Regulators and computations of Mahler measures

Matilde N. Lalín

Institute for Advanced Study
mlalin@math.ias.edu
http://www.math.ias.edu/~mlalin

January 5th, 2006

Mahler measure of multivariate polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n}$$

$$\mathbb{T}^n = S^1 \times \cdots \times S^1$$

Mahler measure of multivariate polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n}$$

$$\mathbb{T}^n = S^1 \times \cdots \times S^1$$

Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1)$$

$$\mathrm{L}(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.
Zagier:

$$\widehat{\mathcal{L}_k}(x) := \widehat{\mathrm{Re}_k} \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number
$\widehat{\mathrm{Re}_k} = \mathrm{Re}$ or $\mathrm{i}\,\mathrm{Im}$ if k is odd or even.
One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

$\widehat{\mathcal{L}_k}$ satisfies lots of functional equations

$$\widehat{\mathcal{L}_k}\left(\frac{1}{x}\right) = (-1)^{k-1}\widehat{\mathcal{L}_k}(x) \qquad \widehat{\mathcal{L}_k}(\bar{x}) = (-1)^{k-1}\widehat{\mathcal{L}_k}(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

Five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

The relation with regulators

Deninger (1997)

$$m(P) = m(P^*) + \frac{1}{(-2i\pi)^{n-1}} \int_\Gamma \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \cdots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$\eta_n(n)(x_1, \dots, x_n)$ is a $\mathbb{R}(n-1)$-valued smooth $n-1$-form in $X(\mathbb{C})$.

Example:

$$\eta_2(2)(x, y) = \log|x| \,\mathrm{di}\arg y - \log|y| \,\mathrm{di}\arg x$$

Example:

$$\eta_2(2)(x,y) = \log|x| \mathrm{d}i \arg y - \log|y| \mathrm{d}i \arg x$$

Smyth (1981)

$$m(1+x+y) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log|1+x+y| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

by Jensen's equality:

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+|1+x| \frac{\mathrm{d}x}{x}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \log|y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \eta_2(2)(x,y)$$

where

$$\Gamma = \{1+x+y=0\} \cap \{|x|=1, |y| \geq 1\}$$

Properties of $\eta_n(n)(x_1, \ldots, x_n)$

- Multiplicative in each variable, anti-symmetric.
  $\eta_n(n)$ is a function on $\bigwedge^n(\mathbb{C}(X)^*)_{\mathbb{Q}}$

Properties of $\eta_n(n)(x_1, \ldots, x_n)$

- Multiplicative in each variable, anti-symmetric.
  $\eta_n(n)$ is a function on $\bigwedge^n(\mathbb{C}(X)^*)_{\mathbb{Q}}$
- $\mathrm{d}\eta_n(n)(x_1, \ldots, x_n) = \widehat{\mathrm{Re}}_n\left(\frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}\right)$

Properties of $\eta_n(n)(x_1, \dots, x_n)$

- Multiplicative in each variable, anti-symmetric.
  $\eta_n(n)$ is a function on $\bigwedge^n(\mathbb{C}(X)^*)_{\mathbb{Q}}$
- $\mathrm{d}\eta_n(n)(x_1, \dots, x_n) = \widehat{\mathrm{Re}_n}\left(\frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}\right)$
- $\eta_n(n)(x, 1-x, x_1, \dots, x_{n-2}) = \mathrm{d}\eta_{n-1}(n)(x, x_1, \dots, x_{n-2})$

Examples

$$\eta_2(2)(x, 1 - x) = \mathrm{d}\widehat{D}(x)$$

Examples

$$\eta_2(2)(x, 1-x) = \mathrm{d}\widehat{D}(x)$$

$$\eta_3(3)(x,y,z) = \log|x| \left(\frac{1}{3}\mathrm{d}\log|y| \wedge \mathrm{d}\log|z| + \mathrm{d}\,\mathrm{i}\arg y \wedge \mathrm{d}\,\mathrm{i}\arg z \right)$$

$$+ \log|y| \left(\frac{1}{3}\mathrm{d}\log|z| \wedge \mathrm{d}\log|x| + \mathrm{d}\,\mathrm{i}\arg z \wedge \mathrm{d}\,\mathrm{i}\arg x \right)$$

$$+ \log|z| \left(\frac{1}{3}\mathrm{d}\log|x| \wedge \mathrm{d}\log|y| + \mathrm{d}\,\mathrm{i}\arg x \wedge \mathrm{d}\,\mathrm{i}\arg y \right)$$

$$\eta_3(3)(x, 1-x, y) = \mathrm{d}\eta_3(2)(x,y)$$

$$\eta_3(2)(x,y)$$

$$= \widehat{D}(x)\mathrm{d}\,\mathrm{i}\arg y + \frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

First variable in $\eta_n(n-1)$ behaves like the five-term relation

$$[x] + [y] + [1 - xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-x}{1-xy}\right]$$

First variable in $\eta_n(n-1)$ behaves like the five-term relation

$$[x]+[y]+[1-xy]+\left[\frac{1-x}{1-xy}\right]+\left[\frac{1-x}{1-xy}\right]$$

Now

$$\eta_n(n-1)(x,x,x_1,\ldots,x_{n-3}) = \mathrm{d}\eta_n(n-2)(x,x_1,\ldots,x_{n-3})$$

First variable in $\eta_n(n-2)$ behaves like rational functional equations of $\mathcal{L}_3$.

...

First variable in $\eta_n(n-1)$ behaves like the five-term relation

$$[x]+[y]+[1-xy]+\left[\frac{1-x}{1-xy}\right]+\left[\frac{1-x}{1-xy}\right]$$

Now

$$\eta_n(n-1)(x,x,x_1,\ldots,x_{n-3}) = \mathrm{d}\eta_n(n-2)(x,x_1,\ldots,x_{n-3})$$

First variable in $\eta_n(n-2)$ behaves like rational functional equations of $\mathcal{L}_3$.

...

$$\eta_n(2)(x,x) = \mathrm{d}\eta_n(1)(x)$$

and

$$\eta_n(1)(x) = \widehat{\mathcal{L}_n}(x)$$

Example in three variables

Smyth (1981)

$$m(1-x+(1-y)z)=\frac{7}{2\pi^2}\zeta(3)$$

$$m(P)=m(1-y)-\frac{1}{(2\pi)^2}\int_\Gamma \eta_3(3)(x,y,z).$$

$$x\wedge y\wedge z=-\ \ x\wedge(1-x)\wedge y-\ \ y\wedge(1-y)\wedge x,$$

in other words,

$$\eta_3(3)(x,y,z)=-\eta_3(3)(x,1-x,y)-\eta_3(3)(y,1-y,x).$$

We have

$$m((1-x)+(1-y)z)=\frac{1}{4\pi^2}\int_\gamma \eta_3(2)(x,y)+\eta_3(2)(y,x).$$

$$\Gamma = \{P(x,y,z)=0\} \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$\partial\Gamma = \{P(x,y,z)=0\} \cap \{|x| = |y| = |z| = 1\}$$

Maillot: $P \in \mathbb{R}[x,y,z]$,

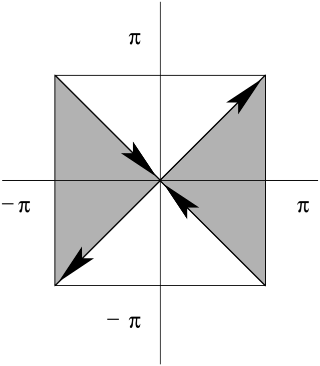
$$\gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x| = |y| = 1\}.$$

$C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$

$$-\{x\}_2 \otimes y - \{y\}_2 \otimes x = \pm 2\{x\}_2 \otimes x.$$



$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2} 8(\mathcal{L}_3(1) - \mathcal{L}_3(-1)) = \frac{7}{2\pi^2}\zeta(3).$$

New examples

Boyd & L. (2005)

$$m(x^2+1+(x+1)y+(x-1)z) = \frac{\mathrm{L}(\chi_{-4},2)}{\pi} + \frac{21}{8\pi^2}\zeta(3)$$

$$m(x^2+x+1+(x+1)y+z) = \frac{\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) + \frac{19}{6\pi^2}\zeta(3)$$

An example in four variables

L.(2003)

$$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 2\pi^2 \mathrm{L}(\chi_{-4},2)+8\sum_{0\le j<k}\frac{(-1)^{j+k+1}}{(2j+1)^3 k}$$

An example in four variables

L.(2003)

$$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 2\pi^2 \mathrm{L}(\chi_{-4},2)+8\sum_{0\le j<k}\frac{(-1)^{j+k+1}}{(2j+1)^3 k}$$

(2005)

$$= 24\mathrm{L}(\chi_{-4},4)$$

An example in four variables

L.(2003)

$$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 2\pi^2 \mathrm{L}(\chi_{-4},2)+8\sum_{0\le j<k}\frac{(-1)^{j+k+1}}{(2j+1)^3 k}$$

(2005)

$$= 24\mathrm{L}(\chi_{-4},4)$$

In general, for m odd,

$$\sum_{0\le j<k}\frac{(-1)^{j+k+1}}{(2j+1)^m k}$$

$$= m\mathrm{L}(\chi_{-4},m+1)+\sum_{h=1}^{\frac{m-1}{2}}\frac{(-1)^h\pi^{2h}(2^{2h}-1)}{(2h)!}B_{2h}\mathrm{L}(\chi_{-4},m-2h+1).$$

Generalized Mahler measure

Gon & Oyanagi (2004)

For $f_1, \ldots, f_r \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$,

$$m(f_1, \ldots, f_r) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \max\{\log|f_1|, \ldots, \log|f_r|\} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}$$

Generalized Mahler measure

Gon & Oyanagi (2004)

For $f_1, \dots, f_r \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$,

$$m(f_1, \dots, f_r) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \max\{\log|f_1|, \dots, \log|f_r|\} \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n}$$

Note

$$m(f_1, f_2) = m(f_1 + zf_2)$$

Examples

The particular case when $f_j = P(x_j)$ for some $P \in \mathbb{C}[x]$.
Gon & Oyanagi (2004)

$$m(1-x_1,\dots,1-x_n) = \sum_{j=1}^{\left[\frac{n}{2}\right]} c_{j,n} \frac{\zeta(2j+1)}{\pi^{2j}}$$

$$m(1-x_1,1-x_2) = \frac{7}{2\pi^2}\zeta(3)$$

$$m(1-x_1,1-x_2,1-x_3) = \frac{9}{2\pi^2}\zeta(3)$$

$$m(1-x_1,1-x_2,1-x_3,1-x_4) = -\frac{93}{2\pi^4}\zeta(5) + \frac{9}{\pi^2}\zeta(3)$$

Can be also computed using regulators.

$|P(x)|$ is montononous when $0 \leq \arg x \leq \pi$.

In this case, $|P(x)| = 2\left|\sin \frac{\arg x}{2}\right|$.

$$m(P(x_1), \ldots, P(x_n)) = \frac{n!}{(\pi \mathrm{i})^n} \int_{0 \leq \arg x_n \leq \cdots \leq \arg x_1 \leq \pi} \eta(P(x_1), x_1, \ldots, x_n)$$

L. (2005)

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_n}{1+x_n}\right)=\sum_{j=1}^{\left[\frac{n}{2}\right]} c'_{j,n}\frac{\zeta(2j+1)}{\pi^{2j}}$$

$$m\left(\frac{1-x_1}{1+x_1},\frac{1-x_2}{1+x_2}\right)=\frac{7}{\pi^2}\zeta(3)$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_3}{1+x_3}\right)=\frac{21}{2\pi^2}\zeta(3)$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_4}{1+x_4}\right)=-\frac{93}{\pi^4}\zeta(5)+\frac{21}{\pi^2}\zeta(3)$$

$m(1+x_1-x_1^{-1},\ldots,1+x_n-x_n^{-1}) =$ combination of polylogarithms.

$$m(1+x_1-x_1^{-1}) = -\log(\varphi),$$

$$m(1+x_1-x_1^{-1}, 1+x_2-x_2^{-1})$$
$$= \frac{1}{\pi^2}\,\mathrm{Re}(\mathrm{Li}_3(\varphi^2) - \mathrm{Li}_3(-\varphi^2) + \mathrm{Li}_3(\varphi^{-2}) - \mathrm{Li}_3(-\varphi^{-2}))$$

for $\varphi = \frac{-1+\sqrt{5}}{2}$.