Problem 1:

(a) Show that 12 is a *unit* in ⏱$_{19}$ and find its *inverse*. [4]

(b) Using part (a), solve the equation [3]

$$12 \cdot x = 4 \text{ in } ⏱_{19}.$$

One solution suffices.

(c) Find a natural number $n \geq 13$ such that all of the numbers [3]

$$2, 3, 5, 7, 11, 13$$

become a unit in ⏱$_n$.
Give at least three other units in this ring.

Hint: Your solutions of (a) and (b) should be elements of ⏱$_{19} = \{1, \ldots, 19\}$. In this sense, -3 or 35 are **not** elements of ⏱$_{19}$.

Total: 10 points

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Problem 3:

Consider the set

$$\text{LinComb}(45, 18) = \{45 \cdot k + 18 \cdot l : k, l \in \mathbb{Z}\}$$

of all *integral linear combinations* of 45 and 18 and denote by

$$I = \text{LinComb}(45, 18) \cap 2\mathbb{Z}$$

the set of all *even* numbers in LinComb(45, 18).

(a) Show that I is an ideal in $\mathbb{Z}$. [7]

(b) Give a generator of I and write it as an integral linear combination of 45 and 18. [3]

Total: 10 points

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Problem 5:

For integers a and b which are not both 0 and integers c and d which are not both 0 assume that

$$\begin{aligned} a, b &\in \text{LinComb}(c, d) \\ c, d &\in \text{LinComb}(a, b), \end{aligned}$$

i.e. a and b can be integrally linearly combined from c and d and vice versa.

(a) Show that [6]

$$\gcd(a, b) = \gcd(c, d).$$

(b) Give an example for a, b, c and d that satisfy the conditions given above with $a \neq \pm c, d$ and $b \neq \pm c, d$. [2]

Total: 8 points

Problem 2:

Consider the set

$$S = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$$

of matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with real entries a, b, d.

(a) Show that S is closed under the usual addition and multiplication of matrices. [4]

(b) Give at least three regular elements in S. [3]

(c) Give the *zero element* and the *identity element* in S. [2]

(d) Decide whether S is a ring, whether it is an integral domain and whether it is a field. [6]

Hint: You can use without further proof that $M_{2x2}(\mathbb{R})$, the ring of 2-by-2-matrices with real entries is a ring.

Further, the inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by the matrix $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $ad - bc \neq 0$ Total: 15 points

Problem 4:

Let a be an integer whose square is divisible by

$$2006 \cdot p^7$$

where p denotes an *irreducible* number.

(a) Show that $p^4 \mid a$. [5]

(b) Show by an example that a need not be divisible by p^2 if its square only is divisible by $2006 \cdot p^2$. [2]

Hint: $2006 = 2 \cdot 17 \cdot 59$.

Total: 7 points