about distribution of values
- Heights
- Volumes in hyperbolic space
- Entropy of certain discrete dynamical systems

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) = L'(\chi_{-3},-1)$$

$$L(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

More examples in several variables

- Boyd & L. (2005)

$$\pi^2 m(x^2+1+(x+1)y+(x-1)z) = \pi L(\chi_{-4},2) + \frac{21}{8}\zeta(3)$$

- L. (2003)

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = 93\zeta(5)$$

- Known formulas for

$$\pi^{n+2} m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

Why do we get nice numbers?

In many cases, the Mahler measure is the special period coming from Beilinson's conjectures!

Deninger (1997) General framework.

Rodriguez-Villegas (1997) 2-variable case.

An algebraic integration for Mahler measure

$$P(x,y) = y + x - 1 \qquad X = \{P(x,y) = 0\}$$

$$m(P) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log|y + x - 1| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

By Jensen's equality:

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{\mathrm{d}x}{x}$$

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$$= \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \eta(x,y)$$

where

$$\gamma = X \cap \{|x| = 1, |y| \geq 1\}$$

$$\eta(x,y) = \log|x| \mathrm{d}\mathrm{i} \arg y - \log|y| \mathrm{d}\mathrm{i} \arg x$$

$$\mathrm{d} \arg x = \mathrm{Im}\left(\frac{\mathrm{d}x}{x}\right)$$

Theorem

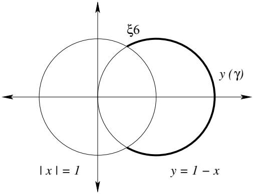
$$\eta(x, 1-x) = \mathrm{di}D(x)$$

dilogarithm

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad |x| < 1$$

$$m(y + x - 1) = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \eta(x, y)$$

$$= -\frac{1}{2\pi} D(\partial\gamma) = \frac{1}{2\pi}(D(\xi_6) - D(\bar{\xi_6})) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$



The three-variable case

Theorem

L. (2005)
$P(x,y,z) \in \mathbb{Q}[x,y,z]$ *irreducible, nonreciprocal,*

$$X = \{P(x,y,z)=0\}, \qquad C = \{\mathrm{Res}_z(P(x,y,z), P(x^{-1},y^{-1},z^{-1}))=0\}$$

$$x \wedge y \wedge z = \sum_i r_i x_i \wedge (1-x_i) \wedge y_i \qquad \text{in} \qquad \bigwedge^3 (\mathbb{C}(X)^*) \otimes \mathbb{Q},$$

$$\{x_i\}_2 \otimes y_i = \sum_j r_{i,j} \{x_{i,j}\}_2 \otimes x_{i,j} \qquad \text{in} \qquad (\mathcal{B}_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$$

Then

$$4\pi^2(m(P^*) - m(P)) = \mathcal{L}_3(\xi) \qquad \xi \in \mathcal{B}_3(\bar{\mathbb{Q}})_{\mathbb{Q}}$$

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$$\{x,y,z\} = 0 \quad \textit{in} \quad K_3^M(\mathbb{C}(X)) \otimes \mathbb{Q}$$

$$\{x_i\}_2 \otimes y_i \quad \textit{trivial in} \quad gr_3^\gamma K_4(\mathbb{C}(C)) \otimes \mathbb{Q}\,(?)$$

Then

$$4\pi^2(m(P^*) - m(P)) = \mathcal{L}_3(\xi) \qquad \xi \in \mathcal{B}_3(\bar{\mathbb{Q}})_{\mathbb{Q}}$$

- Explains all the known cases involving $\zeta(3)$ (by Borel's Theorem).
- It is constructive (no need of "happy idea" integrals).
- Integration sets hard to describe.
- Conjecture for n-variables using Goncharov's regulator currents. Provides motivation for Goncharov's construction.

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Elliptic curves

$$E : Y^2 = X^3 + aX + b$$

Group structure!

Example:

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

$$x = \frac{kX - 2Y}{2X(X-1)} \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

L-function

$$L(E,s)=\prod_{\text{good } p}(1-a_p p^{-s}+p^{1-2s})^{-1}\prod_{\text{bad } p}(1-a_p p^{-s})^{-1}$$

$$a_p = 1+p-\#E(\mathbb{F}_p)$$

Back to Mahler measure in two variables

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd (1998)

$$m(k) \stackrel{?}{=} \frac{L'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

$$m\left(4\sqrt{2}\right) = L'(E_{4\sqrt{2}}, 0)$$

- Rogers & L. (2006)
  For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h+\frac{1}{h}\right)\right)+m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right)=m\left(\frac{4}{h^2}\right).$$

- Kurokawa & Ochiai (2005)
  For $h \in \mathbb{R}^*$,

$$m(4h^2)+m\left(\frac{4}{h^2}\right)=2m\left(2\left(h+\frac{1}{h}\right)\right).$$

Corollary

- *Rogers & L. (2006)*

$$m(8) = 4m(2)$$

- *L. (2008)*

$$m(5) = 6m(1)$$

Combining results of Bloch, Beilinson:

$E/\mathbb{Q}$
Regulator is given by a Kronecker–Eisenstein series that depends on the divisors of x, y.

$$\int_\gamma \eta(x,y) = \mathrm{covol}(\Lambda)\Omega \sum_{\lambda \in \Lambda}{}' \frac{(x-y,\lambda)\bar{\lambda}}{|\lambda|^4}$$

$$E/\mathbb{C} = \mathbb{C}/\Lambda \qquad \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

$$(\cdot,\cdot) : \mathbb{C}/\Lambda \times \Lambda \to \mathbb{S}^1$$

If $x = \sum \alpha_i (P_i)$, then

$$(x,\lambda) := \sum \alpha_i (P_i, \lambda).$$

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A three-variable example

Boyd (2005)

$$m(z+(x+1)(y+1)) \stackrel{?}{=} 2L'(E_{15},-1).$$

$$E_{15} : Y^2 = X^3 - 7X^2 + 16X.$$

$$m(z-(x+1)(y+1)) = -\frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z)$$

$$x\wedge y\wedge z = x\wedge y\wedge (1+x)(1+y) = -x\wedge(1+x)\wedge y + y\wedge(1+y)\wedge x$$

$$\eta(x,1-x,y) = \mathrm{d}\omega(x,y)$$

$$= -\frac{1}{4\pi^2}\int_\gamma -\omega(-x,y)+\omega(-y,x).$$

under the condition

$$z = (x+1)(y+1)$$

$$z^{-1} = (x^{-1}+1)(y^{-1}+1)$$

$$(x+1)^2y^2 + (2(x+1)^2 - x)y + (x+1)^2 = 0.$$

L (2008)

By a result of Goncharov,

$$\int_\gamma \omega(x,y) =$$

$$= (\mathrm{covol}(\Lambda))^2 \Omega \sideset{}{'}\sum_{\lambda_1+\lambda_2+\lambda_3=0} \frac{(y,\lambda_1)(x,\lambda_2)(1-x,\lambda_3)(\overline{\lambda_3}-\overline{\lambda_2})}{|\lambda_1|^2|\lambda_2|^2|\lambda_3|^2}$$

the L-function can be related to the Kronecker–Eisenstein series.

Working with the divisors

Let $(x) = \sum \alpha_j (P_j), (1-x) = \sum \beta_k (Q_k)$, and $(y) = \sum \gamma_l (R_l)$ divisors in E.
Then

$$\diamond : (Div(E) \wedge Div(E)) \otimes Div(E) \to Div(E) \wedge Div(E) / \sim$$

$$((x) \wedge (1-x)) \diamond (y) = \sum \alpha_j \beta_k \gamma_l (P_j - R_l, Q_k - R_l)$$

Here

$$(P, Q) \sim -(-P, -Q).$$

Note that

$$(P, Q) = -(Q, P).$$

$$m(z-(x+1)(y+1)) = -\frac{1}{4\pi^2}\int_\gamma -\omega(-x,y)+\omega(-y,x).$$

$$Y^2 = X^3 - 7X^2 + 16X.$$

Let $P = (4,4)$ (point of order 4).

$$(x) = 2(2P) - 2O$$

$$(1+x) = (P) + (3P) - 2O$$

$$(y) = 2(P) - 2(3P)$$

$$(1+y) = (2P) + O - 2(3P)$$

$$-((x)\wedge(1+x))\diamond(y) + ((y)\wedge(1+y))\diamond(x) = -32((P,O)+(P,2P)-(P,-P))$$

Boyd & L. (in progress)
This relationship may be used to compare with other Mahler measure formulas.

$$m(x+1+(x^2+x+1)y+(x+1)^2z) \stackrel{?}{=} \frac{1}{3}L'(\chi_{-3},-1)+\frac{13}{3\pi^2}\zeta(3) = m_1+m_2$$

with the exotic relation

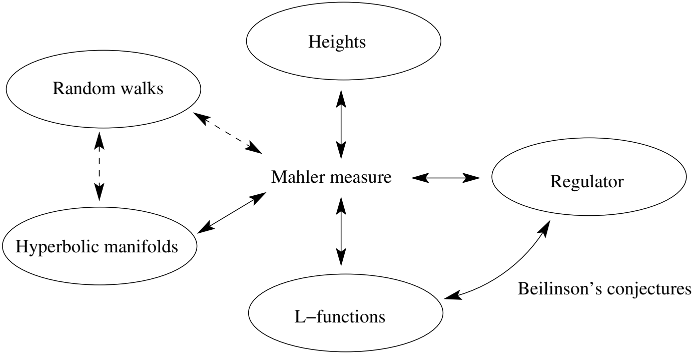
$$m_1 - m_2 \stackrel{?}{=} 3L'(\chi_{-3},-1) - L'(E_{15},-1) \qquad (1)$$

$$m(z+(x+1)(y+1)) \stackrel{?}{=} 2L'(E_{15},-1) \qquad (2)$$

We can prove that the coefficient of $L'(E_{15},-1)$ in (1) is $-\frac{1}{2}$ of the coefficient in (2)

In conclusion...

- Natural examples of Beilinson conjectures in action
- Examples of nontrivial identities between periods
- Hope of better understanding of special values of L-functions



Merci de votre attention!