Regulators and computations of Mahler measures

Mahler Measure in Mobile, University of South Alabama, Mobile, AL

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Mahler measure

Definition 1 *For $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic)* Mahler measure *is defined by*

$$m(P) \quad = \quad \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n}. \tag{1}$$

The simplest example in several variables is due to Smyth [11]

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1)$$

$$\mathrm{L}(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

Polylogarithms

Many examples should be understood in the context of polylogarithms.

Definition 2 *The* kth polylogarithm *is the function defined by the power series*

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1. \tag{2}$$

This function can be continued analytically to $\mathbb{C} \setminus [1, \infty)$.

In order to avoid discontinuities, and to extend polylogarithms to the whole complex plane, several modifications have been proposed. Zagier [12] considers the following version:

$$\widehat{\mathcal{L}_k}(x) := \widehat{\mathrm{Re}_k} \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right), \tag{3}$$

where B_j is the jth Bernoulli number, and $\widehat{\mathrm{Re}_k}$ denotes Re or i Im depending on whether k is odd or even.

This function is one-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ and continuous in $\mathbb{P}^1(\mathbb{C})$. Moreover, $\widehat{\mathcal{L}_k}$ satisfy very clean functional equations. The simplest ones are

$$\widehat{\mathcal{L}_k}\left(\frac{1}{x}\right) = (-1)^{k-1} \widehat{\mathcal{L}_k(x)} \qquad \widehat{\mathcal{L}_k}(\bar{x}) = (-1)^{k-1} \widehat{\mathcal{L}_k}(x).$$

There are also lots of functional equations which depend on the index k. For instance, for $k = 2$, we have the Bloch–Wigner dilogarithm,

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

which satisfies the well-known five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0. \tag{4}$$

The relation with regulators

We write the Mahler measure as an integral of a certain $\mathbb{R}(n-1)$-valued smooth $n-1$-form in $X(\mathbb{C})$, the variety determined by the zeroes of the polynomial.

$$m(P) = m(P^*) + \frac{1}{(-2\pi \mathrm{i})^{n-1}}\int_\Gamma \eta_n(n)(x_1,\dots,x_n)$$

where

$$\Gamma = \{P(x_1,\dots,x_n)=0\} \cap \{|x_1| = \ldots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

This was an idea of Deninger [3].

As an example, let us look at Smyth's case in two variables ([11]). The two-variable differential form is

$$\eta_2(2)(x,y) = \log|x| \mathrm{di}\arg y - \log|y|\mathrm{di}\arg x.$$

Then

$$m(1+x+y) = \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathbb{T}^2}\log|1+x+y|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}.$$

By Jensen's equality:

$$= \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}^1}\log^+|1+x|\frac{\mathrm{d}x}{x}$$

$$= \frac{1}{2\pi\mathrm{i}}\int_\Gamma \log|y|\frac{\mathrm{d}x}{x} = -\frac{1}{2\pi\mathrm{i}}\int_\Gamma \eta_2(2)(x,y),$$

where

$$\Gamma = \{1+x+y=0\} \cap \{|x|=1, |y|\geq 1\}.$$

Here are some properties of $\eta_n(n)(x_1,\dots,x_n)$:

- $\eta_n(n)$ is multiplicative in each variable and anti-symmetric. Hence it can be thought as a function on $\bigwedge^n(\mathbb{C}(X)^*)_\mathbb{Q}$.
- $\mathrm{d}\eta_n(n)(x_1,\dots,x_n) = \widehat{\mathrm{Re}_n}\left(\frac{\mathrm{d}x_1}{x_1}\wedge\ldots\wedge\frac{\mathrm{d}x_n}{x_n}\right)$
- There is an $\mathbb{R}(n-2)$-valued smooth $n-2$-form in $X(\mathbb{C})$ such that

$$\eta_n(n)(x, 1-x, x_1,\dots,x_{n-2}) = \mathrm{d}\eta_{n-1}(n)(x,x_1,\dots,x_{n-2})$$

In the two-variable case we have

$$\eta_2(2)(x, 1-x) = \mathrm{d}\widehat{D}(x).$$

The forms for $n = 3$ are

$$\eta_3(3)(x, y, z) = \log|x| \left(\frac{1}{3} \mathrm{d}\log|y| \wedge \mathrm{d}\log|z| + \mathrm{d}i\arg y \wedge \mathrm{d}i\arg z \right)$$

$$+ \log|y| \left(\frac{1}{3} \mathrm{d}\log|z| \wedge \mathrm{d}\log|x| + \mathrm{d}i\arg z \wedge \mathrm{d}i\arg x \right) + \log|z| \left(\frac{1}{3} \mathrm{d}\log|x| \wedge \mathrm{d}\log|y| + \mathrm{d}i\arg x \wedge \mathrm{d}i\arg y \right),$$

$$\eta_3(3)(x, 1-x, y) = \mathrm{d}\eta_3(2)(x, y),$$

$$\eta_3(2)(x, y) = \widehat{D}(x)\mathrm{d}i\arg y + \frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|).$$

Now the first variable of $\eta_n(n-1)$ behaves like the five-term relation.

As before, there is a $\mathbb{R}(n-3)$-valued smooth $n-3$-form in $X(\mathbb{C})$ such that

$$\eta_n(n-1)(x, x, x_1, \ldots, x_{n-3}) = \mathrm{d}\eta_n(n-2)(x, x_1, \ldots, x_{n-3}).$$

The first variable in $\eta_n(n-2)$ behaves like the functional equations of the trilogarithm.

And so on...

Finally, the second to last form satisfies

$$\eta_n(2)(x, x) = \mathrm{d}\eta_n(1)(x),$$

with

$$\eta_n(1)(x) = \widehat{\mathcal{L}_n}(x).$$

Let us take a look at Smyth's case for three variables. We can express the polynomial as $P(x, y, z) = (1-x) + (1-y)z$. We get:

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} = -\frac{1}{(2\pi)^2} \int_\Gamma \eta_3(3)(x, y, z).$$

$$x \wedge y \wedge z = - \; x \wedge (1-x) \wedge y - \; y \wedge (1-y) \wedge x,$$

in other words,

$$\eta_3(3)(x, y, z) = -\eta_3(3)(x, 1-x, y) - \eta_3(3)(y, 1-y, x).$$

We have

$$m((1-x) + (1-y)z) = \frac{1}{4\pi^2} \int_\gamma \eta_3(2)(x, y) + \eta_3(2)(y, x).$$

On the other hand,

$$\eta_3(2)(x, x) = \mathrm{d}\widehat{\mathcal{L}_3}(x).$$

We would like to apply Stokes' Theorem again. Observe that $\partial\Gamma = \{P(x,y,z) = 0\} \cap \{|x| = |y| = |z| = 1\}$. When $P \in \mathbb{R}[x,y,z]$, Γ can be thought as

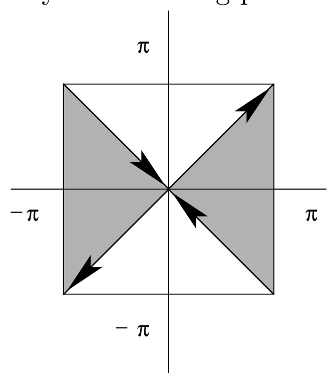
$$\gamma = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}.$$

Note that we are integrating now on a path inside the curve $C = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$. The differential form ω is defined in this new curve (this way of thinking the integral over a new curve has been proposed by Maillot). Now it makes sense to try to apply Stokes' Theorem.

Back to Smyth's case, in order to compute C we set $\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$ and we get $C = \{x = y\} \cup \{xy = 1\}$ in this example, and

$$-\{x\}_2 \otimes y - \{y\}_2 \otimes x = \pm 2\{x\}_2 \otimes x.$$

We integrate in the set described by the following picture



Then

$$m((1-x) + (1-y)z) = \frac{1}{4\pi^2} 8(\mathcal{L}_3(1) - \mathcal{L}_3(-1)) = \frac{7}{2\pi^2}\zeta(3).$$

New examples

Using this method we have been able to prove the following examples which were originally computed numerically by Boyd

$$m(x^2 + 1 + (x+1)y + (x-1)z) = \frac{\mathrm{L}(\chi_{-4}, 2)}{\pi} + \frac{21}{8\pi^2}\zeta(3),$$

$$m(x^2 + x + 1 + (x+1)y + z) = \frac{\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3}, 2) + \frac{19}{6\pi^2}\zeta(3).$$

An example in four variables

In [8] we computed this example

$$\pi^3 m\left(1 + x + \left(\frac{1 - x_1}{1 + x_1}\right)(1 + y)z\right) = 2\pi^2 \mathrm{L}(\chi_{-4}, 2) + 8 \sum_{0 \le j < k} \frac{(-1)^{j+k+1}}{(2j+1)^3 k}.$$

With this method we have been able to prove that

$$= 24\mathrm{L}(\chi_{-4}, 4).$$

In particular this implies

$$\sum_{0\leq j<k} \frac{(-1)^{j+k+1}}{(2j+1)^3 k} = 3\mathrm{L}(\chi_{-4},4) - \frac{\pi^2}{4}\mathrm{L}(\chi_{-4},2)$$

More generally, by using the Hurwitz zeta function we have been able to prove

$$\sum_{0\leq j<k} \frac{(-1)^{j+k+1}}{(2j+1)^m k} = m\mathrm{L}(\chi_{-4},m+1) + \sum_{h=1}^{\frac{m-1}{2}} \frac{(-1)^h \pi^{2h}(2^{2h}-1)}{(2h)!} B_{2h}\mathrm{L}(\chi_{-4},m-2h+1),$$

for m odd.

Generalized Mahler measure

Introduced by Gon & Oyanagi [4]

For $f_1,\dots,f_r \in \mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$,

$$m(f_1,\dots,f_r) = \frac{1}{(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n} \max\{\log|f_1|,\dots,\log|f_r|\}\frac{\mathrm{d}x_1}{x_1}\cdots\frac{\mathrm{d}x_n}{x_n}$$

Note that in particular,

$$m(f_1,f_2) = m(f_1 + zf_2).$$

Examples

There is a particular case. Fix $P\in\mathbb{C}[x]$ and set $f_j = P(x_j)$.

Gon & Oyanagi [4] computed the following example

$$m(1-x_1,\dots,1-x_{2m}) = \frac{(-1)^{m+1}(2m)!}{\pi^{2m}}\zeta(2m+1)$$

$$+(2m)!\sum_{j=1}^{m}(-1)^j \frac{(1-2^{2j})}{(2m-2j)!(2\pi)^{2j}}\zeta(2j+1),$$

$$m(1-x_1,\dots,1-x_{2m-1}) = (2m-1)!\sum_{j=1}^{m-1}(-1)^j\frac{(1-2^{2j})}{(2m-2j-1)!(2\pi)^{2j}}\zeta(2j+1).$$

Some particular cases are:

$$m(1-x_1,1-x_2) = m(1-x_1+z(1-x_2)) = \frac{7}{2\pi^2}\zeta(3),$$

$$m(1-x_1,1-x_2,1-x_3) = \frac{9}{2\pi^2}\zeta(3),$$

$$m(1-x_1,1-x_2,1-x_3,1-x_4) = -\frac{93}{2\pi^4}\zeta(5) + \frac{9}{\pi^2}\zeta(3).$$

This example can be also computed using regulators. Using that $|P(x)|$ is montononous when $0\leq \arg x\leq\pi$ (in this case, $|P(x)| = 2\left|\sin\frac{\arg x}{2}\right|$)

$$m(P(x_1),\dots,P(x_n)) = \frac{n!}{(\pi\mathrm{i})^n}\int_{0\leq\arg x_n\leq\dots\leq\arg x_1\leq\pi}\eta(P(x_1),x_1,\dots,x_n)$$

We have been able to also compute this example

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_{2m}}{1+x_{2m}}\right)=\frac{(-1)^{m+1}(2m)!(2^{2m+1}-1)}{(2\pi)^{2m}}\zeta(2m+1)$$

$$+(2m)!\sum_{j=1}^{m}(-1)^j\frac{(1-2^{2j+1})}{(2m-2j)!(2\pi)^{2j}}\zeta(2j+1),$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_{2m-1}}{1+x_{2m-1}}\right)=(2m-1)!\sum_{j=1}^{m-1}(-1)^j\frac{(1-2^{2j+1})}{(2m-2j-1)!(2\pi)^{2j}}\zeta(2j+1).$$

Some particular cases:

$$m\left(\frac{1-x_1}{1+x_1},\frac{1-x_2}{1+x_2}\right)=m\left(\frac{1-x_1}{1+x_1}+z\left(\frac{1-x_2}{1+x_2}\right)\right)=\frac{7}{\pi^2}\zeta(3),$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_3}{1+x_3}\right)=\frac{21}{2\pi^2}\zeta(3),$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_4}{1+x_4}\right)=-\frac{93}{\pi^4}\zeta(5)+\frac{21}{\pi^2}\zeta(3).$$

Finally, we computed the following

$m(1+x_1-x_1^{-1},\dots,1+x_n-x_n^{-1})=$ combination of polylogarithms.

Some particular cases include

$$m(1+x_1-x_1^{-1})=-\log(\varphi),$$

$$m(1+x_1-x_1^{-1},1+x_2-x_2^{-1})=\frac{1}{\pi^2}\operatorname{Re}(\mathrm{Li}_3(\varphi^2)-\mathrm{Li}_3(-\varphi^2)+\mathrm{Li}_3(\varphi^{-2})-\mathrm{Li}_3(-\varphi^{-2}))$$

for $\varphi=\frac{-1+\sqrt{5}}{2}$.

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